ON EUCLIDEAN RINGS

Ahmet Goksel Agargun
Yildiz Technical University, Sisli-Istambul, Turkey

Abstract

In this article we extend the concept of Euclidean rings and Motzkin's work to commutative rings (not necessarily with an identity).

1. Introduction

In 1949 Motzkin [8] had proved a necessary and sufficient condition for an integral domain to be a Euclidean ring. Two generalizations of this concept came in 1969 and 1971. Firstly, Fletcher gave the Euclidean property for a commutative ring with identity in which zero divisors may appear [3]. Then samuel postulated whether a mapping into a general well-ordered set would increase the number of Euclidean rings [12]. In 1977, the first example of a Euclidean domain without any integer-valued algorithm was given by Hiblot [4], [5]. In 1978, Nagata [9] also proved that there exists an integral domain which is a Euclidean ring with a mapping into a well-ordered set but not a Euclidean ring with a mapping into $\mathbb{Z}^+ \cup \{0\}$. In 1985 Nagata [10], gave his definition which uses an algorithm which maps into a partially ordered set with the minimum condition. But this definition does not increase the family of Euclidean rings given by Samuel's definition.

We generalize the concept of Euclidean rings to commutative rings which do not necessarily have an identity. We give seven new definitions and the relations among them. Our definitions are equivalent to the usual definitions in the case of a ring with identity. We generalize Motzkin's work to give a useful result for finding the Euclidean algorithm (smallest algorithm) for commutative rings. Also some examples of Euclidean rings and the smallest algorithms are given in this work.

2. The Euclidean Algorithms

In generalising the concept of a Euclidean ring to commutative rings we take the opportunity of examining the connections between the many different definitions. Let R and R' be commutative rings, R' with identity and let $\theta : R \rightarrow R'$ be a monomorphism where $\theta(R)$ is an ideal of R' . Let us call this kind of monomorphism as a G-monomorphism. Suppose $a, b \in R$, W is a well-ordered set and φ is mapping into W . The three areas for choice are

- (i) $\varphi : R \setminus \{0\} \rightarrow W$ or $\varphi : R \rightarrow W$,
- (ii) b any element of R or $b \neq 0$,
- (iii) $\theta(a) = q'\theta(b) + \theta(r)$ where $r = 0$ or $\varphi(r) < \varphi(b)$, or $r = b$ or $\varphi(r) < \varphi(b)$, or $\varphi(r) < \varphi(b)$.

Therefore we have twelve possible definitions, but some of these do not give an algorithm.

Definition 2.1 A map $\varphi : R \setminus \{0\} \rightarrow W$ is called a Euclidean algorithm (with respect to $\theta : R \rightarrow R'$) if for all $a, b \in R, b \neq 0$, there exist $q \in R$ and $r \in R$ such that

$$\theta(a) = q' \theta(b) + \theta(r) \text{ where } r = 0 \text{ or } \varphi(r) < \varphi(b).$$

Similarly we have the other definitions by consideration the three areas for choice. .

Definition 2.2 $\varphi : R \setminus \{0\} \rightarrow W, b \neq 0, r = b, \text{ or } \varphi(r) < \varphi(b).$

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Definition 2.5 $\varphi : R \rightarrow W, b \neq 0, \varphi(r) < \varphi(b).$

Definition 2.6 $\varphi : R \rightarrow W, \text{ any } b, r = 0, \text{ or } \varphi(r) < \varphi(b).$

Definition 2.7 $\varphi : R \rightarrow W, \text{ any } b, r = b, \text{ or } \varphi(r) < \varphi(b).$

Here, we note that if R has an identity then by taking to be the identity homomorphism our definitions are the same as the definitions from Amano [2], Lenstra [7], Kanemitsu and Yoshida [6], Nagata [10] [11], Samuel [12].

There are five other possible definitions with the maps $\varphi : R \setminus \{0\} \rightarrow W$ and $\varphi : R \rightarrow W$, but we can not define an algorithm for any of these.

Definition 2.8 $\varphi : R \setminus \{0\} \rightarrow W, \text{ any } b, r = 0, \text{ or } \varphi(r) < \varphi(b).$

Definition 2.9 $\varphi : R \setminus \{0\} \rightarrow W, \text{ any } b, r = b, \text{ or } \varphi(r) < \varphi(b).$

Definition 2.10 $\varphi : R \setminus \{0\} \rightarrow W, \text{ any } b, \varphi(r) < \varphi(b).$

Definition 2.11 $\varphi : R \setminus \{0\} \rightarrow W, b \neq 0, \varphi(r) < \varphi(b).$

Definition 2.12 $\varphi : R \rightarrow W, \text{ any } b, \varphi(r) < \varphi(b).$

Here the first three definitions may be eliminated by taking $a \neq 0$ and $b = 0$ since $\varphi(0)$ is not defined. The last two cases are not given algorithms by choosing b where $\varphi(b)$ is minimal in W .

Let us mean by the word "equivalent" that if φ is an Euclidean algorithm according to one definition then it will be a Euclidean algorithm in the sense of the other. Then the connections between these seven new definitions are the same with the results of our paper on Euclidean rings [1]. Here we only need to consider the elements of R with θ according to the definitions. By allowing W and φ to change all these definitions become equivalent. Then we have the following picture

$$1 \Leftrightarrow 2 \Leftrightarrow 3 \Leftrightarrow 4 \Leftrightarrow 5 \Leftrightarrow 6 \Leftrightarrow 7$$

where $i \in \{1, 2, \dots, 7\}$ is Definition 2.i. Hence we need to use only one of these definitions, say Definition 2.6 to define a Euclidean ring.

