MAXIMAL SEMIGROUPS IN FINITELY GENERATED NILPOTENT GROUPS

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Abstract

We obtain a classification of maximal subsemigroups of finitely generated nilpotent groups. In the principal results of this paper we show that there is a one-to-one correspondence between these subsemigroups and the non trivial homomorphisms of the group in R .

1. Introduction

One of the problems that have improved the development of the theory of semigroups is to find under what conditions a semigroup S of a group G is a subgroup. This problem is closed related to finding the maximal semigroups of G . There are well known results in the theory for maximal semigroups with non-void interior in topological groups (see the Hilgert, Hofmann and Lawson [1] and Lawson [2]). For these semigroups an ultimate criterion is provided by the basic fact that the semigroup generated by a neighborhood of the identity is the identity component of the group.

In [7] it was proved that a generating proper subsemigroup of a finitely generated group is contained in a maximal subsemigroup. This allows us to consider, in the study of semigroups in finitely generated groups, those arguments used for maximal semigroups. The goal of this paper is to obtain a classification of maximal subsemigroups of finitely generated nilpotent groups. In the principal results of this paper (see Theorem 5) we show that there is a one-to-one correspondence between these subsemigroups and the non trivial homomorphisms of the group in R .

2. Generalities

In the theory of maximal semigroups an elementary, but useful result is:

Lemma 1.

(The Reduction Lemma) Let $\theta : G \rightarrow H$ be a onto homomorphism of groups and let S be a submonoid of H . Then, S is a maximal (resp. total resp. invariant) in H if and only if $\theta^{-1}(S)$ is maximal (resp. total resp. invariant) in G .

The proof of this lemma can be found in [1]. Taking into account that $T = \theta^{-1}(\theta(S))$, for any subset $T \subset G$, the assumption that θ is onto is really needed here.

From this Lemma we get a family of maximal semigroups in the free finitely generated groups. In fact: let F_n be the free group in the n generators $x_1, x_2, \dots, x_n$. If W is a word in $x_1, x_2, \dots, x_n$, and

$$W = x_{\nu_1}^{\alpha_1} x_{\nu_2}^{\alpha_2} \cdots x_{\nu_r}^{\alpha_r},$$

where the α_i are integers and $\nu_i \in \{1, 2, \dots, n\}$, then the exponent sum of

W on x_ν is the integer

$$\sigma_\nu = \sum_{\nu_i = \nu} \alpha_i.$$

The mapping

$$(2.1) \quad W \longmapsto \sigma(W) = (\sigma_1(W), \sigma_2(W), \dots, \sigma_n(W))$$

is a homomorphism from F_n , onto the additive group Z^n of all vectors

$$(\alpha_1, \alpha_2, \dots, \alpha_n)$$

with $\alpha_i \in Z$. Now, the maximal semigroups in Z^n are of the form

$$\{x \in Z^n : f(x) \geq 0\}$$

where $f : R^n \rightarrow R$ is a non trivial homomorphism (see [6], Corollary 4.3). Composing the homomorphisms σ with $f|_{Z^n}$ we get from the reduction Lemma the following result.

Proposition 2.

Let F_n be the free group on the n generators $x_1, x_2, \dots, x_n$. If

$$f : Z^n \longrightarrow R$$

is a non trivial homomorphism then

$$\{W \in F_n : f(\sigma(W)) \geq 0\}$$

is a maximal semigroup in F_n .

The purpose of this article is to study the converse of this proposition. In this case the limitation in using the Reduction Lemma comes from the fact that if M is a maximal semigroup in G nothing is known about $\theta(M)$ to be maximal in H .

The next lemma, useful for the analysis of the converse of Proposition 2, deals with the case where θ is not onto.

Lemma 3.

Let $f : G \rightarrow H$ be a non trivial homomorphism of groups and let M be a maximal G -invariant semigroup in H . Then

$$S = \{g \in G : f(g) \in M\}$$

is a maximal semigroup of G .

Proof. We have $S = f^{-1}(f(S))$ because M is maximal and $1 \in M$. By the Reduction Lemma the result is proved if we show that $f(S) = f(G) \cap M$ is a maximal semigroup in $f(G)$. Taking into account that M is maximal and G -invariant we obtain that M is total in H (cf. [2], Corollary 3.10) and M is archimedean (cf. [7], Lemma 3.2). Therefore, $f(S)$ is total and H -invariant in $f(G)$. Now, let us consider $x \in S$ and $g \in G$ such that $x^{-1} \notin S$. Since M is archimedean there exists $n \in \mathbb{N}$ such that $f(g)f(x)^n = f(gx^n) \in M$. Then $gx^n \in S$ which shows that $f(S)$ is archimedean in $f(G)$. Therefore $f(S)$ is maximal in $f(G)$ (cf. [7], Proposition 3.5).

3. The abelian case

The kernel of the homomorphism σ define in (2.1) is the commutator subgroup of F_n . Hence, the free abelian group A_n of rank n is isomorphic with Z^n . Now, Z^n is a lattice of R^n , hence their maximal semigroups are of the form $\tilde{S} \cap Z^n$, where $\tilde{S}$ is a maximal closed semigroup of R^n with non-void interior, (see [6], Corollary 4.3). By [1], Corollary V.5.24, the maximal closed semigroups of R^n , with non-void interior, are their half-spaces. Hence we obtained the following

Proposition 4.

