SEMI-GROUP GENERATED BY K-LIMITS

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Abstract

We use the definitions of limitoid, support and grill to present a semi-group generated by the composition of some-k-limits characterized by l-limits, and establish an order relationship between the elements that are obtained.

KEYWORDS : Limitoid, support, grill, K-limits, semi-groups generated.

1. Introduction

The idea of studying semi-groups generated by the operators of *closure* and *interior* initially appeared in the work of Kuratowski, in 1920 [12], where the relationship among the fourteen elements are presented. Hamner later (in 1960, [11]) presented an extension of Kuratowski's work, simplifying the initial hypotheses. Greco and Dolecki, following the ideas of Kuratowski, presented in 1981 [5] a semi-group generated by *isotonization* and *antitonization* of the generalization of closure and interior operators. In 1975 De Giorgi and Franzoni [4] developed the theory of Γ -convergence that has been successfully employed in approximation problems for Elliptic Partial Differential Equations, Variational Calculus, Minimax Problems, Convex Analysis and so forth. The definition of Γ -limit is closely related to the theory of **lim inf** and **lim sup** operators used by Francaviglia and Franzoni in 1983 [6] in the study of the semi-group generated by

$$\Gamma(X^-), \Gamma(X^+), \Gamma(Y^+, X^-) \text{ and } \Gamma(X^-, Y^+).$$

In this work, we will present a semi-group generated by the composition of some K -limits characterized by Γ -limits, and we will establish an order relationship between the elements that are obtained. We use the definition of the *limitoid*, introduced by Greco in 1983 [7], and, in a more precise way, the definitions of *limitoid support and grill* that have been shown to be efficient tools for studying Kuratowski's K -limits.

2. Preliminaries

2.1 K -operators and Systems of semifilter

A topological space X can be characterized by a system of neighborhoods or with the use of the interior operator. Let $\mathcal{N}(x)$ be the system of neighborhoods of x , $x \in X$. Then the following hold:

- a) $\mathcal{N} \subset \mathcal{N}_t = \{A \subset X : x \in A\}$,
- b) $\mathcal{N}^2 = \mathcal{N}$,
- c) $\mathcal{N}$ is a system of filters, i.e.:

$$\mathcal{N} \neq \emptyset, \quad \emptyset \notin \mathcal{N}, \quad A \subset B \subset \bigcup_{A \in \mathcal{N}} A \Rightarrow B \in \mathcal{N}.$$

On the other hand, if, for every set $A \subset X$ the interior of A is indicated as **int**(**A**), then we have:

- a) $A \supset \text{int}(A)$.
- b) $\text{int}^2(A) = \text{int}(A)$.
- c) $\text{int}(A \cap B) \equiv \text{int}(A) \cap \text{int}(B)$, with the $\text{int}(\phi) = \phi$ and $\text{int}(X) = X$.

The correspondence between $\mathcal{N}$ and int is given by:

$$A \in \mathcal{N}(x) \Leftrightarrow x \in \text{int}(A).$$

Using this topological concept, we now give the relationship between the definitions of systems of semifilters (sfX) and K -operators (Kuratowski operators).

Let X be a set and let $\mathcal{A} \subset \mathcal{P}(X)$ with $\mathcal{A} \neq \phi$ and $\phi \notin \mathcal{A}$. $\mathcal{A}$ is a semifilter if

$$A \in \mathcal{A}, A \subset B \subset \bigcup_{A \in \mathcal{A}} \Rightarrow B \in \mathcal{A}.$$

A semifilter on X is a semifilter $\mathcal{A}$ such that $X = \bigcup\{A : A \in \mathcal{A}\}$.

We indicate by sfX the set of the semifilters on X .

Let X and Y be two sets. A mapping $\Gamma : Y \rightarrow sfX$ is called systems of semifilters on X with domain Y . A set mapping given by:

$$K : \mathcal{P}(X) \rightarrow \mathcal{P}(Y)$$

is denominated a K - operator from X to Y , if

- i) $K(\phi) \equiv \phi$ and $K(X) \equiv Y$.
- ii) $A \subset B \subset X \Rightarrow K(A) \subset K(B)$.