Definition 2 13 Let R be a commutative ring. If there exists a commutative ring with identity R' , a G -monomorphism $\theta : R \rightarrow R'$, a well-ordered set W , an algorithm $\varphi : R \rightarrow W$ satisfying one of Definitions 2.1-2.7, say Definition 2.6, then we say that R is a Euclidean ring with respect to $\theta : R \rightarrow R'$ and $\varphi : R \rightarrow W$.

We consider now the special case where the well-ordered set is taken to be $Z^+ \cup \{0\}$. In general Definition 2.5 does not imply Definition 2.6 and Definition 2.3 does not imply Definition 2.6 since the well-ordered set cannot change. However Definition 2.3 does imply Definition 2.5 since $Z^+ \cup \{0\}$ does not have maximal element. Therefore we have the following picture in the case of $Z^+ \cup \{0\}$.

$$1 \Leftrightarrow 2 \Leftrightarrow 3 \Leftrightarrow 4 \Leftrightarrow 5$$

$$\uparrow$$

$$6 \Leftrightarrow 7$$

Therefore, the case of $Z^+ \cup \{0\}$ splits up in the two cases corresponding to Definitions 2.1-2.5 and Definitions 2.6-2.7.

Hence overall we have three types of Euclidean ring as.

- (i) Definition Group A: Euclidean rings with respect to a general well-ordered set using Definitions 2.1-2.7.
- (ii) Definition Group B: Euclidean rings with respect to $Z^+ \cup \{0\}$ using Definitions 2.1-2.5.
- (iii) Definition Group C: Euclidean rings with respect to $Z^+ \cup \{0\}$ using Definitions 2.6-2.7.

Here the Group C Euclidean rings come naturally into the picture, and these are equivalent to bounded Euclidean rings (defined by Fletcher when the ring has an identity [3]).

Now we give some examples:

- (i) Consider Z . $\varphi : Z \setminus \{0\} \rightarrow Z^+ \cup \{0\}$ is an algorithm (with respect to $1 : Z \rightarrow Z$) given by $\varphi(r) = |r|$ according to Group B. But $\varphi : Z \setminus \{0\} \rightarrow Z^+ \cup \{0\}$ is not an algorithm (with respect to $1 : Z \rightarrow Z$) according to Group C since we have a contradiction when $a \neq 0$ and $b = 0$ since $\varphi(a) = \varphi(a) > \varphi(b)$.

(ii) Consider $n\mathbb{Z}$ with $\text{inc.} : n\mathbb{Z} \rightarrow \mathbb{Z}$ and $\varphi : n\mathbb{Z} \setminus \{0\} \rightarrow \mathbb{Z}^+ \cup \{0\}$ given by $\varphi(nr) = |nr|$ is an algorithm according to Group B. But $\varphi : n\mathbb{Z} \rightarrow \mathbb{Z}^+ \cup \{0\}$ with the same mapping is not an algorithm according to Group C. The proof of this is immediate by following the case of $\mathbb{Z}$ in (i).

(iii) We know that $n\mathbb{Z} \times \mathbb{Z}$ is commutative ring with identity with respect to the following operations, for $(a_1, b_1), (a_2, b_2) \in n\mathbb{Z} \times \mathbb{Z}$,

$$\begin{aligned}(a_1, b_1) + (a_2, b_2) &= (a_2 + b_1, a_2 + b_2) \quad \text{and} \quad (a_1, b_1)(a_2, b_2) \\ &= (a_1a_2 + b_2a_1 + b_1a_2 + b_1a_2)\end{aligned}$$

Now, consider $n\mathbb{Z}$ with a G-monomorphism $\theta : n\mathbb{Z} \rightarrow n\mathbb{Z} \times \mathbb{Z}$ given by $\theta(nr) = (nr, 0)$ and $\varphi : n\mathbb{Z} \setminus \{0\} \rightarrow \mathbb{Z}^+ \cup \{0\}$ is an algorithm (with respect to $\theta : n\mathbb{Z} \rightarrow n\mathbb{Z} \times \mathbb{Z}$) given by $\varphi(nr) = |nr|$ according to Group B. For all $na, nb \in n\mathbb{Z}$, $nb \neq 0$, by (ii) there exist $(0, q) \in n\mathbb{Z} \times \mathbb{Z}$, $nr \in n\mathbb{Z}$ such that

$$\theta(na) = (0, q) \theta(nb) + \theta(nr) \quad \text{where } nr = 0 \text{ or } \varphi(nr) < \varphi(nb).$$

But $\varphi : n\mathbb{Z} \rightarrow \mathbb{Z}^+ \cup \{0\}$ given by $\varphi(nr) = |nr|$ is not an algorithm (with respect to $\theta : n\mathbb{Z} \rightarrow n\mathbb{Z} \times \mathbb{Z}$) according to Group C since we cannot have $|nr| = \varphi(nr) < \varphi(0) = 0$ for $nb = 0$, $nr = na \neq 0$.