Let A_n be the free abelian group of rank n and M a maximal semigroup of A_n . Then

$$M = \{W \in A_n : f(W) \geq 0\},$$

where $f : A_n \rightarrow R$ is a non-trivial homomorphism.

4. The nilpotent case

Let a and b be elements in a group G . The element

$$a^{-1}b^{-1}ab$$

is called the commutator of a and b and is denoted by (a, b) . If H and K are normal subgroups of G then the set of all elements (h, k) , where $h \in H$ and $k \in K$ generates a normal subgroup of G , denoted by (H, K) .

The subgroups G_n of the lower central series of G are defined recursively by

$$G_1 = G, G_{n+1} = (G_n, G), n = 1, 2, \dots$$

A group G for which $G_c \neq 1$ and $G_{c+1} = 1$ for some positive integer c is called nilpotent of class c .

The subgroup G_n is the verbal subgroup of G generated by the set of all commutators

$$((\cdots(g_1, g_2), g_3), \cdots), g_n)$$

where $g_i \in G$, $i = 1, 2, \dots, n$. A commutators of this type is a simple n -fold which we denote simply by

$$(g_1, g_2, \cdots g_n).$$

If F_n denote the free group on the n free generators $x_1, x_2, \dots, x_n$ and $c > 1$ is a positive integer then, the quotient group

$$\frac{F_n}{(F_n)_{c+1}}$$

is a nilpotent group of class c .

This group, which is denoted by $F_n^{(c)}$, is the free nilpotent group of rank n and class of nilpotence c . A presentation of $F_n^{(c)}$ is

$$\langle x_1, x_2, \dots, x_n, (x_1, x_2, \dots, x_{c+1}) \rangle$$

where $(x_1, x_2, \dots, x_{c+1})$ is a simple $(c+1)$ -fold commutator.

Keeping fixed c and n we denote $F_n^{(c)}$ simply by F . Now we are giving the characterization for the maximal semigroups of F . In section 5 we consider this problem for a general nilpotent finitely generated group.

If $F' = (F, F)$ denotes the commutator subgroup of F then F/F' is the free abelian group of rank n .

Let M be a maximal semigroup of F . By [1], Theorem V.5.29, we have that $F' = (F, F) \subset M$. Let

$$\theta : F \longrightarrow F/F'$$

be the canonical homomorphism. Since $M = \theta^{-1}(\theta(M))$, by the Reduction Lemma we have that $\theta(M)$ is a maximal semigroup in F/F' . Then, from Proposition 3, there is a non-trivial homomorphism

$$f : F/F' \longrightarrow \mathbf{R}$$

such that

$$\theta(M) = \{y \in F/F' : f(y) \geq 0\}.$$

If we consider the homomorphism $f \circ \theta$ we get:

Lemma 5.

Let F be a free finitely generated nilpotent group. If M is a maximal semigroup of F there is a non trivial homomorphism

$$\phi : F \longrightarrow R$$

such that

$$M = \{x \in F : \phi(x) \geq 0\}.$$

□

In the above lemma we note that $F' \subset \ker \phi$.

Now, let

$$f : F \longrightarrow \mathbf{R}$$

be a non trivial homomorphism and

$$M = \{x \in F : f(x) \geq 0\}.$$

Since $\mathbf{R}^+$ is a maximal invariant semigroup of R , by Lemma 2, M is maximal in F . From this and Lemma 4, we can deduce the following.

Theorem 6.

Let F be a free finitely generated nilpotent group and M a maximal semigroup in F . Then

$$M = \{x \in F : f(x) \geq 0\},$$

where $f : F \rightarrow \mathbf{R}$ is a non-trivial homomorphism.

□

5. Maximal semigroups in nilpotent finitely generated groups

Let H be a normal subgroup of the nilpotent group F of rank n and class of nilpotence c . If

$$\theta : F \longrightarrow F/H$$

is the canonical homomorphism and S is a maximal semigroup in F/H then $\theta^{-1}(S)$ is a maximal semigroup in F and, by Proposition 2.1, there is a non trivial homomorphism

$$f : F \longrightarrow \mathbf{R}$$

such that

$$\theta^{-1}(S) = \{x \in F : f(x) \geq 0\}$$

Since S is maximal $H \subset \ker f$. Therefore, there is a homomorphism

$$\bar{f} : F/H \longrightarrow \mathbf{R}$$

such that $f = \bar{f} \circ \theta$. Thus

$$\begin{aligned} S = \theta(\theta^{-1}(S)) &= \{\theta(x) : x \in F, f(x) \geq 0\} \\ &= \{\theta(x) : x \in F, \bar{f}(\theta(x)) \geq 0\} \end{aligned}$$

From this, Lemma 2 and the fact that every nilpotent finitely generated group is of the form F/H for some free nilpotent group F and some normal subgroup $H \subset F$, we get.

Theorem 7.

Let Γ be a nilpotent finitely generated group and let $S \subset \Gamma$ be a semigroup. Then, S is a maximal semigroup of Γ if and only if there is a non trivial homomorphism

$$f : \Gamma \longrightarrow \mathbf{R}$$

such that

$$S = \{x \in \Gamma : f(x) \geq 0\}.$$

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