To any system of semifilters σ there is associated the K -operator $K(\sigma)$ defined by

$$K(\sigma)(A) = \{y \in Y : A \in \sigma(y)\}, \quad \text{for all } A \subset X.$$

This correspondence between systems of semifilters on X with a domain Y and K -operators from X to Y is one-to-one (*injective*), onto and it preserves the order as well, that is to say:

$$A \in \Gamma(y) \Leftrightarrow y \in K(A),$$

besides,

$$\Gamma \subset \mu \Leftrightarrow K(\Gamma) \subset K(\mu).$$

2.2 Grill

If $\mathcal{A}$ and $\mathcal{B}$ are two families of sets, we will say that $\mathcal{A}$ is *finer* than $\mathcal{B}$ and we will write $\mathcal{A} \leq \mathcal{B}$ if for any $A \in \mathcal{A}$ there exists $B \in \mathcal{B}$ such that $B \subset A$. We will also say that $\mathcal{A}$ and $\mathcal{B}$ are equivalent and we will write $\mathcal{A} \equiv \mathcal{B}$ if $\mathcal{A} \leq \mathcal{B}$ and $\mathcal{B} \leq \mathcal{A}$.

Let X be a set and let $\mathcal{A} \subset \mathcal{P}(X)$ be a non-degenerate family, that is, $\mathcal{A} \neq \emptyset$ and $\emptyset \notin \mathcal{A}$. The grill of $\mathcal{A}$ is defined by

$$\mathcal{A}^\# = \{A \subset X : A \cap F \neq \emptyset \text{ for all } F \in \mathcal{A}\}.$$

With respect to the grill operation, the following properties hold:

- i) If $\mathcal{A} \subset \mathcal{P}(X)$ is a non-degenerate family, then $\mathcal{A}^{\#\#} \equiv \mathcal{A}$.
- ii) If $\mathcal{A} \subset \mathcal{P}(X)$ and $\mathcal{B} \subset \mathcal{P}(Y)$ are semifilters of X and Y respectively, then $(\mathcal{A} \times \mathcal{B})^\# \supset \mathcal{A}^\# \times \mathcal{B}^\#$.
- iii) If $\mathcal{A}$ and $\mathcal{B}$ are semifilters of X such that $\mathcal{A} \subset \mathcal{B}$, then $\mathcal{B}^\# \subset \mathcal{A}^\#$.
- iv) If $\mathcal{A} \subset \mathcal{P}(X)$ is a filter and $\mathcal{B} \subset \mathcal{P}(Y)$ is a semfilter such that

$$\bigcap \mathcal{B} \neq \emptyset, \text{ then } (\mathcal{A}^\# \times \mathcal{B})^\# \subset \mathcal{A}^\# \times \mathcal{B}^\#.$$

We can prove with respect to the grill operation, that

$$K(\sigma^\#)(A) = (K(\sigma)(A^c))^c.$$

Therefore, if σ is a system of neighborhoods of a topological space, then $K(\sigma)$ is the interior operator and $K(\sigma^\#)$ is the closure operator and we indicate $K(\sigma)$ by int_σ and $K(\sigma^\#)$ by cl_σ .

For the system of semifilters $\sigma : Y \rightarrow sfX$ the following statements are equivalent:

- a) σ is a system of filters;
- b) $K(\sigma)(A \cap B) = K(\sigma)(A) \cap K(\sigma)(B)$ for all $A, B \in \mathcal{P}(Y)$;
- c) $K(\sigma^\#)(A \cup B) = K(\sigma^\#)(A) \cup K(\sigma^\#)(B)$ for all $A, B \in \mathcal{P}(Y)$.

2.3 Limitoids and Lattices

The pair $(L, \leq)$ made up by a set L and an order relation “ $\leq$ ” is an *ordered set*. An ordered set is a complete lattice if each of its subsets possesses both an upper and a lower extreme.

Let L_1 and L_2 be complete lattice. A mapping $\Phi : L_1 \rightarrow L_2$ is a complete homomorphism if $\Phi(\sup W) = \sup(\Phi W)$ and $\Phi(\inf W) = \inf(\Phi W)$ for every subset $W \neq \emptyset$ of L_1 .