(iv) Consider the subring $R = \{0, 2, 4, 6\} \subset \mathbb{Z}_8$. R is a Euclidean ring in the sense of Group C with respect to $\text{inc.} : R \rightarrow \mathbb{Z}_8$ and algorithm $\varphi : R \rightarrow \mathbb{Z}^+ \cup \{0\}$ given by $\varphi(r) = r (r \neq 0), \varphi(0) = 7$.

If R is a commutative ring with identity then R is a Euclidean ring if and only if it is a Euclidean ring (with respect to $1 : R \rightarrow R$). Therefore we see that Group A (with respect to $1 : R \rightarrow R$) is equivalent to a class of the definitions from Amano, Lenstra, Nagata, Samuel. Group B and Group C (with respect to $1 : R \rightarrow R$) are equivalent to Fletcher's definition and his bounded Euclidean ring respectively .

We have seen that there are fundamentally three types of Euclidean ring R . These three groups are distinct. An example of a Euclidean ring Group B which is not Group C is given by $\mathbb{Z}$ (see example (i), above). Group A is in fact a wider class than Group B. An example of this given by $\mathbb{Z} \oplus \mathbb{Z}$ that is a Euclidean ring Group A but not Euclidean ring Group B (see [11] and [3].).

3. The generalization of Motzkin's work

We generalize Motzkin's work to commutative rings (not necessarily with identity) [53]. Suppose R is any commutative ring, R' a commutative ring with identity, $\theta : R \rightarrow R'$ a G -monomorphism and W a well-ordered set. Let R_0 let be the non-zero elements of R and S a subset of R_0 . We define S' as follows

$$S' = \{s \in S : \exists a \in R \text{ such that } \theta(a) + \theta(s)R' \subseteq \theta(S)\}.$$

Then $S' \subseteq S$ and if $S_1 \subseteq S$ we have $S'_1 \subseteq S'$ since $x \in S'_1$ implies $\theta(a) + \theta(x)R' \subseteq \theta(S_1) \subseteq \theta(S)$ and $x \in S'$. Here, also we note that $a \in S$ necessarily since $0 \in R'$.

The following two theorems give us a necessary and sufficient condition for a commutative ring to be Euclidean in the sense of the three groups.

Theorem 3 1 Suppose R is a Euclidean ring Group A (Definition 2.6) with respect to $\theta : R \rightarrow R'$ and the algorithm $\varphi : R \rightarrow W$. Let P_λ be the set of $x \in R_0$ such that $\varphi(x) \geq \lambda$ ($\lambda \in W, 0 \leq \lambda \leq \mu$ where $\varphi(0_R) = \mu$ and 0 denotes the least element of W). Then the subsets $\{P_\lambda\}$ have the following properties : (i) $P_\lambda \subseteq P_{\lambda-1}$, but $P_\lambda \subseteq \bigcap_{\alpha < \lambda} P_\alpha$ if $\lambda - 1$ does not exist, (ii) $P'_\lambda \subseteq P_{\lambda-1}$, (iii) P_μ is empty. Conversely, suppose R is a commutative ring, $\theta : R \rightarrow R'$ a G -monomorphism and W a well-ordered set. If there are subsets $\{P_\lambda\}$ of R_0 ($\lambda \in W$) satisfying (i)-(iii) then R is Euclidean Group A (with respect to $\theta : R \rightarrow R'$ and there exists an algorithm $\varphi : R \rightarrow W$ given by $\varphi(x) = \lambda$ if and only if $x \in P_\lambda \setminus P_{\lambda+1}$.

Proof (i) Clearly if $\lambda - 1 \in W$ then $P_\lambda \subseteq P_{\lambda-1}$ for all $\lambda \in W$ such that $0 \leq \lambda \leq \mu = \varphi(0)$. If $\lambda - 1$ does not exist then for $x \in P_\lambda, \varphi(x) \geq \lambda$. This implies $\varphi(x) \geq \alpha$ for $\alpha < \lambda$. Therefore $x \in P_\alpha$, for $\alpha < \lambda$ and hence $x \in \bigcap_{\alpha < \lambda} P_\alpha$.

(ii) Suppose $b \in P'_\lambda$ and $a \in R$ such that $\theta(a) + \theta(b)R' \subseteq \theta(P_\lambda)$. Since R is a Euclidean ring, there exist $q' \in R'$ and $r \in R$ such that

$$\theta(a) = q'\theta(b) + \theta(r) \text{ where } r = 0 \text{ or } \varphi(r) < \varphi(b).$$

If $r = 0$ then $0 \in \theta(a) + \theta(b)R' \subseteq \theta(P_\lambda)$. This is a contradiction. Therefore $\varphi(r) < \varphi(b)$. Since $\theta(a) + \theta(b)R' \subseteq \theta(P_\lambda)$, $\theta(r) \in \theta(P_\lambda)$ and this implies $\varphi(r) \geq \lambda$. Therefore $\varphi(b) \geq \lambda + 1$ and hence $b \in P_{\lambda+1}$.

(iii) Finally, suppose $x \in P_\mu$, then $\varphi(x) \geq \mu$. Since R is a Euclidean ring for $x, 0 \in R$ there exist $q' \in R'$ and $r \in R$ such that

$$\theta(x) = q'0 + \theta(r) \text{ where } \varphi(r) < \varphi(0) \text{ and } r = x \neq 0.$$

Therefore $\varphi(x) < \varphi(0) = \mu$ which is a contradiction. Hence P_μ is empty.

