A SHIL'NIKOV'S THEOREM IN AN EXTENDED NEIGHBORHOOD OF A SADDLE POINT OF FOCUS - FOCUS TYPE IN BANACH SPACES *

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Abstract

In this work, we will study the existence of chaotic solutions in an extended neighborhood of a homoclinic orbit for a kind of evolutions equations in Banach spaces of infinite dimension. First, we will prove that the solutions admit an exponential expansion and then we will imply the existence of a subsystem of solutions which is in one-to-one correspondence with the set of doubly infinite sequences.

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1. Introduction.

The first researchs connected with this problem, were made by Shil'nikov [10] [11], where he proved that in a neighborhood of a trajectory which is doubly asymptotic to a hyperbolic equilibrium point of a saddle-focus type, a dynamical system under certain conditions for the spectrum of the linear part, has a subsystem where the trajectories are in a one-to-one correspondence with a countable set of doubly infinite sequences of positive integers (Symbolic Dynamic), but, this was done only for a finite dimensional space. That is, if the spectrum of the linear part of the system has three eigenvalues, one of them real and the other two complex conjugates, such that their real parts are closer to the origin than the real parts of the other eigenvalues, then there exists a subsystem of solutions in one-to-one correspondence with sequences of the type $(\dots, k_n, k_{n+1}, \dots)$ with $k_n < \rho k_{n+1}$, $\rho > 1$, such that when it passes by a neighborhood of the fixed point, the orbit has k_n semi-turns around the point; it continues by a tubular neighborhood of a homoclinic orbit and when it returns to the neighborhood of the equilibrium point it has k_{n+1} semi-turns and so on. This result is known as Shil'nikov's Theorem.

Later, Blázquez and Tuma [2] extended these results to a Banach space. They showed an exponential expansion in the unstable manifold and proved Shil'nikov's Theorem.

Here, we will prove the existence of chaotic solutions in an extended neighborhood of the homoclinic orbit for an infinite dimensional dynamical system, such that, the linear part of the system has an infinite spectrum, and there exist four complex conjugates eigenvalues with their real parts closer to the origin, such that two of them have positive real part and the other two have negative real part. We will construct a Poincaré map T whose domain of definition is representable as a countable union of mutually disjoint domains on which T is a composition of two maps T_0 and T_1 , where T_0 is defined in a neighborhood of the origin and T_1 in a tubular neighborhood of the homoclinic orbit. Then, we prove Shil'nikov's Theorem in this case.

2. Presentation of the Problem.

We consider equation

$$(2.1) \quad \dot{z} + Az = f(z)$$

where A is a sectorial operator in a Banach Space X , and $f \in C^k(X^\alpha, X)$, $k \geq 3$, $0 \leq \alpha < 1$. We will assume that system (2.1) has the origin as an isolated equilibrium point, and its linear part $L = A - Df(0)$ satisfies the following conditions:

- (H1) The system has a finite number of eigenvalues $\gamma_1, \gamma_2, \dots, \gamma_n$ of L with negative real parts, γ_1 and γ_2 complex conjugates, and

$$\operatorname{Re} \gamma_i < \gamma = \operatorname{Re} \gamma_1 < 0 \quad i = 3, \dots, n.$$

- (H2) The remainder of the spectrum $\sigma(L)$ has positive real parts, and there exist two eigenvalues λ_1 and λ_2 complex conjugates, such that:

$$0 < \operatorname{Re} \lambda_1 = \lambda < \operatorname{Re} \mu$$

with $\mu \in \sigma(L) \setminus \{\gamma_i, \lambda_1, \lambda_2\}$ $i = 1, 2, \dots, n$.

- (H3) $\gamma + \lambda > 0$.

- (H4) There exists an isolated homoclinic orbit Γ_0 , issuing from 0 and returning to it, as $t \rightarrow +\infty$, i.e., $\Gamma_0 \subset (M^+ \cap M^-)$, where M^+ and M^- are the stable and unstable manifolds, respectively. Furthermore, it satisfies:

$$(2.2) \quad \dim(TM_p^+ \cap TM_p^-) = 1$$

where TM_p^+ and TM_p^- are the tangent spaces at $p \in \Gamma_0$ to the stable and unstable manifolds, respectively.

Definition 1. A focus-focus point is a saddle point which satisfies conditions (H1), and (H2). A small enough tubular neighborhood of the homoclinic orbit is called an extended neighborhood of the focus-focus point.

We consider the set of doubly infinite sequences

$$(\dots, j_i, j_{i+1}, \dots)$$

consisting of the symbols $0, 1, 2, \dots$ satisfying that for arbitrary adjacent indices $j_{i+1} < \rho j_i$ for some $\rho > 1$. We denote this set by $\Omega(\rho)$; i.e.:

$$\Omega(\rho) = \{ (\dots, j_i, j_{i+1}, \dots) / j_i \in \mathbf{N} \cup \{0\}, j_{i+1} < \rho j_i, \text{ for some } \rho > 1 \}.$$

Under conditions (H1)–(H5), where (H5) is defined in section 5, we prove the following theorem:

Theorem 1. For $\lambda + \gamma > 0$ there exists a subsystem of solutions of (2.1) in one-to-one correspondence with the set $\Omega(\rho)$ with $\rho > -\frac{\lambda}{\gamma} > 1$.

3. Exponential Expansion.

It is known that X^α can be decomposed as a direct sum of the spaces X_1^α , X_2^α , X_3^α , and X_4^α , corresponding to the spectral sets

$$\begin{aligned}\sigma_1 &= \{\lambda \pm i\omega\} & \sigma_2 &= \{\sigma(L) \cap \{Re\mu > 0\}\} \setminus \sigma_1 \\ \sigma_3 &= \{\gamma \pm i\Omega\} & \text{and } \sigma_4 &= \{\sigma(L) \cap \{Re\mu < 0\}\} \setminus \sigma_3\end{aligned}$$

System (2.1) can be reduced to

$$\begin{aligned}(3.1) \quad \dot{x} + L_1 x &= P_1(x, y, z, w) \\ \dot{y} + L_2 y &= P_2(x, y, z, w) \\ \dot{z} + L_3 z &= P_3(x, y, z, w) \\ \dot{w} + L_4 w &= P_4(x, y, z, w)\end{aligned}$$

where $L_i = L/X_i^\alpha$ with $i = 1, 2, 3$ and 4.

We consider the integral form

$$\begin{aligned}(3.2) \quad x(t) &= e^{-L_1 t} x_0 + \int_0^t e^{-L_1(t-s)} P_1(x(s), y(s), z(s), w(s)) ds \\ y(t) &= e^{-L_2 t} y_0 + \int_0^t e^{-L_2(t-s)} P_2(x(s), y(s), z(s), w(s)) ds \\ z(t) &= e^{-L_3(t-t_0)} z_1 + \int_{t_0}^t e^{-L_3(t-s)} P_3(x(s), y(s), z(s), w(s)) ds \\ w(t) &= e^{-L_4(t-t_0)} w_1 + \int_{t_0}^t e^{-L_4(t-s)} P_4(x(s), y(s), z(s), w(s)) ds\end{aligned}$$

Definition 2. The variables admit an exponential expansion of regularity r if there exist C^r functions, $\varphi_1 = \varphi_1(x_0, y_0, z_1, w_1)$, $\varphi_2 = \varphi_2(x_0, y_0, z_1, w_1)$, $\varphi_3 = \varphi_3(t_0, x_0, y_0, z_1, w_1)$, $\varphi_4 = \varphi_4(t_0, x_0, y_0, z_1, w_1)$, $R_1 = R_1(t, x_0, y_0, z_1, w_1)$, $R_2 = R_2(t, x_0, y_0, z_1, w_1)$, $R_3 = R_3(t, t_0, x_0, y_0, z_1, w_1)$, $R_4 = R_4(t, t_0, x_0, y_0, z_1, w_1)$, with $r \geq 1$ such that

$$\begin{aligned}x(t) &= e^{-L_1 t}(\varphi_1(x_0, y_0, z_1, w_1) + R_1(t, x_0, y_0, z_1, w_1)) \\ y(t) &= e^{-\lambda t}(\varphi_2(x_0, y_0, z_1, w_1) + R_2(t, x_0, y_0, z_1, w_1)) \\ z(t) &= e^{L_3(t_0-t)}(\varphi_3(t_0, x_0, y_0, z_1, w_1) + R_3(t, t_0, x_0, y_0, z_1, w_1)) \\ w(t) &= e^{\gamma(t_0-t)}(\varphi_4(t_0, x_0, y_0, z_1, w_1) + R_4(t, t_0, x_0, y_0, z_1, w_1))\end{aligned}$$