If $S \subset L_1$ and S contains the $\inf D$ and the $\sup D$ for every subset $D \neq \emptyset$ of S , we say that S is a *closed sublattice* of L_1 .

The most important complete lattice is the set $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$. A closed sublattice of $\overline{\mathbb{R}}$ and a complete homomorphism from $\overline{\mathbb{R}}$ to $\overline{\mathbb{R}}$ coincides with continuous non-decreasing functions.

Let X be a nonempty set and L a complete lattice. Indicating by L^X the set of applications from X into L , then a *Limitoid* is defined by a functional $T : L^X \rightarrow L$, such that for every pair $(f, g) \in L^X$, and Φ a complete homomorphism from L into L , then the following hold:

- 1) $f \leq g \Rightarrow T(f) \leq T(g)$;
- 2) $T(\Phi \circ f) = \Phi(T(f))$;
- 3) $T(f) \in \overline{f(X)}^L$, where $\overline{f(X)}^L$ is the smallest closed sublattice containing $f(X)$.

Some important examples of limitoids can be exhibited by using finite combinations of supremes and infima:

Example:

$$T_1(f) = \inf_{y \in Y} \sup_{x \in X} f(x, y) = \inf_Y \sup_X f$$

is a limitoid of $L^{X \times Y}$ into L .

Analogously,

$$T_2(f) = \sup_{x \in X} \inf_{y \in Y} f(x, y) = \sup_X \inf_Y f$$

is a limitoid.

In our work we center our interest on limitoids from $L^{X \times Y}$ into L given by the Γ -limits, as in the following:

For $f : X \times Y \rightarrow \mathbb{R}$ we define

$$\begin{aligned} (\Gamma(X^+)f)(x, y) &= \limsup_{x' \rightarrow x} f(x', y) \\ (\Gamma(Y^-)f)(x, y) &= \liminf_{y' \rightarrow y} f(x, y') \\ (\Gamma(Y^-, X^+)f)(x, y) &= \inf_{A \in \mathcal{N}(x)} \liminf_{y' \rightarrow y} \sup_{x' \in A} f(x', y') \\ (\Gamma(X^+, Y^-)f)(x, y) &= \sup_{V \in \mathcal{N}(y)} \limsup_{x' \rightarrow x} \inf_{y' \in V} f(x', y') \end{aligned}$$

for X and Y topological spaces and for $\mathcal{N}(x)$ and $\mathcal{N}(y)$ as systems of neighborhoods on X and Y respectively, or, equivalently, as bases of filters.

It is important to note that Γ -limits are fundamental for the study of minimax problems, due to the fact that the saddle-points can be defined by means of the topologies of X and Y . We have the following results:

$$\begin{aligned}\Gamma(Y^-)f &\leq f \leq \Gamma(X^+)f \quad \text{and} \\ \Gamma(Y^-)f &\leq \Gamma(X^+, Y^-)f \leq \Gamma(Y^-, X^+)f \leq \Gamma(X^+)f\end{aligned}$$

for all extended real function f . Besides, f is lower semi-continuous on Y if $f = \Gamma(Y^-)f$, and, dually, f is upper semi-continuous on X if $f = \Gamma(X^+)f$. $\Gamma(Y^-)f$ and $\Gamma(X^+, Y^-)f$ are always lower semi-continuous on Y , while $\Gamma(X^+)f$ and $\Gamma(Y^-, X^+)f$ are upper semi-continuous on X .

In a general way, the Γ -limit of a function $f \in L^X$ is defined by a mapping of the form:

$$\Gamma(\mathcal{A}_1^{\alpha_1}, \mathcal{A}_2^{\alpha_2} \dots \mathcal{A}_n^{\alpha_n}) : L^X \rightarrow L^Y$$

$$\text{with} \quad \Gamma(\mathcal{A}_1^{\alpha_1}, \mathcal{A}_2^{\alpha_2}, \dots, \mathcal{A}_n^{\alpha_n})f = \underset{A_n \in \mathcal{A}_n}{\text{ext}^-}{}^{\alpha_n} \dots \underset{A_1 \in \mathcal{A}_1}{\text{ext}^-}{}^{\alpha_1} \underset{x \in A_1}{\text{ext}^+}{}^{\alpha_1} \dots \underset{x \in A_n}{\text{ext}^+}{}^{\alpha_n}$$

where $\mathcal{A}_i$ is a non-degenerate family of X_i , for each sub index i , $\{\alpha_n\}$ is a finite sequence of $+$ or $-$ signs and, moreover, $\text{ext}^- = \inf$ and $\text{ext}^+ = \sup$.