Conversely, take $a, b \in R$. If $b \mid a$ then

$$\theta(a) = \theta(b)\theta(q) + \theta(r) \text{ where } r = 0 \text{ and } q \in R.$$

So, suppose b does not divide a and choose $q' \in R'$ such that $\varphi(a + c)$ is minimal and $\theta(c) = \theta(b)q'$. Let $\varphi(a + c) = \alpha$. Thus if $\theta(d) = \theta(b)s$ for some $s \in R'$ then $\varphi(a + d) \geq \alpha$ and $\theta(a) + \theta(b)R' \subseteq \theta(P_\alpha)$. If $\varphi(b) = \gamma$ we must prove $\alpha < \gamma$. Suppose not, then $\alpha \geq \gamma$ and $P_\alpha \subseteq P_\gamma$. But $b \in P_\gamma$ and $\theta(a) + \theta(b)R' \subseteq \theta(P_\alpha) \subseteq \theta(P_\gamma)$, which implies $b \in P'_\gamma \subseteq P_{\gamma+1}$. This is a contradiction, i.e. $\alpha < \gamma$.

Remark In the case of $W = Z^+ \cup \{0\}$ we have Theorem 3.1 with the same proof for Group C. Here, (i) becomes $P_i \subseteq P_{i-1}$ for $i \in Z^+$ since $i - 1$ always exists.

Theorem 3.2 Suppose $\varphi : R \setminus \{0\} \rightarrow Z^+ \cup \{0\}$ defines Euclidean algorithm in the sense of Group B (with respect to $\theta : R \rightarrow R'$). Let P_i be the set of $x \in R_0$ where $\varphi(x) \geq i$ ($i = 0, 1, \dots$). Then subsets $\{P_i\}$ have the following properties :

(i) $R_0 = P_0 \supseteq P_1 \supseteq P_2 \supseteq \dots$, (ii) $P'_i \subseteq P_{i+1}$, (iii) $\bigcap_{i=0}^\infty P_i$ is empty. Conversely, suppose R is a commutative ring and $\theta : R \rightarrow R'$ a G-monomorphism. If there are subsets $\{P_i\}$ of R_0 ($i \in Z^+ \cup \{0\}$) which satisfying (i)-(iii) then R is Euclidean Group B (with respect to $\theta : R \rightarrow R'$) and there exists an algorithm $\varphi : R \setminus \{0\} \rightarrow Z^+ \cup \{0\}$ given by $\varphi(x) = i$ if and only if $x \in P_i \setminus P_{i+1}$.

Proof Suppose first we have a Euclidean algorithm in the sense of Group B. Clearly $P_{i+1} \subseteq P_i$ for all $i \geq 0$. Now suppose $x \in \bigcap_{i=0}^\infty P_i$ and let $\varphi(x) = n$ then $x \notin P_{n+1}$. This is a contradiction. Hence $\bigcap_{i=0}^\infty P_i$ is empty. Finally, suppose $b \in P'_i$ and let $a \in R$ be such that $\theta(a) + \theta(b)R' \subseteq \theta(P_i)$. Now there exist $q' \in R'$ and $r \in R$ with property that

$$\theta(a) = q'\theta(b) + \theta(r) \text{ where } r = 0 \text{ or } \varphi(r) < \varphi(b).$$

If $r = 0$ then we have $0 = \theta(a) - \theta(b)q' \in \theta(a) + \theta(b)R' \subseteq \theta(P_i)$ and hence $0 \in P_i$, this is a contradiction. Therefore $\varphi(r) < \varphi(b)$. Since $\theta(a) + \theta(b)R' \subseteq \theta(P_i)$, $\theta(r) \in \theta(P_i)$ and this implies $\varphi(r) \geq i$. Therefore $\varphi(b) > i$ and $b \in P_{i+1}$.

For the converse we have a similar proof as in the previous proof.

The subsets $\{P_\lambda\}$, like the mapping $\varphi : R \rightarrow W$ can be defined in a number of ways. To make this idea more useful we define one set of

subsets directly from R . For convenience we change the notation and write $R_0^{(1)} = R'_0$ and $R_0^{(\lambda)} = (R_0^{(\lambda-1)})'$, but $R_0^{(\lambda)} = \bigcap_{\alpha < \lambda} (R_0^{(\alpha)})'$ if $\lambda - 1$ does not exist. The following two theorems and remark give the necessary and sufficient conditions for the three groups.

Theorem 3.3 R is a Euclidean ring Group A (with respect to $\theta : R \rightarrow R'$) with an algorithm $\varphi : R \rightarrow W$ if and only if $R_0^{(\mu)}$ is empty for some $\mu \in W$. Further if R is Euclidean Group A (with respect to $\theta : R \rightarrow R'$ and if $\{P_\lambda\}$ is the corresponding set sequence of an algorithm $\varphi : R \rightarrow W$ with respect to $\theta : R \rightarrow R'$ (defined in Theorem 3.1), then $R_0^{(\lambda)} \subseteq P_\lambda$ for all $\lambda \in W$ and the sequence $R_0^{(\lambda)}$ itself determines a Euclidean algorithm (as in Theorem 3.1).

