ON COMMUTATIVITY OF RINGS WITH CONSTRAINTS INVOLVING A NIL SUBSET

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Abstract

The main theorem of this paper is that a ring R with unity is commutative if and only if there is a nil subset B of R such that

- 1. for each $x \in R$, either $x \in Z(R)$ or there is a polynomial f over $\mathbf{Z}$ with $x - x^2 f(x) \in B$;*
- 2. for each $x, y \in R$, there are non-negative integers $n > 1, m, r, s$ depending on a pair of ring elements x, y with $x(x^m y \pm x^r y^n x^s) - (x^m y \pm x^r y^n x^s)x = 0$.*

A related result for a nil commutative subset of R is given and the restrictions on the hypothesis of our result are justified by examples.

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0. Introduction

Throughout, R will denote an associative ring (not necessarily with unity 1), $N(R)$ the set of nilpotent elements of R , $Z(R)$ the center of R , and B a non empty subset (not necessarily a subring) of R . $\mathbf{Z}[\lambda]$ is the totality of polynomials in λ with coefficients in $\mathbf{Z}$, the ring of integers. By $(GF(q))_2$, we mean the ring of 2×2 matrices over the Galois field $GF(q)$ with q elements. As usual, $[x, y]$ will denote the commutator $xy - yx$.

Following [8], R is called a left (resp. right) ring s-unital if for each $x \in R$ there holds $x \in Rx$ (resp. $x \in xR$). Further, R is called s-unital if it is both left and right s-unital, that is $x \in xR \cap Rx$ for each $x \in R$. If R is an s-unital (resp. a left s-unital or a right s-unital) ring, then for any finite subset F of R there exists an element $e \in R$ such that $ex = xe = x$ (resp. $ex = x$ or $xe = x$) for all $x \in F$ (see [8, Theorem 1] and [7, Lemma 1]). Such an element e will be called a pseudo-identity (pseudo left-identity or pseudo right-identity) of F in R .

In the present paper, we consider the following properties: (*-B): For each x in R , there exists a polynomial $f(\lambda)$ in $\mathbf{Z}[\lambda]$ such that $x - x^2 f(x) \in B$. (**-B): For each x in R , either $x \in Z(R)$, or there exists a polynomial $f(\lambda)$ in $\mathbf{Z}[\lambda]$ such that $x - x^2 f(x) \in B$. (***-B): For every $a \in B$ and $x \in R$, $[[a, x], x] = 0$. (p₁): For each x, y in R there exists integers $m = m(x, y) \geq 0, n = n(x, y) > 1, r = r(x, y) \geq 0, s = s(x, y) \geq 0$ such that $[x, x^m y \pm x^r y^n x^s] = 0$. (p₂): For each x, y in R there exist integers $m = m(x, y) \geq 0, n = n(x, y) > 1, r = r(x, y) \geq 0$ and $s = s(x, y) \geq 0$ such that $[x, yx^m \pm x^r y^n x^s] = 0$.

In 1953, Herstein [4] proved a classical theorem of commutativity of rings satisfying the condition $(*-Z(R))$. Also a nice theorem of Bell [3] establishes commutativity of all rings satisfying $(*-B)$ under some restrictions on B . Many authors have studied the commutativity of rings satisfying the property $(*-B)$, but always under some restriction on B . Several special cases of (p₁) and (p₂) are known to imply commutativity of rings (see [1], [2] and [9]). For instance, if the integral indices in the underlying conditions are global. The major purpose of this paper is to investigate the commutativity of R when the integral indices are local (i.e. depending on pair of ring elements x and y). In section 1 of this paper, we explore the commutativity of R when $(**-B)$ holds for a nil subset B together with either of the properties (p₁) and (p₂). Our motivation for studying the commutativity of a ring R satisfying $(**-B)$ for a commutative nil subset B was an attempt to prove a theorem extending a recent result of Tominaga and Yaqub [10] and we present this theorem in section 2. In fact, several

commutativity theorems can be obtained as a corollaries to our results (see [1], [2] and [5]).

1. Rings satisfying (p_1) or (p_2) with $(**B)$ for non-commutative B .

Theorem 1. If R is a ring with unity 1, then following statements are equivalent:

- (a) R is commutative
- (b) R satisfies the condition (p_1) and there exists a nil subset B of R for which R satisfies $(**B)$.
- (c) R satisfies the condition (p_2) and there exists a nil subset B of R for which R satisfies $(**B)$.