We then have that every Γ -limit is a limitoid [7].

The support of a limitoid, indicated by $St(T)$, is a family of sets defined by

$$St(T) = \{A \subset X : T(\chi_A) = 1, \chi_A \in L^X\}$$

where χ_A is the characteristic step function of A .

We remark that the support of a limitoid is a semifilter [9].

Theorem 1: (The Representation of Limitoids)

If L is a completely distributed complete lattice, then every limitoid is an inferior limit along a non-degenerate family. More precisely,

$$T(f) = \lim_{st(T)} \inf f \quad [7].$$

Theorem 2: (Reticular Structure of Limitoids)

If L is a completely distributed complete reticulate then the mapping

$$St : \text{Lim}(X, L) \rightarrow sfX,$$

which associates to each limitoid its support, is an isomorphism between the two reticulates [7].

If the limitoid is a Γ -limit, we then indicate the support of $\Gamma(\mathcal{A}_1^{\alpha_1}, \dots, \mathcal{A}_n^{\alpha_n})$ by $(\mathcal{A}_1^{\alpha_1}, \dots, \mathcal{A}_n^{\alpha_n})$. We will then have the following results:

a) $(\mathcal{A}^-) \equiv \mathcal{A}$ and $(\mathcal{A}^+) \equiv \mathcal{A}^\#$

b) $(\mathcal{A}_1^{\alpha_1}, \dots, \mathcal{A}_{n-1}^{\alpha_{n-1}}, \mathcal{A}_n^-) = (\mathcal{A}_1^{\alpha_1}, \dots, \mathcal{A}_{n-1}^{\alpha_{n-1}}) \times \mathcal{A}_n$

c) $(\mathcal{A}_1^{\alpha_1}, \dots, \mathcal{A}_{n-1}^{\alpha_{n-1}}, \mathcal{A}_n^+) = (\mathcal{A}_1^{\alpha_1}, \dots, \mathcal{A}_{n-1}^{\alpha_{n-1}}) \times \mathcal{A}_n^\#$,

where, for any system of semifilters $\mathcal{A}$ and $\mathcal{B}$, we indicate by $\mathcal{A} \times \mathcal{B}$ the semifilter generated by the family $\{A \times B : A \in \mathcal{A} \text{ and } B \in \mathcal{B}\}$. Some particular cases can be shown as follows:

$$(\mathcal{A}^-, \mathcal{B}^-) = \mathcal{A} \times \mathcal{B},$$

$$(\mathcal{A}^+, \mathcal{B}^-) = \mathcal{A}^\# \times \mathcal{B},$$

$$(\mathcal{A}^+, \mathcal{B}^+) = (\mathcal{A} \times \mathcal{B})^\#.$$

$$(\mathcal{A}^-, \mathcal{B}^+) = (\mathcal{A}^\# \times \mathcal{B})^\#.$$

3 Semi-Groups generated by K -Limits

Let $f : X \times Y \rightarrow R$, with X and Y as topological spaces, with the topologies τ and σ , respectively. For $x \in X$ and $y \in Y$, let $\mathcal{N}_\tau(x)$ and $\mathcal{N}_\sigma(y)$ be the families of neighborhoods of x and y , respectively. For any pair (α, β) of signs $+$ or $-$, the following Γ -limits

$$\Gamma(\tau^\alpha, \sigma^\beta)f : X \times Y \rightarrow R$$

can be defined by:

$$(\Gamma(\tau^\alpha, \sigma^\beta)f)(x, y) = \Gamma(\mathcal{N}_\tau(x)^\alpha, \mathcal{N}_\sigma(y)^\beta)f$$

$$(\Gamma(\tau^{\# \alpha}, \sigma^{\# \beta})f)(x, y) = \Gamma(\mathcal{N}_\tau^\#(x)^\alpha, \mathcal{N}_\sigma^\#(y)^\beta)f$$