Proof We use Theorem 3.1. Let us suppose $R_0^{(\mu)}$ is empty for some $\mu \in W$. Then (iii) is satisfied. Since $S' \subseteq S$ for any subset S , if $\lambda - 1 \in W$ by the definition $R_0^{(\lambda)} = (R_0^{(\lambda-1)})' \subseteq R_0^{(\lambda-1)}$. If $\lambda - 1$ does not exist, then by the definition $R_0^{(\lambda)} = (\bigcap_{\alpha < \lambda} R_0^{(\alpha)})' \subseteq \bigcap_{\alpha < \lambda} R_0^{(\alpha)}$ and hence (i) satisfied. Also, by the definition we have $(R_0^{(\lambda)})' = R_0^{(\lambda-1)}$ and therefore (ii) satisfied with equality.

Conversely, suppose R is Euclidean Group A (with respect to $\theta : R \rightarrow R'$ and φ which gives subsets P_α). Then $R_0 = P_0, R_0^{(1)} = R'_0 = P'_0 \subseteq P_1$ and by induction $R_0^{(\lambda)} \subseteq P_\lambda$ for all W . Therefore $R_0^{(\mu)} \subseteq P_\mu$ and by Theorem 3.1 property (iii) $R_0^{(\mu)}$ is empty. Hence from Theorem 3.1 the subsets $\{R_0^{(\lambda)}\}$ determine a Euclidean algorithm in the sense of Group A (with respect to $\theta : R \rightarrow R'$).

Remark By taking $W = \mathbb{Z}^+ \cup \{0\}$, we have the equivalence of Theorem 3.3 for Group C.

Theorem 3.4 R is a Euclidean ring Group B (with respect to $\theta : R \rightarrow R'$) if and only if $(\bigcap_{i=0}^{\infty} R_0^{(i)})$ is empty. Further if R is Euclidean Group B (with respect to $\theta : R \rightarrow R'$ and if $\{P_i\}$ are the corresponding sets of an algorithm of R with respect to $\theta : R \rightarrow R'$ (defined in Theorem 3.2) then $R_0^{(i)} \subset P_i$ for all $i \geq 0$ and the sequence $\{R_0^{(i)}\}$ itself determines a Euclidean algorithm (as in Theorem 3.2).

Proof Use Theorem 3.2 and the same way as in the previous proof.

Definition 3.5

(i) Two ordered sets S, T are said to be order-isomorphic if there exists a bijection $f : S \rightarrow T$ which preserves the ordering, i.e. $a \leq b \Leftrightarrow f(a) \leq f(b)$.

(ii) let $\varphi_1 : R \rightarrow W_1$ and $\varphi_2 : R \rightarrow W_2$ be two algorithms on a ring R (with respect to $\theta : R \rightarrow R'$). If there exists an order-isomorphism map $f : \varphi_1(R) \rightarrow \varphi_2(R)$ such that $\varphi_2 = f \circ \varphi_1$ then we say that φ_1 and φ_2 are isomorphic algorithms.

Since every well-ordered set with cardinal $\leq \text{card}(R)$ is order-isomorphic to an initial segment of any well-ordered set with cardinal $> \text{card}(R)$ all algorithms on a ring R may be construed to take their values in a fixed well-ordered set. Let W_R be a well-ordered set of ordinals whose cardinality does not exceed the cardinality of R .

Definition 3.6 Suppose R is Euclidean ring Groups A, C (or Group B). Then the algorithm defined by the $R_0^{(i)}$ which was given in Theorem 3.3 (or in Theorem 3.4) is called the smallest algorithm in the sense of Groups A, C (or Group B).

Note that we must define $\varphi(0)$ for Definitions 2.3-2.7. For Definitions 2.3 and 2.4 we can define $\varphi(0)$ as the least element of W . For Definitions 2.6-2.7 we define $\varphi(0) = \mu$ as in Theorem 3.3. In fact this is the unique maximal element of $\varphi(R)$. The only problem is in the sense of Definition 2.5. In this case, to get the algorithm with the smallest values we increase each $\varphi(r)$ by 1 and put $\varphi(0) = 0$ which is the least element of W . Some examples follow shortly.

The following theorem shows us the connection between the smallest algorithms with respect to Definitions 2.3, 2.6 and 2.5.

Theorem 3.7 Suppose R is a Euclidean ring Group A (with respect to $\theta : R \rightarrow R'$).

(i) If $\varphi_3 : R \rightarrow W_R$ and $\varphi_6 : R \rightarrow W_R$ are the smallest algorithms wrt. Definitions 2.3 and 2.6 respectively, then $\varphi_3(r) = \varphi_6(r)$ for all non-zero $r \in R$.

(ii) If $\varphi_3 : R \rightarrow W_R$ and $\varphi_5 : R \rightarrow W_R$ are the smallest algorithms wrt. Definitions 2.3 and 2.5 respectively, then for all $r \in R$, $\varphi_3(r) \leq \varphi_5(r) \leq \varphi_3(r) + 1$ when right hand side exists.

Therefore we can define the smallest algorithm either in terms of Definition 2.3 or Definition 2.6.

Proof (i) It is clear that every algorithm with respect to Definition 2.6 is an algorithm with respect to Definition 2.3. Therefore φ_6 is an algorithm with respect to Definition 2.3 and hence $\varphi_3(r) \leq \varphi_6(r)$ for all $r \in R$ (1).

Define $\zeta_3 : R \rightarrow W_R$ by $\zeta_3(r) = \varphi_3(r)$ ($r \neq 0$), $\zeta_3(0) = \varphi_6(0)$.

