CHARACTERIZATIONS OF A CLASS OF MATRIX TRANSFORMATIONS

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Abstract

One of the important investigations in the theory of summability is that of finding characterizations on an infinite matrix in order that the matrix should transform one sequence into another sequence space. In this note we present an abstract matrix transformation theorem. From it we can obtain the characterizations of a class of matrix transformations.

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Let Ω be an abstract non-empty set and G an Abelian topological group. If $f : \Omega \longrightarrow G$, we say that f satisfies condition (A) provided :

there exists $\omega_0 \in \Omega$ such that $f(\omega_0) = 0$.

If Ω is a Abelian group and f is a homomorphism, then f satisfies condition (A) with $\omega_0 = 0$.

Let $s(\Omega)$ be the set of all Ω -valued sequences. If Ω is a topological vector space $(TVS) X$, then $s(X)$ is a vector space, where the operations of addition and scalar multiplication are coordinatewise.

If $i, j \in N$ with $i \leq j$, let $[i, j] = \{k \in N; i \leq k \leq j\}$ be the interval in N induced by i and j . If $\{I_j\}$ is a sequence of intervals in N with $\max I_j < \min I_{j+1}$ for all j , we call $\{I_j\}$ an increasing sequence of intervals.

Let $E(\Omega) \subseteq s(\Omega)$, we say that $E(\Omega)$ has the ω_0 -Weak Gliding Humps Property (ω_0 -WGHP) if $\omega = (\omega_j) \in E(\Omega)$ and $\{I_k\}$ is an increasing sequence of intervals, then there exists a subsequence $\{p_k\}$ such that $\tilde{\omega} = (\tilde{\omega}_j) \in E(\Omega)$, where

$$\tilde{\omega}_j = \begin{cases} \omega_j, & j \in \bigcup_{k=1}^{\infty} I_{p_k}, \\ \omega_0, & \text{otherwise.} \end{cases}$$

Note that if $\Omega = X \in TVS$ and $E(X)$ is a vector subspace of $s(X)$, then the 0-WGHP is just the WGHP was introduced in ([11] or [15]).

Example 1. Let $E(\Omega) = s(\Omega)$, then for each $\omega_0 \in \Omega$, $E(\Omega)$ has the ω_0 -WGHP.

Example 2. Any monotone sequence space has 0-WGHP. So, if $X \in TVS$, then $c_{00}(X)$, the vector space of all X -value sequences which are 0 eventually, or $c_0(X)$, the vector space of all X -valued sequences which converge to 0, or $m_0(X)$, the vector space of all X -valued sequences with finite range, or $l^\infty(X)$, the vector space of all X -valued bounded sequences, all have the 0-WGHP. If $(X, |\cdot|)$ is a quasi-norm space and $0 < p < \infty$, then the space $l^p(X)$ consisting of all X -valued sequences such that $\sum_{k=1}^{\infty} |x_k|^p < \infty$ also has the 0-WGHP.

Recall that a sequence $\{x_k\}$ in a topological vector space X is K -convergent if each subsequence of $\{x_k\}$ has a further subsequence $\{x_{n_k}\}$ such that the series $\sum_k x_{n_k}$ is convergent to an element of X ([1]). A K -convergent sequence obviously converges to 0, but the converse is false in general although it does hold in complete metric linear spaces ([1] 3). X is said to be a K -space if each sequence which converges to 0 is K -convergent ([1] 3). Let $X \in TVS$ and $cs(X)$ be the vector space of all X -valued sequences $\{x_k\}$ such that $\sum_k x_k$ converges; if X is the scalar field, we write $cs(X) = cs$.

Example 3. ([16] 12.4). If $X \in TVS$ is a K -space, then $cs(X)$ has the 0-WGHP.

Let $f_i : \Omega \longrightarrow G$ be a sequence of functions, we say that the series $\sum_j f_j$ is $E(\Omega)$ -convergent if the series $\sum_j f_j(\omega_j)$ converges in G for each $(\omega_j) \in E(\Omega)$. If $E(\Omega)$ has the ω_0 -WGHP, we show that $E(\Omega)$ -convergence is equivalent to many of the familiar forms of convergence for series.