Definition: K -limits

Let A and A' be two subsets of the topological sets X and Y . We then state that:

$$K(\tau^\alpha, \sigma^\beta)(A) = A' \Leftrightarrow \Gamma(\tau^\alpha, \sigma^\beta)\chi_A = \chi_{A'}$$

therefore,

- (i) $(x, y) \in K(\tau^\alpha, \sigma^\beta)(A) \Leftrightarrow A \in ((\mathcal{N}_\tau(x)^\alpha, \mathcal{N}_\sigma(y)^\beta)$
- (ii) $(x, y) \in K(\tau^{\#\alpha}, \sigma^{\#\beta})(A) \Leftrightarrow A \in ((\mathcal{N}_\tau^\#(x)^\alpha, \mathcal{N}_\sigma^\#(y)^\beta)$

Examples:

1. $K(\tau^-, \sigma^-)(A) = \text{int } A$ and $K(\tau^+, \sigma^+) = \text{cl } A$
2. For $A \subset X \times Y$, and for any pairs of signs (α, β) and $(\bar{\alpha}, \bar{\beta})$, we have that

$$(x, y) \in \left(K(\tau^\alpha, \sigma^\beta) K(\tau^{\bar{\alpha}}, \sigma^{\bar{\beta}})(A) \right) \iff A \in \bigcup_{\Omega \in (\mathcal{N}_\tau(x)^{\bar{\alpha}}, \mathcal{N}_\sigma(y)^{\bar{\beta}})} \bigcap_{(\bar{x}, \bar{y}) \in \Omega} (\mathcal{N}_\tau(\bar{x})^\alpha, \mathcal{N}_\sigma(\bar{y})^\beta).$$

Concluding this part we may state that there exists a reticular isomorphism between the K -limits, the systems of semilters, and the Γ -limits, on the same set X [10].

We will now establish the semi-group generated by the K -limits: $K(\tau^-, 0^-), K(\tau^+, 0^-), K(\tau^-, 0^+), K(\tau^+, 0^+), K(0^+, \tau^-), K(0^-, \tau^-), K(\tau^{\#-}, 0^{\#-})$ and $K(\tau^{\#+}, 0^{\#+})$ where τ is topology in X and 0 the chaotic topology in Y .

To obtain this semi-group we will need some results related to the decomposition of the K -limits.

To simplify our notation, we will use $K(\tau^\alpha)$ to represent $K(\tau^\alpha, i^-)$, where i stands for the discrete topology in X .

Lemma 1: (Decomposition of the K -limits)

- 1) $K(\tau^-, 0^-) = K(\tau^-)K(0^-)$
- 2) $K(\tau^+, 0^-) = K(\tau^+)K(0^-)$
- 3) $K(\tau^-, 0^+) = K(\tau^-)K(0^+)$
- 4) $K(\tau^+, 0^+) = K(\tau^+)K(0^+)$
- 5) $K(0^+, \tau^-) = K(0^+)K(\tau^-)$
- 6) $K(0^-, \tau^+) = K(0^-)K(\tau^+)$
- 7) $K(\tau^{\#-}, 0^{\#-}) = K(0^+)K(\tau^+)$
- 8) $K(\tau^{\#+}, 0^{\#+}) = K(0^-)K(\tau^-)$

Lemma 2: For any sequence $\alpha_1, \alpha_2, \alpha_3$ and α_4 of $+$ or $-$ signs, we have:

- 1) $K(0^{\alpha_1})K(\sigma^{\alpha_2})K(0^{\alpha_3})K(\sigma^{\alpha_4}) = K(\sigma^{\alpha_2})K(0^{\alpha_3})K(\sigma^{\alpha_4})$
- 2) $K(\sigma^{\alpha_1})K(0^{\alpha_2})K(\sigma^{\alpha_3})K(0^{\alpha_4}) = K(\sigma^{\alpha_1})K(\sigma^{\alpha_3})K(0^{\alpha_4})$