Then ζ_3 is an algorithm with respect to Definition 2.6. For suppose $a, b \in R$. If $b \neq 0$ then there exist $q' \in R'$ and $\in R$ such that

$\theta(a) = \theta(b)q' + \theta(r)$ where $r = 0$ or $\zeta_3(r) = \varphi_3(r) < \varphi_3(b) = \zeta_3(b)$.

If $b = 0$ then easily we have

$\theta(a) = 0 + \theta(r)$ where $r = a = 0$ or $\zeta_3(r) = \varphi_3(r) \leq \varphi_6(r) < \varphi_6(0) = \zeta_3(0)$.

Since φ_6 is the smallest algorithm with respect to Definition 2.6, we have for all non-zero $r \in R$ $\varphi_6(r) \leq \zeta_3(r) = \varphi_3(r)$ (2)

Therefore, by (1) and (2) $\varphi_3(r) = \varphi_6(r)$ for all non-zero $r \in R$.

(ii) It is straightforward to check that φ_5 is an algorithm with respect to Definition 2.3. Therefore $\varphi_3(r) \leq \varphi_5(r)$ for all $r \in R$. Suppose $\varphi_3(r) + 1$ exists in W_R for all $r \in R$. Define $\zeta_3 : R \rightarrow W_R$ by $\zeta_3(r) = \varphi_3(r) + 1$ ($r \neq 0$), $\zeta_3(0) = \min.$ of W_R . Then ζ_3 is an algorithm with respect to Definition 2.5. For suppose $a, b \in R$. If $b \neq 0$, there exist $q' \in R'$ and $r \in R$ such that $\theta(a) = \theta(b)q' + \theta(r)$ where $r = 0$ or $\varphi_3(r) < \varphi_3(b)$.

Therefore by the definition of ζ_3 we have $\theta(a) = \theta(b)q' + \theta(r)$ where $\zeta_3(r) < \zeta_3(b)$. Since φ_5 is the smallest algorithm with respect to Definition 2.5, we have $\varphi_5(r) \leq \zeta_3(r)$ for all $r \in R$. In fact we write $\zeta_3(r) \leq \varphi_3(r) + 1$ for all $r \in R$. Therefore $\varphi_5(r) \leq \varphi_3(r) + 1$ and hence

$\varphi_3(r) \leq \varphi_5(r) \leq \varphi_3(r) + 1$ for all $r \in R$.

We show that the smallest algorithm satisfies an inequality which is given by the following theorem. Here, we give Theorem 3.8 with respect to Group A, but also the same theorem with the same proof is given wrt. Groups B and C by taking $W = Z^+ \cup \{0\}$.

Theorem 3.8 If R is a Euclidean ring Group A (with respect to $\theta : R \rightarrow R'$) then the smallest algorithm $\varphi : R \rightarrow W$ defined by the subsets $\{R_0^{(\lambda)}\}$ satisfies the following property

$\varphi(b) \geq \varphi(a)$ for all $a, b \in R$ where $b \neq 0$ and $\theta(b) = \theta(a)q'$ for $q' \in R'$.

Proof Suppose $\varphi(b) = \lambda$ and $\varphi(a) = \alpha$. Therefore $b \in R_0^{(\lambda)} \setminus R_0^{(\lambda+1)}$ and $a \in R_0^{(\alpha)} \setminus R_0^{(\alpha+1)}$. If $\varphi(b) < \varphi(a)$ then $\lambda < \alpha$ and $R_0^{(\lambda)} \supset R_0^{(\alpha)}$ with strict inclusion since $\bigcap_{\lambda=0}^{\infty} R_0^{(\lambda)}$ is empty (because $R_0^{(\mu)}$ is empty for some $\mu \in W$). Suppose $\alpha - 1$ exists in W . Since $a \in R_0^{(\alpha)}$ there exists $y_1 \in R$ such that $\theta(y_1) + \theta(a)R' \subseteq \theta(R_0^{(\alpha-1)}) \subseteq \theta(R_0^{(\lambda)})$. Suppose now, $\alpha - 1$ does not exist. Since $a \in R_0^{(\alpha)} = (\bigcap_{\gamma < \alpha} R_0^{(\gamma)})'$ there exists $y_2 \in R$ such that $\theta(y_2) + \theta(a)R' \subseteq \theta(\bigcap_{\gamma < \alpha} R_0^{(\gamma)}) \subseteq \theta(R_0^{(\lambda)})$. Therefore, there exists $y \in R$ such that $\theta(y) + \theta(b)R' = \theta(y) + \theta(a)q'R' \subseteq \theta(R_0^{(\lambda)})$ and $b \in R_0^{(\lambda+1)}$. This is a contradiction. Hence $\varphi(b) \geq \varphi(a)$.

Corollary If R is a Euclidean ring Group A (with respect to $\theta : R \rightarrow R'$) then the smallest algorithm $\varphi : R \rightarrow W$ defined by the subsets $\{R_0^{(\lambda)}\}$ satisfies the following property

$\varphi(ab) \geq \varphi(a)$ for all $a, b \in R$ where $ab \neq 0$.

Theorem 3.8 therefore generalises the inequality property given by Fletcher [3].