Example 4. Let $\sum x_j$ be a series in G . If $\Omega = \{0, 1\}$ and $f_i : \Omega \longrightarrow G$ is defined by $f_j(0) = 0$ and $f_j(1) = x_j$ then $\sum_j f_j$ being $s(\Omega)$ -convergent is equivalent to subseries convergence for the series $\sum x_j$.

Example 5. Let $X \in TVS$ and $\{x_j\} \subseteq X$. If Ω is the unit ball of scalar field C and $f_j : \Omega \longrightarrow X$ is defined by $f_j(t) = tx_j$, then $\sum_j f_j$ being $s(\Omega)$ -convergent is equivalent to the series $\sum x_j$ being bounded multiplier convergent ([6], [12], [13]). And, the series $\sum_j f_j$ being c_0 -convergent is equivalent to the series $\sum x_j$ being a C -series ([3], [5], [12]).

Example 6. Let X, Y be normed spaces and $\{T_j\}$ a sequence of continuous linear operators from X into Y . Let Ω be the unit ball of X

and define $f_j : \Omega \longrightarrow Y$ by $f_j(x) = T_j x$. Then $s(\Omega)$ -convergent for the series $\sum f_j$ is equivalent to the series $\sum T_j$ being bounded multiplier convergent ([2], [9], [10], [14]). And $c_0(X)$ -convergent for the series $\sum f_j$ is the series convergent considered in ([17]).

Example 7. Let X be a normed space and $\{x_j\} \subseteq X$. If Ω is the unit ball of X' and $f_j : \Omega \longrightarrow C$ is defined by $f_j(x') = x'(x_j)$, then $\sum f_j$ being $s(\Omega)$ -convergent is equivalent to the series $\sum x_j$ being absolutely convergent.

One of the tools used in the proofs below is the matrix theorem of Antosik-Mikusinski. We state it for the convenience of the reader.

Theorem 1. Let G be an Abelian topological group and $x_{ij} \in G$ for $i, j \in N$. If

(i) $\lim_i x_{ij} = x_j$ exists for each j and

(ii) for each increasing sequence of positive integers $\{m_k\}$ there is a subsequence $\{n_k\}$ of $\{m_k\}$ such that the sequence $\left\{ \sum_k x_{in_k} \right\}_i$ is Cauchy, then $\lim_i x_{ij} = x_j$ uniformly with respect to $j \in N$. In particular, $\lim_i x_{ii} = 0$.

A matrix $M = [x_{ij}]$ satisfying conditions (i) and (ii) is called a K -matrix. For proofs of Theorem 1 see ([1], [7]).

Our main results are as follows :

Theorem 2. Let $f_{ij} : \Omega \longrightarrow G$ for $i, j \in N$ and assume that each function f_{ij} satisfies condition (A) for the same $\omega_0 \in \Omega$ and $E(\Omega)$ has the ω_0 -WGHP. If for each $i, j \in N$ and $\omega \in \Omega$, the series $\sum_j f_{ij}$ is $E(\Omega)$ -convergent and $\lim_i f_{ij}(\omega)$ exists, then the following conditions are equivalent :

(1) If $\omega = (\omega_j) \in E(\Omega)$ and $\{I_k\}$ is an increasing sequence of intervals, then $\lim_i \sum_{j \in I_k} f_{ij}(\omega_j)$ exists for each $k \in N$ and uniformly with respect to $k \in N$.

(2) For each $\omega = (\omega_j) \in E(\Omega)$, the series $\sum_j f_{ij}(\omega_j)$ converge uniformly for $i \in N$.

(3) For each $\omega = (\omega_j) \in E(\Omega)$, $\left\{ \sum_j f_{ij}(\omega_j) \right\}_i$ is Cauchy.