where

$$L_1 = \begin{pmatrix} \lambda & -\omega \\ \omega & \lambda \end{pmatrix} \quad L_3 = \begin{pmatrix} \gamma & -\Omega \\ \Omega & \gamma \end{pmatrix}$$

$$\frac{\partial \varphi_1}{\partial z_1}(0, 0, 0, 0) = 0 = \frac{\partial \varphi_1}{\partial w_1}(0, 0, 0, 0)$$

$$\frac{\partial \varphi_2}{\partial z_1}(0, 0, 0, 0) = 0 = \frac{\partial \varphi_2}{\partial w_1}(0, 0, 0, 0)$$

$$\frac{\partial \varphi_3}{\partial x_0}(t_0, 0, 0, 0, 0) = 0 = \frac{\partial \varphi_3}{\partial y_0}(t_0, 0, 0, 0, 0)$$

$$\frac{\partial \varphi_4}{\partial x_0}(t_0, 0, 0, 0, 0) = 0 = \frac{\partial \varphi_4}{\partial y_0}(t_0, 0, 0, 0, 0)$$

$$\frac{\partial \varphi_1}{\partial x_0}(0, 0, 0, 0) = Id_2 = \frac{\partial \varphi_3}{\partial z_1}(t_0, 0, 0, 0, 0)$$

and R_i , and their derivatives tend exponentially to zero as $t_0 \rightarrow +\infty$ with $i \in \{1, 2, 3, 4\}$.

Lemma 1. *There exists a change of variable such that system (3.1) can be reduced to*

$$\begin{aligned} \dot{x} + L_1 x &= P_{11}(x, y, z, w)x + P_{12}(x, y, z, w)y \equiv \tilde{P}_1(x, y, z, w) \\ \dot{y} + L_2 y &= P_{21}(x, y, z, w)x + P_{22}(x, y, z, w)y \equiv \tilde{P}_2(x, y, z, w) \end{aligned} \quad (3.3)$$

$$\dot{z} + L_3 z = P_{31}(x, y, z, w)z + P_{32}(x, y, z, w)w \equiv \tilde{P}_3(x, y, z, w)$$

$$\dot{w} + L_4 w = P_{41}(x, y, z, w)z + P_{42}(x, y, z, w)w \equiv \tilde{P}_4(x, y, z, w)$$

where P_{ij} satisfy the following conditions for $i \in \{1, 2, 3, 4\}$ and $j \in \{1, 2\}$

$$P_{21}(0, 0, z, w) = 0 \quad \left. \frac{\partial^{k+l} P_{21}}{\partial z^k \partial w^l}(x, y, z, w) \right|_{x=y=0} = 0$$

$$P_{3j}(0, 0, z, w) = 0 \quad \left. \frac{\partial^{k+l} P_{3j}}{\partial z^k \partial w^l}(x, y, z, w) \right|_{x=y=0} = 0$$

$$P_{i1}(x, y, 0, 0) = 0 \quad \left. \frac{\partial^{k+l} P_{i1}}{\partial z^k \partial w^l}(x, y, z, w) \right|_{z=w=0} = 0$$

for $i = 3, 4$ and $j = 1, 2$.

Proof: To obtain these conditions, we make the following change of variables:

$$\eta = x, \quad \theta = y - g(z, w)x, \quad \xi = z, \quad \nu = w$$

So, we have

$$\begin{aligned} \dot{\theta} &= [P_{21} - \dot{g} + gL_1 - gP_{11} - L_2g + P_{22}g - gP_{12}g]\eta + \\ &\quad + [-L_2 + P_{22} - gP_{12}]\theta \\ \dot{\theta} + L_2\theta &= [-\dot{g} + gL_1 - L_2g + P_{21} - gP_{11} + P_{22}g - gP_{12}g]\eta + \\ &\quad + [P_{22} - gP_{12}]\theta. \end{aligned}$$

Then, for $x = y = 0$, $P_{21}(0, 0, z, w)$ vanishes if

$$\begin{aligned} \dot{g} + L_2g - gL_1 &= P_{21}(0, 0, z, w) - gP_{11}(0, 0, z, w) + \\ &\quad + P_{22}(0, 0, z, w)g - gP_{12}(0, 0, z, w)g. \end{aligned}$$

The real part of $\sigma(L_2 - L_1)$ is positive, whereas the equations of z and w for $x = y = 0$ give that L_3 and L_4 have negative real part of their spectrum. So, there exists a local manifold given by $g = g(z, w)$, with $g(0, 0) = 0$, which satisfy the conditions required.

The other conditions can be found in a similar way in [2].

□

Now, for the new system, we have the following integral form:

$$\begin{aligned} x(t) &= e^{-L_1 t} x_0 + \int_0^t e^{-L_1(t-s)} \tilde{P}_1(x(s), y(s), z(s), w(s)) ds \\ y(t) &= e^{-L_2 t} y_0 + \int_0^t e^{-L_2(t-s)} \tilde{P}_2(x(s), y(s), z(s), w(s)) ds \\ (3.4) \quad z(t) &= e^{-L_3(t-t_0)} z_1 + \int_{t_0}^t e^{-L_3(t-s)} \tilde{P}_3(x(s), y(s), z(s), w(s)) ds \\ w(t) &= e^{-L_4(t-t_0)} w_1 + \int_{t_0}^t e^{-L_4(t-s)} \tilde{P}_4(x(s), y(s), z(s), w(s)) ds \end{aligned}$$

There exists a unique solution of this system as we mention in the following theorem, and an analogous proof can be found in [1].

Theorem 2. *Given $\delta > 0$, there exists a unique solution of (3.4), which satisfies $\|(x(t), y(t), z(t), w(t))\|_\alpha < \delta$, for $\|x_0\|_\alpha, \|y_0\|_\alpha, \|z_1\|_\alpha, \|w_1\|_\alpha < \frac{\delta}{4K}$ for all $0 \leq t \leq t_0$ where K is a constant depending on L .*

These solutions can be represented by:

$$\begin{aligned} x(t) &= \bar{x}(t, t_0, x_0, y_0, z_1, w_1) \\ y(t) &= \bar{y}(t, t_0, x_0, y_0, z_1, w_1) \\ z(t) &= \bar{z}(t, t_0, x_0, y_0, z_1, w_1) \\ w(t) &= \bar{w}(t, t_0, x_0, y_0, z_1, w_1) \end{aligned}$$

such that for $t = 0$ it passes through the point $M_0(x_0, y_0, z_0, w_0)$ where $z_0 = \bar{z}(0, t_0, x_0, y_0, z_1, w_1)$ and $w_0 = \bar{w}(0, t_0, x_0, y_0, z_1, w_1)$ and, for $t = t_0$ it passes through the point $M_1(x_1, y_1, z_1, w_1)$ where $x_1 = \bar{x}(t_0, t_0, x_0, y_0, z_1, w_1)$ and $y_1 = \bar{y}(t_0, t_0, x_0, y_0, z_1, w_1)$.

Lemma 2. *For $\|x_0\|_\alpha, \|y_0\|_\alpha, \|z_1\|_\alpha, \|w_1\|_\alpha \leq \delta$ and $t_0 > 0$, we have:*

$$\begin{aligned} \|x(t)\|_\alpha &\leq K e^{-\lambda t} \\ \|y(t)\|_\alpha &\leq K e^{-(\lambda+\beta)t} \\ (3.5) \quad \|z(t)\|_\alpha &\leq K e^{\gamma(t_0-t)} \\ \|w(t)\|_\alpha &\leq K e^{(\gamma+\sigma)(t_0-t)} \end{aligned}$$

for $0 \leq t \leq t_0$, $K > 0$ is a constant, $Re \gamma_i - \gamma < \sigma < 0$, $0 < \beta < -\lambda + Re \mu$, for $i = 3, 4, \dots, n$.

Proof: Let us define

$$I = \{ (x(t), y(t), z(t), w(t)) / x(t), y(t), z(t), w(t) \text{ satisfy inequalities (3.5)} \}.$$

I is a complete metric space with metric given by

$$\begin{aligned} d((x_1, y_1, z_1, w_1); (x_2, y_2, z_2, w_2)) &= \\ &= \|x_1 - x_2\|_\alpha + \|y_1 - y_2\|_\alpha + \|z_1 - z_2\|_\alpha + \|w_1 - w_2\|_\alpha \end{aligned}$$

We define an operator N on I by

$$N : I \longrightarrow I$$

such that

$$N(x, y, z, w) = (N_1(x, y, z, w), N_2(x, y, z, w), N_3(x, y, z, w), N_4(x, y, z, w))$$

where N_1 , N_2 , N_3 , and N_4 are the right-hand sides of (3.4).