Suppose R is a commutative ring with identity. Consider the G -monomorphism $1 : R \rightarrow R$. Then we have that $R_0^{(1)} = \{s \in R_0 : \exists a \in R \text{ with } a + sR \subseteq R_0\}$. Therefore $a + sr \neq 0$ and $sr \neq -a$ for all $r \in R$. If s is a nonunit then we take $a = 1$ and $s \in R_0^{(1)}$. If s is a unit then for $r = -s^{-1}a$ we have $a + sr = 0$ and $s \notin R_0^{(1)}$. Hence $R_0^{(1)} = R_0 \setminus \{\text{units}\}$.

Here we give some examples.

Example 1 (i) First, we may take the ring $R = \mathbb{Z}_{12}$, show it is a Euclidean ring Group B (and Group C) with respect to $1 : \mathbb{Z}_{12} \rightarrow \mathbb{Z}_{12}$ and determine its smallest algorithm φ . Therefore $R_0 = \{1, 2, 3, \dots, 10, 11\}$.

As we showed above, $R_0^{(1)} = \{2, 3, 4, 6, 8, 9, 10\}$.

It is easy to show that $R_0^{(2)} = \{4, 6, 8\}$. Because $2 + 4R, 2 + 6R, 2 + 8R \subseteq R_0^{(1)}$ and $a + sR \subseteq R_0^{(1)}$ for all $a \in R_0^{(1)}$ and $s \in \{2, 3, 9, 10\}$. Also $R_0^{(3)}$ is empty since $x + 4R \subseteq R_0^{(2)}, y + 6R \subseteq R_0^{(2)}$ and $z + 8R \subseteq R_0^{(2)}$ for all $x, y, z \in R_0^{(2)}$. Thus $\mathbb{Z}_{12}$ is a Euclidean ring in the sense of Group B (and Group C) with respect to $1 : \mathbb{Z}_{12} \rightarrow \mathbb{Z}_{12}$ from Theorem 3.4 (and Theorem 3.3).

(ii) Now, let us consider the subring $R = \{0, 2, 4, 6, 8, 10\} \subset \mathbb{Z}_{12}$ with the inclusion homomorphism $\text{inc.} : R \rightarrow \mathbb{Z}_{12}$. By the definition

$R_0^{(1)} = \{s \in R_0 : \exists a \in R \text{ with } \theta(a) + \theta(s)R' \subseteq \theta(R_0)\}$ and $R_0 = \{2, 4, 6, 8, 10\}$.

Clearly, there is no $a \in R$ with $a + 2\mathbb{Z}_{12} \subseteq R_0$. Therefore $2 \notin R_0^{(1)}$. For $2 \in \mathbb{Z}_{12}$ we have $2 + 4\mathbb{Z}_{12} = \{2, 6, 10\} \subset R_0$ and $4 \in R_0^{(1)}$. Similarly, $2 + 6\mathbb{Z}_{12} = \{2, 8\} \subset R_0$, $2 + 8\mathbb{Z}_{12} = \{2, 6, 10\} \subset R_0$ and hence $6, 8 \in R_0^{(1)}$. There is no $a \in R$ such that $a + 10\mathbb{Z}_{12} \subset R_0$, therefore $10 \notin R_0^{(1)}$. Hence $R_0^{(1)} = \{4, 6, 8\}$.

Since there is no $x, y, z \in R$ such that $x + 4\mathbb{Z}_{12} = x + \{2, 6, 10\} \subset R_0^{(1)}$, $y + 6\mathbb{Z}_{12} = y + \{0, 6\} \subset R_0^{(1)}$ and $z + 8\mathbb{Z}_{12} = z + \{0, 4, 8\} \subset R_0^{(1)}$, $4, 6, 8 \notin R_0^{(2)}$. Therefore $R_0^{(2)}$ is empty. Hence R is uclidean ring Group B (and Groupe) with respect to $\text{inc.} : R \rightarrow \mathbb{Z}_{12}$ and its smallest algorithm φ is given by

$$\varphi(2) = 0, \varphi(4) = 1, \varphi(6) = 1, \varphi(8) = 1, \varphi(10) = 0,$$

(I) For an algorithm in the sense of Definitions 2.3-2.4, we take $\varphi(0) = 0$,

$$\varphi(0) = 0, \varphi(2) = 0, \varphi(4) = 1, \varphi(6) = 1, \varphi(8) = 1, \varphi(10) = 0,$$

(II) For an algorithm in the sense of Definitions 2.6-2.7, we take $\varphi(0) = 2$,

$$\varphi(0) = 2, \varphi(2) = 0, \varphi(4) = 1, \varphi(6) = 1, \varphi(8) = 1, \varphi(10) = 0.$$

Also in the sense of Definition 2.5 to get the algorithm with smallest values we increase each $\varphi(r)$ by 1 and put $\varphi(0) = 0$,

$$\varphi(0) = 0, \varphi(2) = 1, \varphi(4) = 2, \varphi(6) = 2, \varphi(8) = 2, \varphi(10) = 1.$$

Example 2 For a positive integer n , consider the ring $n\mathbb{Z}$ with the G-monomorphisms

(i) $\text{inc.} : n\mathbb{Z} \rightarrow \mathbb{Z}$ and (ii) $\theta : n\mathbb{Z} \rightarrow n\mathbb{Z} \times \mathbb{Z}$.

The smallest algorithms (with respect to these G-monomorphisms) are given by $\varphi(nr) = k$ for $nr \neq 0$ where $n2^k \leq |nr| < n2^{k+1}$ (i.e $2^k \leq |r| < 2^{k+1}$).