Lemma 3: For α representing $+$ or $-$ signs, we have

$$K(\sigma^\alpha)K(\sigma^\alpha) = K(\sigma^\alpha).$$

Lemma 4: For every pair α and β as $+$ or $-$ signs, we have:

- 1) $K(\sigma^\alpha)K(0^\beta)K(\sigma^\alpha) = K(0^\beta)K(\sigma^\alpha)$
- 2) $K(\sigma^\alpha)K(\sigma^\beta)K(\sigma^\alpha)K(\sigma^\beta) = K(\sigma^\alpha)K(\sigma^\beta)$

These lemmas are easily proven. We will, for example, illustrate this with the proof of $K(\tau^-)K(0^+)K(\tau^-) = K(0^+)K(\tau^-)$. To show this, we must prove that

$$K(\tau^-)K(0^+)K(\tau^-)A = K(0^+)K(\tau^-)A, \text{ that is,}$$

$$\Gamma(\tau^-)\Gamma(0^+)\Gamma(\tau^-)\chi_A = \Gamma(0^+)\Gamma(\tau^-)\chi_A.$$

In fact, since we have:

$$\begin{aligned} \Gamma(\tau^-)\chi_A(x, y) &= \sup_{U \in \mathcal{N}(x)} \inf_{\bar{x} \in U} \chi_A(\bar{x}, y), \\ \Gamma(0^+)\Gamma(\tau^-)\chi_A(x, y) &= \sup_{y \in Y} \sup_{U \in \mathcal{N}(x)} \inf_{\bar{x} \in U} \chi_A(\bar{x}, y) = h(x, y) \quad \text{and} \\ \Gamma(\tau^-)\Gamma(0^+)\Gamma(\tau^-)\chi_A(x, y) &= \Gamma(\tau^-)h(x, y) = \sup_{U \in \mathcal{N}(x)} \inf_{\bar{x} \in U} h(\bar{x}, y) = \\ &= \sup_{U \in \mathcal{N}(x)} \inf_{\bar{x} \in U} \sup_{y \in Y} \sup_{\bar{y} \in \mathcal{N}(\bar{x})} \inf_{\bar{y} \in \bar{y}} \chi_A(\bar{x}, \bar{y}) = j(x, y), \end{aligned}$$

then $j(x, y) \leq h(x, y)$.

Now

$$\begin{aligned} j(x, y) &\geq \sup_{U \in \mathcal{N}(x)} \inf_{\bar{x} \in U} \sup_{y \in Y} \inf_{\bar{y} \in U} \chi_A(\bar{x}, \bar{y}) \\ &\geq \sup_{U \in \mathcal{N}(x)} \sup_{y \in Y} \inf_{\bar{x} \in U} \chi_A(\bar{x}, y) \\ &\geq \sup_{y \in Y} \sup_{U \in \mathcal{N}(x)} \inf_{\bar{x} \in U} \chi_A(\bar{x}, y) = h(x, y). \end{aligned}$$

Therefore: $\Gamma(\tau^-)\Gamma(0^+)\Gamma(\tau^-)\chi_A = \Gamma(0^+)\Gamma(\tau^-)\chi_A$.

With the results of the previous lemmas, we can obtain the following:

a) eight elements resulting from two compositions:

$$\begin{array}{ll}
 K(\tau^+)K(0^-)K(\tau^-) & K(\tau^-)K(\tau^+)K(0^+) \\
 K(\tau^+)K(0^+)K(\tau^-) & K(\tau^+)K(\tau^-)K(0^+) \\
 K(\tau^-)K(0^-)K(\tau^+) & K(\tau^-)K(\tau^+)K(0^-) \\
 K(\tau^-)K(0^+)K(\tau^+) & K(\tau^+)K(\tau^-)K(0^-)
 \end{array}$$

b) eight elements resulting from three compositions:

$$\begin{array}{ll}
 K(\tau^+)K(\tau^-)K(0^+)K(\tau^+) & K(\tau^+)K(\tau^-)K(\tau^+)K(0^+) \\
 K(\tau^+)K(\tau^-)K(0^-)K(\tau^+) & K(\tau^+)K(\tau^-)K(\tau^+)K(0^-) \\
 K(\tau^-)K(\tau^+)K(0^+)K(\tau^-) & K(\tau^-)K(\tau^+)K(\tau^-)K(0^+) \\
 K(\tau^-)K(\tau^+)K(0^-)K(\tau^-) & K(\tau^-)K(\tau^+)K(\tau^-)K(0^-)
 \end{array}$$

c) four elements resulting from four compositions:

$$\begin{array}{l}
 K(\tau^-)K(\tau^+)K(\tau^-)K(0^+)K(\tau^+) \\
 K(\tau^-)K(\tau^+)K(\tau^-)K(0^-)K(\tau^+) \\
 K(\tau^+)K(\tau^-)K(\tau^+)K(0^+)K(\tau^-)
 \end{array}$$

which, added to the eight previously generated elements of the semi-group, make up the 28 elements.

We can also establish the following order relations in the semi-group:

a) among the generating elements of the semi-group:

$$K(\tau^-, 0^-) \leq K(\tau^{\# +}, 0^{\# +}) \leq K(\tau^+, 0^-) \leq K(0^-, \tau^+) \leq K(\tau^{\# -}, 0^{\# -}) \leq K(\tau^+, 0^+)$$

and

$$K(\tau^{\# +}, 0^{\# +}) \leq K(0^+, \tau^-) \leq K(\tau^-, 0^-) \leq K(\tau^{\# -}, 0^{\# -}).$$

These inequalities are easily obtained using grill properties.

b) among the elements resulting from two compositions:

$$K(\tau^+)K(\tau^-)K(0^-) \leq K(\tau^+)K(0^-)K(\tau^-) \leq K(\tau^+)K(0^+)K(\tau^-) \leq K(\tau^+)K(\tau^-)K(0^+)$$

and

$$K(\tau^-)K(\tau^+)K(0^-) \leq K(\tau^-)K(0^-)K(\tau^+) \leq K(\tau^-)K(0^+)K(\tau^+) \leq K(\tau^-)K(\tau^+)K(0^+).$$

c) among the elements resulting from triple compositions:

$$K(\tau^-)K(\tau^+)K(\tau^-)K(0^-) \leq K(\tau^-)K(\tau^+)K(0^-)K(\tau^-) \leq \\ \leq K(\tau^-)K(\tau^+)K(0^+)K(\tau^-) \leq K(\tau^-)K(\tau^+)K(\tau^-)K(0^+)$$

and

$$K(\tau^+)K(\tau^-)K(\tau^+)K(0^-) \leq K(\tau^+)K(\tau^-)K(0^-)K(\tau^+) \leq \\ \leq K(\tau^+)K(\tau^-)K(0^+)K(\tau^+) \leq K(\tau^+)K(\tau^-)K(\tau^+)K(0^+)$$

d) among those elements resulting from four compositions:

$$K(\tau^-)K(\tau^-)K(\tau^+)K(0^-)K(\tau^-) \leq K(\tau^+)K(\tau^-)K(\tau^+)K(0^+)K(\tau^-)$$

and

$$K(\tau^-)K(\tau^+)K(\tau^-)K(0^-)K(\tau^+) \leq K(\tau^-)K(\tau^+)K(\tau^-)K(0^+)K(\tau^+).$$

These relations are placed in the following table, where each line joining any two elements define an order relation between them, defining the "largest element" as the one above the other in the figure 1.

Other relationships between the elements of the semi-group are obtained by using the lemma that follows:

Lemma 5: If A and B are elements of the generated semi-group and $A \subset B$, then:

a) $K(\tau^-)A \leq K(\tau^+)B$

b) $A \leq BK(\tau^-)$

c) $A \leq K(\tau^+)B$

d) $AK(\tau^-)B \leq B$

e) $K(\tau^-)A \leq B$

f) $K(\tau^\alpha)A \leq K(\tau^\alpha)B$

g) $AK(\tau^\alpha) \leq BK(\tau^\alpha)$, where α represents the $+$ or $-$ signs.

Figure 1: Order relations of Semi-Group's elements

Finally, we present a geometric example of one of the order relations between the elements of the semi-group.