Proof. If (1) holds and (2) fails, then there exist $\omega(\omega_j) \in E(\Omega)$ such that the series $\sum_j f_{ij}(\omega_j)$ do not converge uniformly with respect to $i \in N$, so there exists a neighbourhood U of 0 in G such that for each $k \in N$ we have n and i satisfying

$$(0.1) \quad \sum_{j=n}^{\infty} f_{ij}(\omega_j) \notin U$$

Se $k = 1$, there exists n_1 and i_1 such that $\sum_{j=n_1}^{\infty} f_{i_1 j}(\omega_j) \notin U$. Pick a symmetric neighbourhood V of 0 in G such that $V + V \subseteq U$. Since $\lim_k \sum_{j=k}^{\infty} f_{i,j}(\omega_j) = 0$ but $\sum_{j=n_1}^{\infty} f_{i_1 j}(\omega_j) \notin U$, there exists an integer $m_1 >$

n_1 such that $\sum_{j=n_1}^{m_1} f_{i_1 j}(\omega_j) \notin V$. Note that since $\sum_j f_{ij}(\omega_j)$ converges for each $i \in N$, there exists $m_0 \in N$ such that for each $m \geq m_0$ and $i \leq i_1$ we have $\sum_{j=m}^{\infty} f_{ij}(\omega_j) \in U$. Set $k = \max\{m_1, m_0\}$, from

(1) there exists $n_2 > k$ and i_2 such that $\sum_{j=n_2}^{\infty} f_{i_2 j}(\omega_j) \notin U$. That $i_2 > i_1$ is obvious. As above, there exists an integer $m_2 > n_2$ such that $\sum_{j=n_2}^{m_2} f_{i_2 j}(\omega_j) \notin V$. Continuing this construction inductively gives two integers sequence $n_1 \leq m_1 < n_2 \leq m_2 < n_3 \leq m_3 < \dots$ and $i_1 < i_2 \dots$ such that

$$(0.2) \quad \sum_{j=n_k}^{m_k} f_{i_k j}(\omega_j) \notin V.$$

Let $I_k = \{j : n_k \leq j \leq m_k\}$, then $\{I_k\}$ is a increasing sequence of intervals. Consider the infinite matrix $M = \left[\sum_{j \in I_k} f_{ipj}(\omega_j) \right] = [x_{pk}]$. For each $p \in N$ since $\sum_j f_{ipj}(\omega_j)$ converges, $\lim_k x_{pk} = 0$. From (1) if

follows that $\lim_p x_{pk}$ exists for each $k \in N$ and uniformly with respect to $k \in N$. Denote $\lim_p x_{pk} = x_k$; we show that $\lim_k x_k = 0$. In fact, pick a neighbourhood V_1 of 0 in G ; there exists a symmetric neighbourhood V_2 of 0 in G such that $V_2 + V_2 \subseteq V_1$. For V_2 , there exists $p_0 \in N$, such that when $p \geq p_0$ for all $k \in N$ we have $x_{pk} - x_k \in V_2$. In particular, $x_{p_0k} - x_k \in V_2$. For p_0 there exists $k_0 \in N$, such that when $k \geq k_0$ we have $x_{p_0k} \in V_2$. Thus we have $x_k = (x_k - x_{p_0k}) +$

(2) $\implies$ (3). Let $\omega = (\omega_j) \in E(\Omega)$. For each neighbourhood V_3 of 0 in G , there exists a symmetric neighbourhood V_4 of 0 such that $V_4 + V_4 + V_4 \subseteq V_3$. For V_4 , there exists an integer j_0 such that for all $i \in N$ we have $\sum_{j=j_0+1}^{\infty} f_{ikj}(\omega_j) \in V_4$. For j_0 , there exists $i_0 \in N$ such

that when $i, l \geq i_0$ we have $\sum_{j=1}^{j_0} f_{ij}(\omega_j) - \sum_{j=1}^{j_0} f_{lj}(\omega_j) \in V_4$. Thus, when

$i, l \geq i_0$ we have $\sum_{j=1}^{\infty} f_{ij}(\omega_j) - \sum_{j=1}^{\infty} f_{lj}(\omega_j) = \sum_{j=1}^{j_0} f_{ij}(\omega_j) - \sum_{j=1}^{j_0} f_{lj}(\omega_j) + \sum_{j=j_0+1}^{\infty} f_{ij}(\omega_j) - \sum_{j=j_0+1}^{\infty} f_{lj}(\omega_j) \in V_4 + V_4 + V_4 \subseteq V_3$. So, $\left\{ \sum_{j=1}^{\infty} f_{ij}(\omega_j) \right\}_i$ is

Cauchy.