1. For we prove by induction with Motzkin's notation that

$$(n\mathbb{Z}_0)^{(k)} = \{\pm n2^k, \pm n(2^k + 1), \pm n(2^k + 2), \dots\}.$$

Note that $n\mathbb{Z}_0 = \{\pm n, \pm 2n, \pm 3n, \dots\}$. For $k = 1$ let us prove $(n\mathbb{Z}_0)^{(1)} = \{\pm 2n, \pm 3n, \dots\}$. Because if there exists $na \in n\mathbb{Z}_0$ such that $na + (\pm n)\mathbb{Z} \subseteq n\mathbb{Z}_0$

(and for (ii) $(na, 0) + (\pm n, 0)(n\mathbb{Z} \times \mathbb{Z}) \subseteq \theta(n\mathbb{Z}_0)$) then $0 \in na \pm n\mathbb{Z}$ (and for (ii)

$(0, 0) \in (na, 0) \pm (n, 0)(n\mathbb{Z} \times \mathbb{Z})$) which is a contradiction. Therefore $\pm n \notin (n\mathbb{Z}_0)^{(1)}$. For

$|nr| > n, nr \in (n\mathbb{Z}_0)^{(1)}$ since $\pm n \pm nr\mathbb{Z} \subseteq n\mathbb{Z}_0$ (and $\pm(n, 0) \pm (nr, 0)(n\mathbb{Z} \times \mathbb{Z}) \subseteq \theta(n\mathbb{Z}_0)$). Suppose now $(n\mathbb{Z}_0)^{(k)} = \{\pm n2^k, \pm n(2^k + 1), \pm n(2^k + 2), \dots\}$ and let us prove $(n\mathbb{Z}_0)^{(k+1)} = \{\pm n2^{k+1}, \pm n(2^{k+1} + 1), \pm n(2^{k+1} + 2), \dots\}$. Take $nb \in (n\mathbb{Z}_0)^{(k)}$ with $|nb| < n2^{k+1}$ and suppose there exists $na \in n\mathbb{Z}$ such that $na + nb\mathbb{Z} \subseteq (n\mathbb{Z}_0)^{(k)}$

$((na, 0) + (nb, 0)(n\mathbb{Z} \times \mathbb{Z}) \subseteq \theta(n\mathbb{Z}_0)^{(k)})$. Since $na, nb \in \mathbb{Z}$ and $nb \neq 0$ there exist $q, nr \in \mathbb{Z}$ such that $na = nbq + nr$ where $nr = 0$ or $|nr| < |nb|$. If $nr = 0$ then $0 \in na + nb\mathbb{Z} \subseteq (n\mathbb{Z}_0)^{(k)}$ ($(0, 0) \in (na, 0) + (nb, 0)(n\mathbb{Z} \times \mathbb{Z}) \subseteq \theta((n\mathbb{Z}_0)^{(k)})$), this is a contradiction. Suppose now $|nr| < |nb|$. Here there are the two cases:

(I) If $|nr| < n2^k$ then we have a contradiction since $nr \in na + nb\mathbb{Z} \subseteq (n\mathbb{Z}_0)^{(k)}$ ($(nr, 0) \in (na, 0) + (nb, 0)(n\mathbb{Z} \times \mathbb{Z}) \subseteq \theta((n\mathbb{Z}_0)^{(k)})$).

(II) If $n2^k < |nr| < |nb|$ then we take

$$na = nbq + nr = nb(q + 1) + nr - nb \text{ such that } |nr - nb| < |nb|$$

(if $|nr - nb| \geq |nb|$ then we consider as $na = nb(q - 1) + nr + nb$ with $|nr + nb| < |nb|$ and continue similarly). In fact $|nr - nb| < n2^{k+1} - n2^k$ and therefore we have

$$nr - nb \in na + nb\mathbb{Z} \subseteq (n\mathbb{Z}_0)^{(k)} ((nr - nb, 0) \in (na, 0) + (nb, 0)(n\mathbb{Z} \times \mathbb{Z}) \subseteq \theta((n\mathbb{Z}_0)^{(k)}))$$

which is a contradiction. Thus for $|nb| < n2^{k+1}$, $nb \notin (nZ_0)^{(k+1)}$. If $|nb| \geq n2^{k+1}$ we easily take $na = n2^k \in nZ$ and we have

$$na + nbZ \subseteq (nZ_0)^{(k)}((na, 0) + (nb, 0)(nZ \times Z) \subseteq \theta((nZ_0)^{(k)}).$$

Hence $(nZ_0)^{(k+1)} = \{\pm n2^{k+1}, \pm n(2^{k+1} + 1), \pm n(2^{k+1} + 2), \dots\}$

For instance, $\varphi(14) = 1, \varphi(42) = 2, \varphi(70) = 3$ are some values of the smallest algorithm $\varphi : 7Z \rightarrow Z^+ \cup \{0\}$ (with respect to $\theta : 7Z \rightarrow 7Z \times Z$). If 175 and 70 are given two elements in $7Z$ then we have $\theta(175) = \theta(70)(3, -1) + \theta(35)$ where $\varphi(35) = 2 < \varphi(70)$.

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Ahmet Goksel Agargun

Yildiz Technical Universitesi

Fen-Ed. Fakultesi Matematik Bolumu

Ogretim Uyesi

Sisli, Istanbul

Turkey