We prove that N is well defined and that it is a contraction. We only show for N_2 , the proof of these facts for the other components are similar.

So, for N_2 , we have:

$$\begin{aligned} N_2(x, y, z, w) &= e^{-L_2 t} y_0 + \int_0^t e^{-L_2(t-s)} \tilde{P}_2(x(s), y(s), z(s), w(s)) ds \\ \|N_2(x, y, z, w)\|_\alpha &\leq K e^{-(\lambda+\beta)t} \|y_0\|_\alpha + \\ &\quad + \int_0^t K e^{-(\lambda+\beta)(t-s)} (t-s)^\alpha \|\tilde{P}_2(x(s), y(s), z(s), w(s))\| ds \\ &\leq e^{-(\lambda+\beta)t} \left(K\delta + K \int_0^t e^{(\lambda+\beta)s} (t-s)^\alpha (\|P_{21}\| \|x\|_\alpha + \|P_{22}\| \|y\|_\alpha) ds \right) \\ &\leq e^{-(\lambda+\beta)t} \left(K\delta + K^2 \int_0^t (t-s)^{-\alpha} \|P_{21}\| e^{\beta s} ds \right) + \\ &\quad + e^{-(\lambda+\beta)t} K^2 \int_0^t (t-s)^{-\alpha} \|P_{22}\| ds \end{aligned}$$

As P_{21} can be factored by x , then

$$\|P_{21}\| \leq M(\delta) \|x\|_\alpha (\|x\|_\alpha + \|z\|_\alpha)$$

So,

$$\begin{aligned} \|N_2(x, y, z, w)\|_\alpha &\leq \\ &\leq e^{-(\lambda+\beta)t} \left(K\delta + K^2 M(\delta) \int_0^t (t-s)^{-\alpha} \|x\|_\alpha (\|x\|_\alpha + \|z\|_\alpha) e^{\beta s} ds \right) \\ &\leq e^{-(\lambda+\beta)t} \left(K\delta + K^4 M(\delta) \int_0^t (t-s)^{-\alpha} (e^{-(\lambda-\beta)s} (e^{-\lambda s} + e^{\gamma(t_0-s)})) ds \right) \end{aligned}$$

as $\lambda - \beta > 0$ and the expression in parenthesis is bounded and less than K , i.e.,

$$\|N_2(x, y, z, w)\|_\alpha \leq K e^{-(\lambda+\beta)t}.$$

Now, we prove that N_2 is a contraction.

$$\begin{aligned}
 & \|N_2(x_1, y_1, z_1, w_1) - N_2(x_2, y_2, z_2, w_2)\|_\alpha = \\
 & = \left\| \int_0^t e^{-L_2(t-s)} (\tilde{P}_2(x_1, y_1, z_1, w_1) - \tilde{P}_2(x_2, y_2, z_2, w_2)) ds \right\|_\alpha \\
 & \leq \int_0^t K e^{-(\lambda+\beta)(t-s)} (t-s)^{-\alpha} M(\delta) d((x_1, y_1, z_1, w_1); (x_2, y_2, z_2, w_2)) ds \\
 & \leq q_2 d((x_1, y_1, z_1, w_1); (x_2, y_2, z_2, w_2))
 \end{aligned}$$

with $0 < q_2 < 1$ for an appropriate small enough δ . Thus, there exists a fixed point for N in I . □

Lemma 3. For $0 \leq t \leq t_0$,

$$\begin{aligned}
 (3.6) \quad & \left\| \frac{\partial x(t)}{\partial t_0} \right\|_\alpha \leq K e^{-\lambda t} \\
 & \left\| \frac{\partial y(t)}{\partial t_0} \right\|_\alpha \leq K e^{-(\lambda+\beta)t} \\
 & \left\| \frac{\partial z(t)}{\partial t_0} \right\|_\alpha \leq K e^{\gamma(t_0-t)} \\
 & \left\| \frac{\partial w(t)}{\partial t_0} \right\|_\alpha \leq K e^{(\gamma+\sigma)(t_0-t)}
 \end{aligned}$$

where K is a constant, $Re \gamma_i - \gamma < \sigma < 0$, $0 < \beta < -\lambda + Re \mu$, for $i = 3, 4, \dots, n$.

Proof: For the sake of brevity we prove it only for the forth variable, since similar procedures apply to the other bounds.

Let us define

$$J = \{(x(t), y(t), z(t), w(t)) / x(t), y(t), z(t), w(t) \text{ satisfy inequalities (3.6)}\}$$

and O an operator on J defined by:

$$O(x, y, z, w) = (O_1(x, y, z, w), O_2(x, y, z, w), O_3(x, y, z, w), O_4(x, y, z, w))$$

where O_1 , O_2 , O_3 , and O_4 are the right hand sides of (3.4).

We show that O_4 is well defined and that it is a contraction.

$$O_4(x, y, z, w) = e^{L_4(t_0-t)} w_1 + \int_{t_0}^t e^{L_4(s-t)} \tilde{P}_4(x(s), y(s), z(s), w(s)) ds$$

$$\begin{aligned} \left\| \frac{\partial O_4}{\partial t_0} \right\|_{\alpha} &\leq \|L_4 e^{L_4(t_0-t)}\| \|w_1\|_{\alpha} + \|e^{L_4(t_0-t)}\| \|\tilde{P}_4(x(t_0), y(t_0), z(t_0), w(t_0))\| + \\ &+ \int_{t_0}^t \|e^{L_4(s-t)}\| \left\{ \left\| \frac{\partial \tilde{P}_4}{\partial x} \right\| \left\| \frac{\partial x}{\partial t_0} \right\|_{\alpha} + \left\| \frac{\partial \tilde{P}_4}{\partial y} \right\| \left\| \frac{\partial y}{\partial t_0} \right\|_{\alpha} + \right. \\ &\left. + \left\| \frac{\partial \tilde{P}_4}{\partial z} \right\| \left\| \frac{\partial z}{\partial t_0} \right\|_{\alpha} + \left\| \frac{\partial \tilde{P}_4}{\partial w} \right\| \left\| \frac{\partial w}{\partial t_0} \right\|_{\alpha} \right\} ds. \end{aligned}$$

Using the fact that $\tilde{P}_4 = P_{41}z + P_{42}w$ and replacing, we have

$$\begin{aligned} \left\| \frac{\partial O_4}{\partial t_0} \right\|_{\alpha} &\leq e^{(\gamma+\sigma)(t_0-t)} \left\{ K\delta + M(\delta) + \int_{t_0}^t K^3(s-t)^{-\alpha} \left\{ \left\| \frac{\partial P_{41}}{\partial x} \right\| e^{-\lambda s + \sigma(s-t_0)} + \right. \right. \\ &+ \left\| \frac{\partial P_{41}}{\partial y} \right\| e^{-(\lambda+\beta)s + \sigma(s-t_0)} + \left\| \frac{\partial P_{42}}{\partial x} \right\| e^{-\lambda s} + \left\| \frac{\partial P_{42}}{\partial y} \right\| e^{-(\lambda+\beta)s} + \\ &+ \left(\left\| \frac{\partial P_{41}}{\partial z} \right\| e^{\sigma(s-t_0)} + \left\| \frac{\partial P_{41}}{\partial w} \right\| \right) e^{\gamma(t_0-s)} + \\ &+ \left(\left\| \frac{\partial P_{42}}{\partial z} \right\| e^{\sigma(s-t_0)} + \left\| \frac{\partial P_{42}}{\partial w} \right\| \right) e^{(\gamma+\sigma)(t_0-s)} \left. \right\} ds + \\ &+ \int_{t_0}^t K^2(s-t)^{-\alpha} \{ \|P_{41}\| e^{\sigma(s-t_0)} + \|P_{42}\| \} ds \}. \end{aligned}$$

But $\|P_{41}\| \leq M(\delta)\|z\|_{\alpha}(\|x\|_{\alpha} + \|z\|_{\alpha})$, so $\left\| \frac{\partial P_{41}}{\partial x} \right\|, \left\| \frac{\partial P_{41}}{\partial y} \right\| \leq M(\delta)\|z\|_{\alpha}(\|x\|_{\alpha} + \|z\|_{\alpha}) \leq M(\delta)\|z\|_{\alpha}$ and as $\gamma - \sigma < 0$, both integrals of the last inequality are bounded, so the expression in parenthesis is less than K , i.e.,

$$\left\| \frac{\partial O_4}{\partial t_0} \right\|_{\alpha} \leq K e^{(\gamma+\sigma)(t_0-t)}$$

O_4 is a contraction, because

$$\begin{aligned}
 & \|O_4(x_1, y_1, z_1, w_1) - O_4(x_2, y_2, z_2, w_2)\|_\alpha \leq \\
 & \leq \int_{t_0}^t KM(\delta) e^{(\gamma+\sigma)(s-t)} (s-t)^\alpha d((x_1, y_1, z_1, w_1); (x_2, y_2, z_2, w_2)) ds \\
 & = KM(\delta) d((x_1, y_1, z_1, w_1); (x_2, y_2, z_2, w_2)) \int_{t_0}^t e^{(\gamma+\sigma)(s-t)} (s-t)^\alpha ds \\
 & = KM(\delta) \frac{\Gamma(1-\alpha)}{-(\gamma+\sigma)^{1-\alpha}} d((x_1, y_1, z_1, w_1); (x_2, y_2, z_2, w_2)) \\
 & = q_4 d((x_1, y_1, z_1, w_1); (x_2, y_2, z_2, w_2))
 \end{aligned}$$

with $0 < q_4 < 1$ for an appropriate small enough δ . Thus, there exists a fixed point for O .