(3) $\implies$ (1). Let $\omega = (\omega_j) \in E(\Omega)$ and $\{l_k\}$ be an increasing sequence of intervals. Consider the matrix $M_1 = \left[\sum_{j \in I_k} f_{ij}(\omega_j) \right] = [y_{ik}]$. From the hypothesis, $\lim_i y_{ik}$ exists for each $k \in N$. If $\{m_k\}$ is an increasing $\{n_k\}$ of $\{m_k\}$ such that $\tilde{\omega} = (\tilde{\omega}_j) \in E(\Omega)$. Where

$$\tilde{\omega}_j = \begin{cases} \omega_j, & j \in \bigcup_{k=1}^{\infty} l_{n_k}, \\ \omega_0, & \text{otherwise.} \end{cases}$$

Since $f_{ij}(\omega_0) = 0$ for $i, j \in N$, $\sum_{k=1}^{\infty} y_{mk} = \sum_{k=1}^{\infty} \sum_{j \in I_{n_k}} f_{ij}(\omega_j) = \sum_{j=1}^{\infty} f_{ij}(\tilde{\omega}_j)$. From (3) we have $\left\{ \sum_{k=1}^{\infty} y_{mk} \right\}_i$ is Cauchy. Thus, M_1 is a K-matrix. It follows from Theorem 1 that $\lim_i y_{ik} = \lim_i \sum_{j \in I_k} f_{ij}(\omega_j)$ exists for each $k \in N$ and uniformly with respect to $k \in N$. Q. E. D.

Let $c_\infty(\Omega)$ be the set of all Ω -valued sequences which are ω_0 eventually. From Theorem 2 we can present an abstract matrix transformation theorem.

Theorem 3. Let $f_{ij} : \Omega \longrightarrow G$ for $i, j \in N$ and assume that each function f_{ij} satisfies condition (A) for the same $\omega_0 \in \Omega$ and $E(\Omega)$ has the ω_0 -WGHP. If the series $\sum_j f_{ij}$ is $E(\Omega)$ -convergent for each $i \in N$, $E(\Omega) \supseteq c_\infty(\Omega)$, and G is sequentially complete, then the following conditions (4) and (5) are equivalent :

(4) (a) $\lim_i f_{ij}(\omega)$ exists for each $\omega \in \Omega$ and $j \in N$,

(b) for each $\omega = (\omega_j) \in E(\Omega)$ the series $\sum_j f_{ij}(\omega_j)$ convergence

uniformly for $i \in N$.

(5) $\lim_i \sum_j f_{ij}(\omega_j)$ exists for each $\omega = (\omega_j) \in E(\Omega)$.

Let E and F be two scalar sequence spaces and $A = (a_{ij})$ a infinite scalar matrix. If for each $x = (x_j) \in E$, $Ax = \left\{ \left(\sum_j a_{ij} x_j \right)_i \right\} \in F$, then we say that $A \in (E, F)$. One of the important investigations in the theory of summability is that of finding characterizations on an infinite matrix A in order that the matrix $A \in (E, F)$. Now, we can obtain the characterizations for (l^∞, c) , (cs, c) , (c_0, c) and (l^p, c) ($0 < p < \infty$).

Corollary 4 ([10]). $A \in (l^\infty, c)$ if and only if $\lim_i a_{ij}$ exists for each j and the series $\sum_j |a_{ij}|$ are uniformly convergent for $i \in N$.

Proof. $\Leftarrow$ is clear.

$\Rightarrow$ From Theorem 3 we have $\lim_i a_{ij}$ exists for each j . It is easy to show that for each bounded subset B of $(l^\infty, \|\cdot\|_\infty)$, the series $\sum_j a_{ij} x_j$ converge uniformly for $(x_j) \in B$ and $i \in N$ ([8]). Let $x^i = (\text{sign } a_{ij})_j$, then $\{x^i\}$ is a bounded subset of $(l^\infty, \|\cdot\|_\infty)$. Hence, $\sum_j a_{ij} \text{sign } a_{ij} = \sum_j |a_{ij}|$ converge uniformly for $i \in N$. Q. E. D.