In order to prove our Theorem 1, we state the following well-known lemma:

Lemma 1. Let R be a ring with unity 1 and suppose that f is any polynomial function of two variables on R with the property that $f(x+1, y) = f(x, y)$ for all x, y in R . If there exists a positive integer n such that $x^n f(x, y) = 0$, then $f(x, y) = 0$ for all x, y in R .

Proof of Theorem 1: Trivially, one can check that (a) $\Rightarrow$ (b) and (a) $\Rightarrow$ (c). It is enough to prove that (b) $\Rightarrow$ (a) and (c) $\Rightarrow$ (a). First, we show that (b) $\Rightarrow$ (a). Let $u \in N(R)$ and x be an arbitrary element of R and let R satisfy (p_1) . Then there exist integers $n_1 = n(x, u) > 1, m_1 = m(x, u) \geq 0, r_1 = r(x, u) \geq 0$ and $s_1 = s(x, u) \geq 0$ such that for all $x, y \in R$,

$$x^{m_1}[x, u] = x^{r_1}[x, u^{n_1}]x^{s_1},$$

$$x^{m_1}[x, u] = x^{r_1}[u^{n_1}, x]x^{s_1}.$$

Now, let $n_2 = n(x, u^{n_1}) > 1, m_2 = m(x, u^{n_1}) \geq 0, r_2 = r(x, u^{n_1}) \geq 0$ and $s_2 = s(x, u^{n_1}) \geq 0$. Then

$$x^{m_1+m_2}[x, u] = x^{m_2}(x^{r_1}[x, u^{n_1}]x^{s_1})$$

$$x^{m_1+m_2}[x, u] = x^{r_1+r_2}[x, u^{n_1 n_2}]x^{s_1+s_2}$$

and

$$x^{m_1+m_2}[x, u] = x^{r_1+r_2}[u^{n_1 n_2}, x]x^{s_1+s_2}.$$

Hence, for any positive integer t , we have integers

$$n_1, n_2, \dots, n_t > 1, m_1, m_2, \dots, m_t \geq 0, r_1, r_2, \dots, r_t \geq 0,$$

and $s_1, s_2, \dots, s_t \geq 0$ such that

$$x^{m_1 + m_2 + \dots + m_t} [x, u] = x^{r_1 + r_2 + \dots + r_t} [x, u^{n_1 n_2 \dots n_t}] x^{s_1 + s_2 + \dots + s_t}$$

and

$$x^{m_1 + m_2 + \dots + m_t} [x, u] = x^{r_1 + r_2 + \dots + r_t} [u^{n_1 n_2 \dots n_t}, x] x^{s_1 + s_2 + \dots + s_t}.$$

But $u^{n_1 n_2 \dots n_t} = 0$ for sufficiently large t . Therefore, $x^{m_1 + m_2 + \dots + m_t} [x, u] = 0$ and by Lemma 1, $[x, u] = 0$ and hence $u \in Z(R)$. Therefore $N(R) \subseteq Z(R)$ and in view of (**-B) R satisfies $(* - Z(R))$. Thus R is commutative by Herstein's Theorem [4]. Next, let (c) $\Rightarrow$ (a) and R satisfies (p_2) . Then by using same steps as above, we obtain the commutativity of R .

The following example shows that it is essential to keep any one of the conditions (p_1) and (p_2) together with (**-B) in the hypotheses of Theorem 1 to get the commutativity of R .

Example 1. Consider

$$R = \{wI + X \mid X = \begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ y & z & 0 \end{pmatrix}, I = \begin{pmatrix} 100 \\ 010 \\ 001 \end{pmatrix}\}$$

where $w, x, y, z \in GF(2)\}$.

Trivially, one can see that $N(R) = \{X\}$ and R satisfies neither (p_1) nor (p_2) . Let $B = N(R)$. Then for any $x \in R$ such that $\lambda - \lambda^2 f(\lambda) \in B$. But, R is non-commutative.

Remark 1. One might observe that in the statement of condition (p_1) and (p_2) , the requirement $n(x, y) > 1$ is necessary in order to prove Theorem 1 since if R is a ring of characteristic 2, then R satisfies (p_1) and (p_2) for global values of $m = r = s = 0$ and $n = 1$, and Example 1 gives a non-commutative ring that satisfies (**-E).

Example 2. Consider $R = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ \alpha & 0 & 0 \\ \beta & \gamma & 0 \end{pmatrix} \mid \alpha, \beta, \gamma \in GF(2) \right\}$. It is easy to see that R is a nilpotent ring of index 3. Hence $R = N(R)$ and R satisfies both conditions (p_1) and (p_2) . Further, let $B = N(R)$. Then R also satisfies (**-B). However, R is non-commutative.