□

If we denote by

$$\begin{pmatrix} Z \\ W \end{pmatrix} = e^{-L^*(t_0-t)} \begin{pmatrix} z \\ w \end{pmatrix} \quad , \quad L^* = \begin{pmatrix} L_3 & 0 \\ 0 & L_4 \end{pmatrix}$$

in these new variables we obtain the following lemma.

Lemma 4. *There exists a constant $K_2 > 0$, such that*

$$\left\| \frac{\partial Z}{\partial t_0} \right\|_\alpha, \left\| \frac{\partial W}{\partial t_0} \right\|_\alpha \leq K_2 e^{\sigma(t_0-t)}$$

Remark: The proof is similar to the one in [2].

Theorem 3. *There exists a change of variables such that the variables admit an exponential expansion of regularity $r - 2$.*

Proof: We have

$$\|x(t)\|_\alpha \leq K e^{-\lambda t} \quad \text{and} \quad \|P_{1j}\| \leq M(\delta)(\|x\|_\alpha + \|z\|_\alpha) \quad j = 1, 2.$$

As

$$x(t) = e^{-L_1 t} x_0 + \int_0^t e^{-L_1(t-s)} \tilde{P}_1(x(s), y(s), z(s), w(s)) ds$$

so

$$\|x(t)\|_\alpha e^{\lambda t} \leq \|x_0\|_\alpha + M(\delta) \int_0^t (\|x\|_\alpha + \|z\|_\alpha) ds$$

We define

$$\varphi_1(x_0, y_0, z_1, w_1) = \lim_{t \rightarrow \infty} x(t) e^{L_1 t}$$

$$\|\varphi_1(x_0, y_0, z_1, w_1)\| \leq \|x_0\|_\alpha + M(\delta) \int_0^\infty (e^{-\lambda s} + e^{\gamma(t_0-s)}) ds$$

and

$$R_1(t_0, x_0, y_0, z_1, w_1) = e^{-L_1 t} (x(t) e^{L_1 t} - \varphi_1(x_0, y_0, z_1, w_1))$$

Analogously

$$\|y(t)\|_\alpha \leq K e^{-(\lambda+\beta)t}$$

and

$$\|P_{21}\| \leq M(\delta) \|x\|_\alpha (\|x\|_\alpha + \|z\|_\alpha) \quad \text{and} \quad \|P_{22}\| \leq M(\delta) (\|x\|_\alpha + \|z\|_\alpha)$$

As

$$y(t) = e^{-L_2 t} y_0 + \int_0^t e^{-L_2(t-s)} \tilde{P}_2(x(s), y(s), z(s), w(s)) ds$$

$$\|y(t)\|_\alpha e^{\lambda t} \leq \|y_0\|_\alpha e^{-\beta t} + M(\delta) \int_0^t e^{-\beta(t-s)} (\|x\|_\alpha + \|z\|_\alpha) ds$$

Let us define

$$\varphi_2(x_0, y_0, z_1, w_1) = \lim_{t \rightarrow \infty} y(t) e^{\lambda t}$$

$$\|\varphi_2(x_0, y_0, z_1, w_1)\| \leq \|y_0\|_\alpha e^{-\beta t} + M(\delta) \int_0^\infty (e^{-\lambda s} + e^{\gamma(t_0-s)}) ds$$

and

$$R_2(t_0, x_0, y_0, z_1, w_1) = e^{-\lambda t} (y(t) e^{\lambda t} - \varphi_2(x_0, y_0, z_1, w_1))$$

Now, from lemma 4 we know that

$$\overline{Z} = \begin{pmatrix} DZ(t) \\ DW(t) \end{pmatrix}$$

is Cauchy in t_0 , and that it converges uniformly for t_0, x_0, y_0, z_1, w_1 on a compact set, because

$$\begin{aligned} \|\bar{Z}(t, t'_0, x_0, y_0, z_1, w_1) - \bar{Z}(t, t_0, x_0, y_0, z_1, w_1)\|_\alpha &\leq \\ &\leq \int_{t_0}^{t'_0} \left\| \frac{\partial}{\partial T} \bar{Z}(t, T, x_0, y_0, z_1, w_1) \right\|_\alpha dT \\ &\leq \int_{t_0}^{t'_0} k_2 e^{\sigma(T-t)} dT \\ &\leq \left| \frac{K_2}{\sigma} \right| e^{\sigma(t_0-t)} \longrightarrow 0 \quad \text{as } t_0 \longrightarrow \infty. \end{aligned}$$

So, there exists

$$\varphi(t, x_0, y_0, z_1, w_1) = \lim_{t_0 \rightarrow \infty} \bar{Z}(t, t_0, x_0, y_0, z_1, w_1)$$

and

$$R(t, t_0, x_0, y_0, z_1, w_1) = e^{L^*(t_0-t)} (\bar{Z}(t, t_0, x_0, y_0, z_1, w_1) - \varphi(t, x_0, y_0, z_1, w_1))$$

where $\varphi = (\varphi_3, \varphi_4)$, and $R = (R_3, R_4)$.

Moreover, if $\bar{Z}_1 = \begin{pmatrix} z_1 \\ w_1 \end{pmatrix}$, then we have that

$$\frac{\partial \bar{Z}}{\partial \bar{Z}_1}(t, t_0, 0, 0, 0, 0) = \begin{pmatrix} e^{(L_1-L_3)(t_0-t)} \\ e^{(L_2-L_4)(t_0-t)} \end{pmatrix} \longrightarrow \text{diag}(Id_2, 0) \quad \text{as } t_0 \longrightarrow \infty.$$

The last property follows from the definitions of R_i , for $i \in \{1, 2, 3, 4\}$.

□

4. The Return Map.

The existence of local stable and unstable manifolds in the equilibrium point, implies the existence of Liapunov functionals $V_s(x, y)$ and $V_u(z, w)$. We define

$$\Sigma^+ = \{ (x, y, z, w) / V_s(x, y) = r_0, V_u(z, w) \leq r_0 \}$$

$$\Sigma^- = \{ (x, y, z, w) / V_s(x, y) \leq r_0, V_u(z, w) = r_0 \}$$

where $r_0 > 0$ is chosen such that their surfaces are without contact for the system.

T_0 will be the correspondence between Σ^+ and Σ^- defined by the trajectories at $t = 0$ passing through the point $M_0(x_0, y_0, z_0, w_0) \in \Sigma^+$, with $(z_0, w_0) \neq (0, 0)$ and for some $t_0 > 0$ they intersect Σ^- in $M_1(x_1, y_1, z_1, w_1)$, i.e.,

$$\begin{aligned}
 T_0(x_0, y_0, z_0, w_0) &= (x_1, y_1, z_1, w_1) \\
 &= (x(t_0, x_0, y_0, z_0, w_0), y(t_0, x_0, y_0, z_0, w_0), \\
 &\quad z(t_0, x_0, y_0, z_0, w_0), w(t_0, x_0, y_0, z_0, w_0)) \\
 (4.1) \quad &\text{with } V_s(x_0, y_0) = V_u(z_1, w_1) = r_0.
 \end{aligned}$$

If we choose x_0, y_0, z_1 , and w_1 , such that they satisfy (4.1), then the relations

$$\begin{aligned}
 x_1 &= \bar{x}(t_0, t_0, x_0, y_0, z_1, w_1) \equiv x^p(t_0, x_0, y_0, z_1, w_1) \\
 y_1 &= \bar{y}(t_0, t_0, x_0, y_0, z_1, w_1) \equiv y^p(t_0, x_0, y_0, z_1, w_1) \\
 z_0 &= \bar{z}(0, t_0, x_0, y_0, z_1, w_1) \equiv z^p(t_0, x_0, y_0, z_1, w_1) \\
 w_0 &= \bar{w}(0, t_0, x_0, y_0, z_1, w_1) \equiv w^p(t_0, x_0, y_0, z_1, w_1)
 \end{aligned}$$

define a parametric form of T_0 from $(t_0, x_0, y_0, z_1, w_1)$ to $(t_0, x_1, y_1, z_0, w_0)$.