We have

$$K(\tau^+)K(\tau^-)K(0^-) \leq K(\tau^+)K(0^-)K(\tau^-).$$

Considering that $A \subset X \times Y$, we can see that

$$(x, y) \in K(\tau^+)K(\tau^-)K(0^-)A \quad \text{if and only if}$$

$$A \in \bigcup_{\Omega \subset (\mathcal{N}_+(x)^+, \mathcal{N}_-(y)^-)} \bigcup_{(x', y') \subset \Omega} \bigcup_{\Omega' \in (\mathcal{N}_+(\bar{x})^-, \mathcal{N}_-(\bar{y})^-)} \bigcap_{(x', y') \in \Omega'} (\mathcal{N}_+(x')^-, \mathcal{N}_-(y')^-)$$

that is, if and only if

$$A \in \bigcup_{\Omega \in \mathcal{N}_\tau(x)^\# \times \mathcal{N}_i(y)} \bigcap_{(\bar{x}, \bar{y}) \in \Omega} \bigcup_{\Omega' \in \mathcal{N}_\tau(\bar{x}) \times \mathcal{N}_i(\bar{y})} \bigcap_{(x', y') \in \Omega'} \mathcal{N}_i(x') \times \mathcal{N}_0(y').$$

which means, in turn, that

A contains $\{(x', R) \in \mathcal{N}_i(x') \times \mathcal{N}_0(y'), \forall (x', y') \in \Omega'\}$ for some Ω', Ω' contains $\{B(\bar{x}, r) \times \bar{y} \in \mathcal{N}_\tau(\bar{x}) \times \mathcal{N}_i(\bar{y}), \forall (\bar{x}, \bar{y}) \in \Omega, (\bar{y} = y)\}$ for some Ω , and Ω contains $\{(x_n, y) \in \mathcal{N}_\tau(x)^\# \times \mathcal{N}_i(y), \text{ with } x_n \rightarrow x\}$.

Geometrically, we can see that A contains strips, as those in figure 2.c. We observe, as well, that

$$(x, y) \in K(\tau^+)K(0^-)K(\tau^-)A \quad \text{if and only if}$$

$$A \in \bigcup_{\Omega \in (\mathcal{N}_\tau(x)^\#, \mathcal{N}_i(y))} \bigcap_{(\bar{x}, \bar{y}) \in \Omega} \bigcup_{\Omega' \in (\mathcal{N}_i(\bar{x}), \mathcal{N}_0(\bar{y})^-)} \bigcap_{(x', y') \in \Omega'} (\mathcal{N}_\tau(x'), \mathcal{N}_i(y')^-)$$

which means that

$$A \in \bigcup_{\Omega \in \mathcal{N}_\tau(x)^\# \times \mathcal{N}_i(y)} \bigcap_{(\bar{x}, \bar{y}) \in \Omega} \bigcup_{\Omega' \in \mathcal{N}_i(\bar{x}) \times \mathcal{N}_0(\bar{y})} \bigcap_{(x', y') \in \Omega'} \mathcal{N}_\tau(x') \times \mathcal{N}_i(y'),$$

which means that A contains $\{B(x', r(y')) \times y' \in \mathcal{N}_\tau(x') \times \mathcal{N}_i(y'), \forall (x', y') \in \Omega'\}$ for some Ω' , Ω' contains $\{(\bar{x}, R) \in \mathcal{N}_i(\bar{x}) \times \mathcal{N}_0(\bar{y}), \forall (\bar{x}, \bar{y}) \in \Omega\}$ for some Ω , and Ω contains $\{(x_n, y) \in \mathcal{N}_\tau(x)^\# \times \mathcal{N}_i(y) \text{ with } x_n \rightarrow x\}$.

Again, we can, from the geometrical point of view, see that A contains strips as those in figure 3.c; this means that if $(x, y) \in K(\tau^+)K(\tau^-)K(0^-)A$ then $(x, y) \in K(\tau^+)K(0^-)K(\tau^-)A$, that is, $K(\tau^+)K(\tau^-)K(0^-) \leq K(\tau^+)K(0^-)K(\tau^-)$.

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