Remark 2. Example 2 shows that the existence of unity is essential in the hypothesis of Theorem 1.

Remark 3. The following example rules out the possibility that $n(x, y) = 1$ in Theorem 1.

Example 3. Let $G = \langle \epsilon_1, \epsilon_2, \dots; \epsilon_i^2 = 0, \epsilon_i \epsilon_j + \epsilon_j \epsilon_i = 0 \rangle$ be a Grassman algebra. Set $x = \epsilon_1$ and $y = \epsilon_2$ with $\epsilon_1^2 = 0$ and $\epsilon_1 \epsilon_2 + \epsilon_2 \epsilon_1 = 0$. It can easily be verified that G satisfies (p₁) and (p₂) for all non-negative values of m, r, s and $n = 1$. But, G is non-commutative with unity.

Now, Theorem 1 may be generalized as follows:

Theorem 2. Let R be a left s-unital ring satisfying (p₁) and suppose there exists a subset B of $N(R)$ for which R satisfies (**-B). Then R is commutative (and conversely).

Theorem 3. Let R be a right s-unital ring satisfying (p₂). Suppose, further that there exists a subset B of $N(R)$ for which R satisfies (**-B). Then R is commutative (and conversely). Following Lemmas are essential in proving our theorem.

Lemma 2 [10]. Let R be a left (resp right) s-unital ring. If for each pair of elements x and y in R , there exists a positive integer $k = k(x, y)$ and an element $e = e(x, y)$ of R such that $x^k e = x^k$ and $y^k e = y^k$ (resp $e x^k = x^k$ and $e y^k = y^k$), then R is an s-unital ring.

Lemma 3. Let R be a left s-unital ring satisfying (p₁). Then R is an s-unital ring.

Proof. Let $x, y \in R$. Then there exists $e = e(x, y) \in R$ such that $ex = x$ and $ey = y$. Further, there exist nonnegative integers $n = n(x, e) > 1, m = m(x, e), r = r(x, e)$, and $s = s(x, e) > 0$ such that $x^m[x, e] = \pm x^r[x, e^n]x^s$. So $x^{m+1}e - x^m ex = \pm(x^{r+1}e^n x^s - x^r e^n x^{s+1})$. Hence $x^{m+1}e = x^{m+1}$. Also, if $n_1 = n(y, e) > 1, m_1 = m(y, e), r_1 = r(y, e)$ and $s_1 = s(y, e) > 0$ are non-negative integers, then $y^{m_1+1}e = y^{m_1+1}$. Hence $x^{m+m_1+2}e = x^{m+1}(x^{m_1+1}e) = x^{m+m_1+2}$ and $y^{m+m_1+2}e = y^{m+m_1+2}$. Therefore, R is s-unital by Lemma 2. Let $s = s(x, y) = 0$. Then $x^{m+1}y - x^m yx = \pm(x^{r+1}y^n - x^r y^n x)$. So $e^{m+1}y - e^m ye = \pm(e^{r+1}y^n - e^r y^n e)$ and thus $y = ye \pm (y^n - y^n e) = y(e \pm (y^{n-1} - y^{n-1}e)) \in yR$. Thus R is right s-unital and hence R is s-unital.

Lemma 4. Let R be right s-unital ring satisfying (p₂). Then R is an s-unital ring.

Proof. Let $x, y \in R$. Then there exists $e = e(x, y) \in R$ such that $xe = x$ and $ye = y$. Also, there exists non-negative integers $n = n(x, e) > 1, m = m(x, e), r = r(x, e) > 0$, and $s = s(x, e)$ such that $x^{r+1} = ex^{r+1}$. Also if

$n_1 = n(y, e) > 1, r_1 = r(y, e) > 0, m_1 = m(y, e)$ and $s_1 = s(y, e)$ are non-negative integers, then we obtain $y^{m_1+1} = ey^{m_1+1}$. Hence $ex^{m+m_1+2} = x^{m+m_1+2}$ and $ey^{m+m_1+2} = y^{m+m_1+2}$. Therefore, R is s-unital by Lemma 2. If $r = r(x, y) = 0$, then $(xy - yx)x^m = \pm(xy^n - y^n x)x^s$ and we can easily see that $y = (e \mp (ey^{n-1} - y^{n-1}))y \in Ry$ for $n = n(e, y) > 1$. Thus R is left s-unital. Hence R is s-unital.