Let $M_0(x^+, y^+, 0, 0)$ be the point of intersection of the orbit Γ_0 with Σ^+ . We define in Σ^+ a neighborhood

$$U_0 = \{ (x_0, y_0, z_0, w_0) / \|x_0 - x^+\|_\alpha, \|y_0 - y^+\|_\alpha, \|z_0\|_\alpha, \|w_0\|_\alpha < \epsilon_0, \} \cap \Sigma^+$$

and from the assumption that Γ_0 tends to the origin tangent to a 2-dimensional manifold when $t \rightarrow -\infty$, we can choose r_0 and ϵ_1 , such that $\Gamma_0 \cap \Sigma^- = M_1(0, 0, z^-, w^-)$ and an ϵ_1 -neighborhood U_1 in Σ^- such that Γ_0 intersects U_1 only in M_1 .

If

$$U_{(x,y)} = \{ (x_0, y_0) / (\exists (z_0, w_0)) (\|z_0\|_\alpha, \|w_0\|_\alpha \leq r_0, (x_0, y_0, z_0, w_0) \in U_0) \}$$

$$U_{(z,w)} = \{ (z_1, w_1) / (\exists (x_1, y_1)) (\|x_1\|_\alpha, \|y_1\|_\alpha \leq r_0, (x_1, y_1, z_1, w_1) \in U_1) \}$$

then, for each given ϵ_0, ϵ_1 , we can find $t(\epsilon_0, \epsilon_1)$ such that for $(x_0, y_0) \in U_{(x,y)}$ and $(z_1, w_1) \in U_{(z,w)}$, we have

$$\|x_1\|_\alpha = \|x^p\|_\alpha \leq \epsilon_1 \quad \text{and} \quad \|y_1\|_\alpha = \|y^p\|_\alpha \leq \epsilon_1$$

$$\|z_0\|_\alpha = \|z^p\|_\alpha \leq \epsilon_0 \quad \text{and} \quad \|w_0\|_\alpha = \|w^p\|_\alpha \leq \epsilon_0$$

for $t_0 > t(\epsilon_0, \epsilon_1)$, i.e.,

$$T_0(U_0) \cap U_1 \neq \emptyset.$$

To define T_1 in a tubular neighborhood of orbit Γ_0 , we consider that the trajectories passing through points on Σ^- close to M_1 will intersect Σ^+ at points close to M_0 . This correspondence is denoted by T_1 . We can choose a suitable $\epsilon_1 \geq 0$, such that $U_1 \subseteq D(T_1)$, i.e.,

$$T_1(x_1, y_1, z_1, w_1) = (\bar{x}_0, \bar{y}_0, \bar{z}_0, \bar{w}_0)$$

where $(x_1, y_1, z_1, w_1) \in U_1$ and $(\bar{x}_0, \bar{y}_0, \bar{z}_0, \bar{w}_0) \in \Sigma^+$, with $\bar{x}_0 = Q_1(x_1, y_1, z_1, w_1)$, $\bar{y}_0 = Q_2(x_1, y_1, z_1, w_1)$, $\bar{z}_0 = Q_3(x_1, y_1, z_1, w_1)$, and $\bar{w}_0 = Q_4(x_1, y_1, z_1, w_1)$.

Now, we introduce a local system of coordinates in U_0 and in $U_1 \subseteq D(T_1)$, respectively, by

$$x_0 = x^+ + \Delta x, y_0 = y^+ + \Delta y, z_0 = z_0 \quad \text{and} \quad w_0 = w_0$$

and

$$x_1 = x_1, y_1 = y_1, z_1 = z^- + \Delta z \quad \text{and} \quad w_1 = w^- + \Delta w$$

Since $\left(\frac{\partial V_u}{\partial(z_1^k, w_1^k)} \right)_{M_1} \neq 0$ for some component z_1^k of z_1 and w_1^k of w_1 , we can express z_1^k and w_1^k in terms of $\Delta' z$ and $\Delta' w$ respectively, corresponding to the other components of Δz and Δw . So, in the new system

$$\begin{aligned} \bar{\Delta}x &= \bar{Q}_1(x_1, y_1, \Delta' z_1, \Delta' w_1) \\ \bar{\Delta}y &= \bar{Q}_2(x_1, y_1, \Delta' z_1, \Delta' w_1) \\ \bar{z}_0 &= \bar{Q}_3(x_1, y_1, \Delta' z_1, \Delta' w_1) \\ \bar{w}_0 &= \bar{Q}_4(x_1, y_1, \Delta' z_1, \Delta' w_1) \end{aligned}$$

or equivalently

$$\begin{aligned} \bar{\Delta}x &= (B_{11} + \cdots)x_1 + (B_{12} + \cdots)y_1 + (C_{11} + \cdots)\Delta' z_1 + (C_{12} + \cdots)\Delta' w_1 \\ \bar{\Delta}y &= (B_{21} + \cdots)x_1 + (B_{22} + \cdots)y_1 + (C_{21} + \cdots)\Delta' z_1 + (C_{22} + \cdots)\Delta' w_1 \\ (4.2) \quad \bar{z}_0 &= (B_{31} + \cdots)x_1 + (B_{32} + \cdots)y_1 + (C_{31} + \cdots)\Delta' z_1 + (C_{32} + \cdots)\Delta' w_1 \\ \bar{w}_0 &= (B_{41} + \cdots)x_1 + (B_{42} + \cdots)y_1 + (C_{41} + \cdots)\Delta' z_1 + (C_{42} + \cdots)\Delta' w_1 \end{aligned}$$

with $V_s(x^+ + \bar{\Delta}x, y^+ + \bar{\Delta}y) = V_u(z_1, w_1) = r_0$ and the dots denote the terms which tend to zero as $x_1, y_1, \Delta'z$, and $\Delta'w$ go to zero.

Then, according to the assumption for ϵ_1 we can define T as the composition

$$T_1 \circ T_0 : D(T_0) \longrightarrow \Sigma^+$$

and it is well defined by

$$\begin{aligned} x_1 &= x(t_0, x_0, y_0, z_0, w_0) & , & \quad y_1 = y(t_0, x_0, y_0, z_0, w_0) \\ z_1 &= z(t_0, x_0, y_0, z_0, w_0) & , & \quad w_1 = w(t_0, x_0, y_0, z_0, w_0) \\ \bar{x}_0 &= Q_1(x_1, y_1, z_1, w_1) & , & \quad \bar{y}_0 = Q_2(x_1, y_1, z_1, w_1) \\ \bar{z}_0 &= Q_3(x_1, y_1, z_1, w_1) & , & \quad \bar{w}_0 = Q_4(x_1, y_1, z_1, w_1) \end{aligned}$$

with

$$V_s(x_0, y_0) = V_s(\bar{x}_0, \bar{y}_0) = V_u(z_1, w_1) = r_0.$$

From the parametric form of T_0 , we obtain that $D(T_0)$ and $R(T_0)$ are the images of $U_{(x, y)} \times U_{(z, w)} \times [t, \infty)$ under τ_0 and τ_1 , which are given by

$$\tau_0(t_0, x_0, y_0, z_1, w_1) = (x_0, y_0, z^p(t_0, x_0, y_0, z_1, w_1), w^p(t_0, x_0, y_0, z_1, w_1))$$

and

$$\tau_1(t_0, x_0, y_0, z_1, w_1) = (x^p(t_0, x_0, y_0, z_1, w_1), y^p(t_0, x_0, y_0, z_1, w_1), z_1, w_1).$$

We consider $\bar{t} = N\pi + s^*$, $s^* \in [0, \pi)$.