Corollary 5 ([10]). $A \in (cs, c)$ if and only if $\lim_i a_{ij}$ exists for each j and

$$\sup_i \sum_j |a_{ij} - a_{i,j+1}| < \infty \cdots$$

Proof. For $x = (x_j) \in cs$, let $\|x\| = \sup_n \left| \sum_{j=1}^n x_j \right|$, then cs is a Banach space. Let bv be the space of all sequences $y = (y_j)$ for which $\sum_j |y_{j+1} - y_j| < \infty$. If $y = (y_j) \in bv$, let $\|y\| = \sum_j |y_{j+1} - y_j| + \lim_j |y_j|$, then bv is also a Banach space and, if $x \in cs$, $y \in bv$, $\sum_{j=1}^{\infty} x_j y_j$ is convergent, so that using this bilinear form bv can be identified with a space of linear functions on cs ; in fact bv corresponds to the topological dual of cs , and the norm on bv is the dual norm ([4]). Therefore, for each $i \in N$, $(a_{ij})_i \in bv$. From Theorem 3 and the uniform boundedness principle it follows that the necessity holds.

$\Leftarrow$ Let $(x_j) \in cs$, we show that $\sum_j a_{ij} x_j$ converge uniformly for $i \in N$. In fact, let $m > n$ and $s_k = \sum_{j=n}^k x_j$, then $\sum_{j=n}^m a_{ij} x_j = s_m a_{im} + \sum_{j=n}^{m-1} s_j (a_{ij} - a_{i,j+1})$. Therefore, for $m > n$ we have $\left| \sum_{j=n}^m a_{ij} x_j \right| \leq |s_m a_{im}| + \sum_{j=n}^{m-1} |s_j| (a_{ij} - a_{i,j+1})$. From the hypothesis it follows from Theorem 3 that $A \in (cs, c)$. Q. E. D.

Theorem 6. $A \in (c_0, c)$ if and only if $\lim_i a_{ij}$ exists for each $j \in N$ and $\sup_i \sum_j |a_{ij}| < \infty$.

Proof. $\Leftarrow$ is clear.

$\Rightarrow$ From Theorem 3 it follows that $\lim_i a_{ij}$ exists for each $j \in N$. Since $A \in (c_0, c)$, it is easily to see that for each $i \in N$, $\sum_j |a_{ij}| < \infty$. That

is $(a_{ij}) \in l^1$ for each $i \in N$. From the uniform boundedness principle it follows that $\sup_i \sum_j |a_{ij}| < \infty$. Q. E. D.

Theorem 7. If $p > 1$, then $A \in (l^p, c)$ if and only if $\lim_i a_{ij}$ exists for each $j \in N$ and $\sup_i \sum_j |a_{ij}|^q < \infty$. Here $q = \frac{p}{p-1}$.

Proof. $\Leftarrow$ Let $(x_j) \in l^p$. Then for each $i \in N$, $(a_{ij}) \in l^p$ and from Holder's inequality it follows that the series $\sum_j a_{ij} x_j$ converge uniformly for $i \in N$. Hence, $A \in (l^p, c)$.

$\Rightarrow$ From Theorem 3 we have $\lim_i a_{ij}$ exists for each $j \in N$. Since $\left\{ \sum_j a_{ij} x_j \right\}$ is a bounded sequence for each $(x_j) \in l^p$, therefore, by using the uniform boundedness principle we know that $\sup_i \sum_j |a_{ij}|^q < \infty$. Q. E. D.

Theorem 8. If $0 < p \leq 1$, then $A \in (l^p, c)$ if and only if $\lim_i a_{ij}$ exists for each $j \in N$ and $\sup_i \sup_j \{|a_{ij}|\} < \infty$.

Proof. $\Leftarrow$ is clear since $l^p (0 < p \leq 1) \subseteq l^1$.

$\Rightarrow$ If $\sup_i \sup_j \{|a_{ij}|\} = \infty$, then we can pick two integers sequences $i_1 < i_2 < \dots$ and $j_1 < j_2 < \dots$ such that $|a_{i_k j_k}| \geq k^{\frac{2}{p}}$. Let

$$x_j = \begin{cases} k^{\frac{2}{p}}, & j = j_k \\ 0, & \text{otherwise} \end{cases}$$

Then $(x_j) \in l^p$. For each $k_0 \in N$, since $|a_{i_{k_0} j_{k_0}} x_{j_{k_0}}| \geq 1$, the series $\sum_j a_{ij} x_j$ is not convergent uniformly for $i \in N$. This contradicts Theorem 3. Q. E. D.

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