Proof of Theorem 2. Let R be left s-unital ring satisfying (p_1) . Then by Lemma 3, R is s-unital. In view of [5, Proposition 1] we may assume that R has unity 1 and thus R is commutative by Theorem 1.

Corollary 1 [2]. Let $n > 1, m, r$ and s be fixed non-negative integers. If R is a left s-unital ring satisfies $[x^m y \pm x^r y^m x^s, x] = 0$ for all x, y in R , then R is commutative.

Proof of Theorem 3. Let R be right s-unital ring satisfying (p_2) . Then by Lemma 4, R is s-unital. In view of proposition of [5], we may assume that R has unity 1 and so R is commutative by Theorem 1.

Corollary 2. Let $m > 1, n, r$ and s be fixed non-negative integers. If R is a right s-unital ring which satisfies $[yx^m \pm x^r y^n x^s, x] = 0$ for all $x, y \in R$, then R is commutative.

Remark 4. Let $m = r = 0$ (resp $m = s = 0$) in (p_1) (resp. (p_2)). Then

$$(1) \quad [x, y] = \pm[y^n, x]x^s \text{ for all } x, y \in R$$

(resp.

$$(2) \quad [x, y] = \pm x^r[y^n, x] \text{ for all } x, y \in R).$$

Let $n > 1$ or $s \geq 1$ in (1) (resp $n > 1$ or $r \geq 1$ in (2)). Then R is a $(\mathbf{Z}, \beta)$ -ring in the sense of Streb [11], hence R is commutative even if R is not assumed to be a left (resp. right) s-unital ring (ring with unity 1), (cf [11, Lemma 9]).

Remark 5. The following examples demonstrate that there are non-commutative left (resp. right) s-unital rings satisfying (p_1) (resp. (p_2)).

Example 4. Consider $S_1 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}$ and

$$S_2 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}$$

be two subrings of 2×2 matrices over $GF(2)$. One can easily see that in both cases $N(R)$ is the set consisting of the matrices $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

Next, let $B = N(R)$. Then for any x in R we can find a polynomial $f(\lambda)$, for instance, $f(\lambda) = \lambda$ such that $x - x^2 f(x)$ in B . But, S_1 is a left s-unital ring that satisfies (p_1) and S_2 is a right s-unital ring that satisfies (p_2) for any $n > 1, m, r, s \geq 0$.

2. Rings satisfying (**-B) for commutative B .

Our final theorem, a companion theorem to Theorem 3 of [10] is

Theorem 4. Let R be a normal ring, and let B be a nil commutative subset of R for which R satisfies (**-B). Then R is commutative. Before beginning the general proof, we need the following lemma which proved in [10].

Lemma 5. (i) Let ϕ be a ring homomorphism of R onto R^* . If R satisfies (*-B), (**-B) or (* * *-B), then R^* satisfies $(* - \phi(B)), (* * - \phi(B))$ or $(*** - \phi(B))$ respectively. (ii) If B is commutative and R satisfies (**-B), then $N(R)$ is a commutative nil ideal of R and contained in a centralizer of B , in particular $N(R)^2 \subseteq Z(R)$. (iii) If there exists a commutative subset B of $N(R)$ for which R satisfies (**-B) and (***-B), then R is commutative.

Proof of Theorem 4. By Lemma 5(i) R assumed to be subdirectly irreducible. If x is an arbitrary element in $R/Z(R)$, using (**-B), we obtain that there exists $y \in \langle x \rangle$ and a positive integer n such that $x^n = x^{n+1}y$. Trivially, $e = x^n y^n$ is an idempotent with $x^n = x^n e$. But idempotents of R are central, thus e is either 0 or 1. Since R has no unity then $e = 0$ and by lemma 5(ii) x is in the commutative ideal N and so $[[a, x], x] = 0$ for all $a \in B$. Thus R is commutative by Lemma 5 (iii).

Remark 6. The above example 4 shows that the property (**-B) alone does not imply commutativity of rings in Theorem 4.

In this regards one might gives the following:

Conjecture. If R is a ring satisfies any one of the properties (p_1) and (p_2) . Moreover, if there exists a nil commutative subset B for which R satisfies (**-B), then R is commutative.

Remark 7. From the proof of Theorem 4, we can observe that the above conjecture is true when R is normal. Example 4 shows that the above conjecture is not true because the centrality of idempotent in S_1 (resp S_2) are not followed by (p_1) (resp. (p_2)) together with (**-B).

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