If we write $[\bar{t}, \infty) = \cup_{k=N}^{\infty} I_k$ with $I_k = \left[\frac{k\pi + s^*}{\Omega}, \frac{(k+1)\pi + s^*}{\Omega} \right)$, then the domain and the range of T_0 are given by

$$D(T_0) = \cup_{k=N}^{\infty} \sigma_k^0 \quad \text{and} \quad R(T_0) = \cup_{k=N}^{\infty} \sigma_k^1$$

where

$$\begin{aligned} \sigma_k^0 &= \tau_0(U_{(x, y)} \times U_{(z, w)} \times I_k) \\ \sigma_k^1 &= \tau_1(U_{(x, y)} \times U_{(z, w)} \times I_k) \end{aligned}$$

σ_k^0 is a collection of surfaces given by

$$\begin{aligned} z_0 &= z^p(\pi k + S_0 + s^*, x_0, y_0, z_1, w_1) \equiv z_{(z_1, w_1)}^k(S_0, x_0, y_0) \\ w_0 &= w^p(\pi k + S_0 + s^*, x_0, y_0, z_1, w_1) \equiv w_{(z_1, w_1)}^k(S_0, x_0, y_0) \end{aligned}$$

which we denote by $P_{(z_1, w_1)}^k$, $(z_1, w_1) \in U_{(z, w)}$ and σ_k^1 is a collection of surfaces given by

$$\begin{aligned} x_1 &= x^p(\pi k + S_1 + s^*, x_0, y_0, z_1, w_1) \equiv x_{(S_1, x_0, y_0)}^k(z_1, w_1) \\ y_1 &= y^p(\pi k + S_1 + s^*, x_0, y_0, z_1, w_1) \equiv y_{(S_1, x_0, y_0)}^k(z_1, w_1) \end{aligned}$$

which we denote by $Q_{(S_1, x_0, y_0)}^k$, $(x_0, y_0) \in U_{(x, y)}$.

5. Proof of Shil'nikov Theorem.

System (4.2) is equivalent to

$$\begin{aligned} \bar{\Delta}x &= \bar{B}_{11}x_1 + \bar{B}_{12}y_1 + \bar{C}_{11}\Delta'z_1 + \bar{C}_{12}\Delta'w_1 \\ \bar{\Delta}y &= \bar{B}_{21}x_1 + \bar{B}_{22}y_1 + \bar{C}_{21}\Delta'z_1 + \bar{C}_{22}\Delta'w_1 \\ \bar{z}_0 &= \bar{B}_{31}x_1 + \bar{B}_{32}y_1 + \bar{C}_{31}\Delta'z_1 + \bar{C}_{32}\Delta'w_1 \\ \bar{w}_0 &= \bar{B}_{41}x_1 + \bar{B}_{42}y_1 + \bar{C}_{41}\Delta'z_1 + \bar{C}_{42}\Delta'w_1 \end{aligned}$$

and we consider $\tilde{C}_3 = (\bar{C}_{31}, \bar{C}_{32})$ and $\tilde{C}_4 = (\bar{C}_{41}, \bar{C}_{42})$. Now, we will assume that the following condition is satisfied

$$(H5) \quad \det \begin{pmatrix} \tilde{C}_3 & (\varphi_3)_1 \\ \tilde{C}_4 & (\varphi_4)_1 \end{pmatrix}^2 + \det \begin{pmatrix} \tilde{C}_3 & (\varphi_3)_2 \\ \tilde{C}_4 & (\varphi_4)_2 \end{pmatrix}^2 \neq 0$$

with $(\varphi_3)_i$ and $(\varphi_4)_i$, $i = 1, 2$, the first and second coordinates of φ_3 and φ_4 respectively, given by the exponential expansion presented in section 3.

If we denote by $D = \{(0, 0)\} \times U_{(z, w)}$ and $D^+ = T_1(D)$, under conditions (H1)–(H5), we obtain the following lemmas.

Lemma 5. *There exists $N_1 \geq N$, such that for $k > N_1$ and for each $(z_1, w_1) \in U_{(z, w)}$ the surfaces $P_{(z_1, w_1)}^k$ intersect D^+ in a unique point.*

Proof: The equations for D^+ are

$$\begin{aligned} \bar{\Delta}x &= (C_{11} + \dots)\Delta'z + (C_{12} + \dots)\Delta'w \equiv \bar{C}_{11}\Delta'z + \bar{C}_{12}\Delta'w \\ \bar{\Delta}y &= (C_{21} + \dots)\Delta'z + (C_{22} + \dots)\Delta'w \equiv \bar{C}_{21}\Delta'z + \bar{C}_{22}\Delta'w \\ \bar{z}_0 &= (C_{31} + \dots)\Delta'z + (C_{32} + \dots)\Delta'w \equiv \bar{C}_{31}\Delta'z + \bar{C}_{32}\Delta'w \\ \bar{w}_0 &= (C_{41} + \dots)\Delta'z + (C_{42} + \dots)\Delta'w \equiv \bar{C}_{41}\Delta'z + \bar{C}_{42}\Delta'w \end{aligned}$$

We can obtain $\Delta'z$ and $\Delta'w$ from the third and forth equations, as $\bar{C}_{31}\bar{C}_{42} - \bar{C}_{32}\bar{C}_{41} \neq 0$. Replacing in the first two equations:

$$\bar{\Delta}x = \frac{(\bar{C}_{11}\bar{C}_{42} - \bar{C}_{12}\bar{C}_{41})\bar{z}_0 + (\bar{C}_{12}\bar{C}_{31} - \bar{C}_{11}\bar{C}_{32})\bar{w}_0}{\bar{C}_{31}\bar{C}_{42} - \bar{C}_{41}\bar{C}_{32}} \equiv \psi_1(\bar{z}_0, \bar{w}_0)$$

(5.1)

$$\bar{\Delta}x = \frac{(\bar{C}_{11}\bar{C}_{42} - \bar{C}_{12}\bar{C}_{41})\bar{z}_0 + (\bar{C}_{12}\bar{C}_{31} - \bar{C}_{11}\bar{C}_{32})\bar{w}_0}{\bar{C}_{31}\bar{C}_{42} - \bar{C}_{41}\bar{C}_{32}} \equiv \psi_1(\bar{z}_0, \bar{w}_0)$$

(5.2)

and the third and forth equations can be written as

$$(5.3) \quad \psi_3(\bar{z}_0, \bar{w}_0) \equiv \det \begin{pmatrix} \tilde{C}_3 & z_0 \\ \tilde{C}_4 & w_0 \end{pmatrix} = 0$$

where $\tilde{C}_3 = (\bar{C}_{31}, \bar{C}_{32})$, $\tilde{C}_4 = (\bar{C}_{41}, \bar{C}_{42})$ and $(\bar{\Delta}x, \bar{\Delta}y, \bar{z}_0, \bar{w}_0) \in P_{(z_1, w_1)}^k$ if

$$(5.4) \quad \bar{z}_0 = z_{(z_1, w_1)}^k(S_0, \Delta x, \Delta y)$$

$$(5.5) \quad \bar{w}_0 = w_{(z_1, w_1)}^k(S_0, \Delta x, \Delta y)$$

So, the points of intersection of D^+ with $P_{(z_1, w_1)}^k$ are going to be the solutions of (5.1)–(5.5).

If we substitute (5.4) and (5.5) in (5.3) we obtain

$$\det \begin{pmatrix} \tilde{C}_3 & z_{z_1}^k \\ \tilde{C}_4 & w_{w_1}^k \end{pmatrix} = 0,$$

or if we use the exponential expansion, then we have

$$\det \begin{pmatrix} \tilde{C}_3 & e^{\gamma(k\pi + S_0 + s^*)}((\varphi_3)_1 \cos \Omega t_0 + (\varphi_3)_2 \sin \Omega t_0 + \dots) \\ \tilde{C}_4 & e^{\gamma(k\pi + S_0 + s^*)}((\varphi_4)_1 \cos \Omega t_0 + (\varphi_4)_2 \sin \Omega t_0 + \dots) \end{pmatrix} = 0$$

or

$$\det \begin{pmatrix} \tilde{C}_3 & (\varphi_3)_1 \cos \Omega t_0 + (\varphi_3)_2 \sin \Omega t_0 + \dots \\ \tilde{C}_4 & (\varphi_4)_1 \cos \Omega t_0 + (\varphi_4)_2 \sin \Omega t_0 + \dots \end{pmatrix} = 0$$

which can be reduced to

$$\det \begin{pmatrix} \tilde{C}_3 & (\varphi_3)_1 \\ \tilde{C}_4 & (\varphi_4)_1 \end{pmatrix} \cos \Omega t_0 + \det \begin{pmatrix} \tilde{C}_3 & (\varphi_3)_2 \\ \tilde{C}_4 & (\varphi_4)_2 \end{pmatrix} \sin \Omega t_0 + \dots = 0$$

where the dots denote the terms that tend to zero as k (t_0) tend to ∞ .

But, if we assume $(H5)$, equation

$$\det \begin{pmatrix} \tilde{C}_3 & (\varphi_3)_1 \\ \tilde{C}_4 & (\varphi_4)_1 \end{pmatrix} \cos \Omega t_0 + \det \begin{pmatrix} \tilde{C}_3 & (\varphi_3)_2 \\ \tilde{C}_4 & (\varphi_4)_2 \end{pmatrix} \sin \Omega t_0 = 0$$

has a unique solution $\theta^* \in [0, \pi]$. Thus, if we choose s^* such that $S_0(z_1, w_1, k)$ converges to $\frac{\pi}{2}$ when $k \rightarrow \infty$, then our system has a unique solution S_0 for $k > N_1$.

□

Corollary 1. $\overline{T_0(D^+ \cap D(T_0))}$ is locally disconnected.

Remark: The following lemma shows the condition to obtain an invariant set.

Lemma 6. For $m, k > N$, such that $\lambda k + \gamma m > N_2$, with $N_2 > N$ we have

$$\sigma_{m,k} = T\sigma_k^0 \cap \sigma_m^0 \neq \emptyset$$

Proof: We will prove that for k and m large enough $T_1(Q_{(S_1, x_0, y_0)}^k) \cap P_{(z_1, w_1)}^m$ is not empty.

From (4.2) we have

$$\begin{aligned} \overline{\Delta}x &= \overline{B}_{11}x_1 + \overline{B}_{12}y_1 + \overline{C}_{11}\Delta'z + \overline{C}_{12}\Delta'w \\ \overline{\Delta}y &= \overline{B}_{21}x_1 + \overline{B}_{22}y_1 + \overline{C}_{21}\Delta'z + \overline{C}_{22}\Delta'w \\ \overline{z}_0 &= \overline{B}_{31}x_1 + \overline{B}_{32}y_1 + \overline{C}_{31}\Delta'z + \overline{C}_{32}\Delta'w \\ \overline{w}_0 &= \overline{B}_{41}x_1 + \overline{B}_{42}y_1 + \overline{C}_{41}\Delta'z + \overline{C}_{42}\Delta'w \end{aligned}$$

Solving the third and forth equations for $\Delta'z$ and $\Delta'w$, and replacing in the first and second equations, we obtain

$$(5.6) \quad \begin{aligned} \overline{\Delta}x &= \chi_1(x_0, y_0, \overline{z}_0, \overline{w}_0, S_1, k) \\ \overline{\Delta}y &= \chi_2(x_0, y_0, \overline{z}_0, \overline{w}_0, S_1, k) \end{aligned}$$

By Lemma 5, we have that

$$\det \begin{pmatrix} \tilde{C}_3 & \overline{z}_0 \\ \tilde{C}_4 & \overline{w}_0 \end{pmatrix} = \det \begin{pmatrix} \tilde{C}_3 & \overline{B}_{31}x_1 \\ \tilde{C}_4 & \overline{B}_{41}x_1 \end{pmatrix} + \det \begin{pmatrix} \tilde{C}_3 & \overline{B}_{32}y_1 \\ \tilde{C}_4 & \overline{B}_{42}y_1 \end{pmatrix}$$

So,

$$\begin{aligned}\bar{\psi}_2(x_0, y_0, \bar{z}_0, \bar{w}_0, S_1, k) &\equiv \det \begin{pmatrix} \tilde{C}_3 & \bar{z}_0 \\ \tilde{C}_4 & \bar{w}_0 \end{pmatrix} - \det \begin{pmatrix} \tilde{C}_3 & \bar{B}_{31}x_1 \\ \tilde{C}_4 & \bar{B}_{41}x_1 \end{pmatrix} \\ &\quad - \det \begin{pmatrix} \tilde{C}_3 & \bar{B}_{32}y_1 \\ \tilde{C}_4 & \bar{B}_{42}y_1 \end{pmatrix} = 0\end{aligned}$$

or

$$(5.7) \bar{\psi}_2(x_0, y_0, \bar{z}_0, \bar{w}_0, S_1, k) = \psi_3(\bar{z}_0, \bar{w}_0) + \bar{\psi}_4(x_0, y_0, \bar{z}_0, \bar{w}_0, S_1, k) = 0$$

with

$$\|\bar{\psi}_4\|_{C^1} \leq K_1 \|x_1\|_\alpha \leq K_2 e^{-\lambda(k\pi/\Omega)}$$

where K_1 and K_2 are constants.

Therefore, the equations for $T_1(Q_{(S_1, x_0, y_0)}^k) \cap P_{(z_1, w_1)}^m$ are (5.6), (5.7) and

$$\begin{aligned}\bar{z}_0 &= z^p(m\pi + S_0 + s^*, \bar{x}_0, \bar{y}_0, \bar{z}_1, \bar{w}_1) \\ \bar{w}_0 &= w^p(m\pi + S_0 + s^*, \bar{x}_0, \bar{y}_0, \bar{z}_1, \bar{w}_1)\end{aligned}$$

From the exponential expansion of $\bar{z}_0$ and $\bar{w}_0$ we obtain

$$\det \begin{pmatrix} \tilde{C}_3 & (\varphi_3)_1 \\ \tilde{C}_4 & (\varphi_4)_1 \end{pmatrix} \cos \Omega t_0 + \det \begin{pmatrix} \tilde{C}_3 & (\varphi_3)_2 \\ \tilde{C}_4 & (\varphi_4)_2 \end{pmatrix} \sin \Omega t_0 + e^{-\gamma t_0} \bar{\psi}_4 + \dots = 0$$

or

$$\begin{aligned}\det \begin{pmatrix} \tilde{C}_3 & (\varphi_3)_1 \\ \tilde{C}_4 & (\varphi_4)_1 \end{pmatrix} \cos(S_0 + s^*) + \det \begin{pmatrix} \tilde{C}_3 & (\varphi_3)_2 \\ \tilde{C}_4 & (\varphi_4)_2 \end{pmatrix} \sin(S_0 + s^*) + \\ e^{-(\gamma/\Omega)(\pi m + S_0 + s^*)} \bar{\psi}_4 + \dots = 0\end{aligned}$$

where the dots denote the terms tending to zero as $t_0 \rightarrow \infty$ and

$$\|e^{-(\gamma/\Omega)(\pi m + S_0 + s^*)} \bar{\psi}_4\| \leq K_3 e^{-(\lambda k + \gamma m)\pi/\Omega}$$

As $\lambda k + \gamma m > N_2$, we have a unique solution $\bar{S}_0$, given by

$$\bar{S}_0 = \frac{\pi}{2} + \alpha(S_1, x_0, y_0, z_1, w_1, m, k)$$

where $\alpha \rightarrow 0$ as $N_2 \rightarrow \infty$.

□

Proof of Theorem 1: We realize that $\bar{\sigma}_{m,k} \subseteq \sigma_m^0$ and for $\lambda k + \gamma m$ large enough, we have that the application

$$T_{m,k} : \sigma_k \longrightarrow \sigma_m$$

is well defined and it is given by

$$T_{m,k}(\Delta x, \Delta y, \bar{\Delta}' z, \bar{\Delta}' w, S) \longrightarrow (\bar{\Delta} x, \bar{\Delta} y, \Delta' z, \Delta' w, \bar{S})$$

where

$$\bar{\Delta} x = \bar{Q}_1(x^p(k\pi + S + s^*, x^+ + \Delta x, y^+ + \Delta y, z^- + \Delta z, w^- + \Delta w), \\ y^p(k\pi + S + s^*, x^+ + \Delta x, y^+ + \Delta y, z^- + \Delta z, w^- + \Delta w), \Delta' z, \Delta' w)$$

$$\bar{\Delta} y = \bar{Q}_2(x^p(k\pi + S + s^*, x^+ + \Delta x, y^+ + \Delta y, z^- + \Delta z, w^- + \Delta w), \\ y^p(k\pi + S + s^*, x^+ + \Delta x, y^+ + \Delta y, z^- + \Delta z, w^- + \Delta w), \Delta' z, \Delta' w)$$

$$z^p(m\pi + \bar{S} + s^*, x^+ + \bar{\Delta} x, y^+ + \bar{\Delta} y, z^- + \bar{\Delta} z, w^- + \bar{\Delta} w) = \\ \bar{Q}_3(x^p(k\pi + S + s^*, x^+ + \Delta x, y^+ + \Delta y, z^- + \Delta z, w^- + \Delta w), \\ y^p(k\pi + S + s^*, x^+ + \Delta x, y^+ + \Delta y, z^- + \Delta z, w^- + \Delta w), \Delta' z, \Delta' w)$$

$$w^p(m\pi + \bar{S} + s^*, x^+ + \bar{\Delta} x, y^+ + \bar{\Delta} y, z^- + \bar{\Delta} z, w^- + \bar{\Delta} w) = \\ \bar{Q}_4(x^p(k\pi + S + s^*, x^+ + \Delta x, y^+ + \Delta y, z^- + \Delta z, w^- + \Delta w), \\ y^p(k\pi + S + s^*, x^+ + \Delta x, y^+ + \Delta y, z^- + \Delta z, w^- + \Delta w), \Delta' z, \Delta' w)$$

Using the above lemma, this system can be reduced to

$$\begin{aligned} \bar{\Delta} x &= \bar{F}_1(\Delta x, \Delta y, \bar{\Delta} z, \bar{\Delta} w, S, k, m) \\ \bar{\Delta} y &= \bar{F}_2(\Delta x, \Delta y, \bar{\Delta} z, \bar{\Delta} w, S, k, m) \\ \Delta z &= \bar{G}_1(\Delta x, \Delta y, \bar{\Delta} z, \bar{\Delta} w, S, k, m) \\ \Delta w &= \bar{G}_2(\Delta x, \Delta y, \bar{\Delta} z, \bar{\Delta} w, S, k, m) \\ \bar{S} &= \frac{\pi}{2} + \bar{\alpha}(\Delta x, \Delta y, \bar{\Delta} z, \bar{\Delta} w, S, k, m) \end{aligned}$$

where $\bar{F}_1, \bar{F}_2, \bar{G}_1, \bar{G}_2$, and $\bar{\alpha}$ are contractions for $\lambda k + \gamma m$ large enough.

So, if Γ is an orbit contained in a neighborhood of Γ_0 , $\Gamma_0 \neq \Gamma$, then, by definition of T_0 and T_1 , we have that Γ intersects Σ^+ in points of $D(T_0)$, which define a sequence of points $\{P_n\}_{-\infty}^{\infty}$ with $P_n \in \sigma_{k_n}^0$, for some k_n , $k_n > \bar{N}$.

Let us define $j_n = k_n - \bar{N}$. This definition determines a unique sequence

$$\{\dots, j_n, j_{n+1}, \dots\} \in \Omega(\rho)$$

associated to Γ_0 .

On the other hand, for a given sequence

$$\beta = \{\dots, j_n, j_{n+1}, \dots\} \in \Omega(\rho)$$

we define $k_n = j_n + \bar{N}$, so, we have $k_n > \bar{N}$, and $\gamma k_{n+1} + \lambda k_n > \bar{N}$ as

$$k_n > -\frac{\gamma}{\lambda} k_{n+1} + \frac{\bar{N}}{\lambda} > \alpha_{n+1}$$

or

$$k_{n+1} < \frac{1}{\alpha} k_n$$

then $\rho = \frac{1}{\alpha}$.

Let $(\Delta x_n, \Delta y_n, \Delta z_n, \Delta w_n)$ be the coordinates of $M_n^0 \in \sigma_k^0$, $(x_n, y_n, \Delta z_n, \Delta w_n)$ the coordinates of $M_n^1 = T_0 M_n^0$ and $t_0 = k_n \pi + S_n + s^*$ the transition time from M_n^0 to M_n^1 .

The relations between the coordinates of the points $M_n^0, M_{n+1}^0, \dots$ can be written as

$$\begin{aligned} \Delta x_{n+1} &= \bar{F}_1(\Delta x_n, \Delta y_n, \Delta z_{n+1}, \Delta w_{n+1}, S_n, k_n, k_{n+1}) \\ \Delta y_{n+1} &= \bar{F}_2(\Delta x_n, \Delta y_n, \Delta z_{n+1}, \Delta w_{n+1}, S_n, k_n, k_{n+1}) \\ \Delta z_n &= \bar{G}_1(\Delta x_n, \Delta y_n, \Delta z_{n+1}, \Delta w_{n+1}, S_n, k_n, k_{n+1}) \\ \Delta w_n &= \bar{G}_2(\Delta x_n, \Delta y_n, \Delta z_{n+1}, \Delta w_{n+1}, S_n, k_n, k_{n+1}) \\ S_{n+1} &= \frac{\pi}{2} + \bar{\alpha}(\Delta x_n, \Delta y_n, \Delta z_{n+1}, \Delta w_{n+1}, S_n, k_n, k_{n+1}) \end{aligned}$$

Applying the fixed point lemma for product spaces given by Shil'nikov in [11] to $(U_{(x,y)} \times \bar{I}) \times U_{(z,w)}$ ($\bar{I} = [0, \pi]$),

$$A_n : \begin{cases} \Delta x_{n+1} &= \bar{F}_1(\Delta x_n, \Delta y_n, \Delta z_{n+1}, \Delta w_{n+1}, S_n, k_n, k_{n+1}) \\ \Delta y_{n+1} &= \bar{F}_2(\Delta x_n, \Delta y_n, \Delta z_{n+1}, \Delta w_{n+1}, S_n, k_n, k_{n+1}) \\ S_{n+1} &= \frac{\pi}{2} + \bar{\alpha}(\Delta x_n, \Delta y_n, \Delta z_{n+1}, \Delta w_{n+1}, S_n, k_n, k_{n+1}) \end{cases}$$

and

$$B_{n+1} : \begin{cases} \Delta z_n &= \bar{G}_1(\Delta x_n, \Delta y_n, \Delta z_{n+1}, \Delta w_{n+1}, S_n, k_n, k_{n+1}) \\ \Delta w_n &= \bar{G}_2(\Delta x_n, \Delta y_n, \Delta z_{n+1}, \Delta w_{n+1}, S_n, k_n, k_{n+1}) \end{cases}$$

we obtain the unique sequence

$$(\dots, (\Delta x_n^i, \Delta y_n^i, \Delta z_n^i, \Delta w_n^i, S_n), (\Delta x_{n+1}^j, \Delta y_{n+1}^j, \Delta z_{n+1}^j, \Delta w_{n+1}^j, S_{n+1}), \dots)$$

which corresponds to a solution Γ in a neighborhood of Γ_0 .

□

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References

- [1] Blázquez, C. M. "Bifurcation from a homoclinic orbit in parabolic differential equation". *Proceedings of the Royal Society of Edinburgh*, **103A**, pp. 265–274, (1986).
- [2] Blázquez, C. M. & Tuma, E. "Chaos in infinite dimensions: A generalization of a Theorem of Shil'nikov". *Nonlinear Analysis, Theory, Methods & Applications*, Vol. **21**, No. 2, pp. 103–121, (1993).
- [3] Blázquez, C. M. & Tuma, E. "Strange Attractors of the Shil'nikov type in Chua's Circuit". *International Journal of Bifurcation and Chaos*, Vol. **3**, No. 5, pp. 1293–1298, (1993).
- [4] Deng, B. "Exponential expansion with Shil'nikov Saddle-Focus". *Journal Differential Equation*, **82**, pp. 156–173, (1989).
- [5] Henry, D. "Geometry Theory of Semilinear Parabolic Equations", Lecture Notes in Mathematics, Vol. **840**. Springer-Verlag, Berlin, (1981).
- [6] Lancaster, P. "Theory of Matrices", Academic Press, San Diego, (1969).
- [7] Ovsyannikov, I. M. & Shil'nikov, L.P. "On Systems with a Saddle-Focus Homoclinic Curve". *Math. USSR Sbornik*, Vol. **58**, No. 2, pp. 557–574, (1987).
- [8] Pazy, A. "Semigroups of linear operators and applications to partial differential equations", Applied Mathematical Sciences, Vol. **44**, Springer-Verlag, Berlin, (1983).
- [9] Shil'nikov, L.P. "On a Poincaré-Birkhoff problem". *Math USSR Sbornik*, Vol. **3**, No. 3, pp. 353–371, (1967).
- [10] Shil'nikov, L.P. "The existence of a denumerable set of Periodic Motions in four-dimensional space in an extended neighborhood of a Saddle-Focus". *Soviet Math. Doklady* Vol. **8**, No. 1, MR 35 # 1872, pp. 54–58, (1967).

- [11] Shil'nikov, L.P. "A contribution to the problem of the structure of an extended neighborhood of a rough equilibrium state of Saddle-Focus type". *Math USSR Sbornik* Vol. **10**, No. 1, pp. 91-102, (1970).
- [12] Tresser, C. "*Modèles simples de transitions vers la turbulence*", Doctorat D'Etat, Université de Nice, (1981).

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