A SHIL'NIKOV'S THEOREM IN $\mathbb{R}^4$ IN AN EXTENDED NEIGHBORHOOD OF A SADDLE POINT OF FOCUS - FOCUS TYPE

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Abstract

In this article we study the existence of chaotic solutions in a neighborhood of a homoclinic orbit of a saddle point of focus-focus type. We will prove that the solutions have an exponential expansion, this fact implies the existence of a subsystem of solutions, which is in one-to-one correspondence with the set of doubly infinite sequences.

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1. Introduction.

In [8] Shil'nikov proved that in a neighborhood of a trajectory which is doubly asymptotic to a hyperbolic equilibrium point of a saddle-focus type, under certain conditions for the spectrum of the linear part, there exists a subsystem of solution where the trajectories are in a one-to-one correspondence with a countable set of doubly infinite sequences of positive integers (Symbolic Dynamic). That is, linear part of the system has three eigenvalues, one real and the other two complex such that their real parts are closer to the origin than the real parts of the other eigenvalues, then there exists a subsystem of solutions in one-to-one correspondence with sequences of the type $(\dots, k_n, k_{n+1}, \dots)$ with $k_n < \rho k_{n+1}$, $\rho > 1$, such that when it passes by a neighborhood of the fixed point, the orbit has k_n semi-turns around the point; it continues by a tubular neighborhood of a homoclinic orbit and when it returns to the neighborhood of the equilibrium point it has k_{n+1} semi-turns and so on. This result is known as the Shil'nikov Theorem.

Furthermore Tresser [9] proved in a more geometric manner the existence of chaotic solutions in a neighborhood of a homoclinic solution, where the presence of an infinite number of Smale horseshoes is illustrated.

Here, we prove the existence of chaotic solutions in a tubular neighborhood of the homoclinic orbit for a 4-dimensional dynamical system, such that the system has two pairs of complex conjugate eigenvalues. In this case, Shil'nikov [7] proved the existence of a denumerable set of periodic motions. The proof of our theorem is similar to that of Shil'nikov [8]. We construct a Poincaré map T whose domain of definition is representable as a countable union of mutually disjoint domains on which T is a composition of two maps T_0 and T_1 where T_0 is defined in a neighborhood of the origin and T_1 in a tubular neighborhood of the homoclinic orbit.

2. Presentation of the Problem.

We consider a system of four differential equations

$$(1) \quad \dot{z} = Z(z)$$

where $Z \in C^k(\mathbb{R}^4, \mathbb{R}^4)$, $k \geq 4$. We will assume that the system (1) has an equilibrium point 0; i.e., $Z(0) = 0$, at which the characteristic equation

$$\left| \frac{\partial Z}{\partial t}(0) - \lambda Id_4 \right| = 0$$

satisfies the following conditions:

(H1) The system has two pairs of complex conjugate eigenvalues $\lambda \pm i\omega$ and $\gamma \pm i\mu$, where $\lambda < 0$ and $\gamma > 0$.

(H2) There exists a homoclinic orbit Γ_0 , issuing from 0 and returning to it, as $t \rightarrow +\infty$, i.e., $\Gamma_0 \subset (M^+ \cap M^-)$, where M^+ and M^- are the stable and unstable manifolds, respectively. Furthermore, it satisfies:

$$(2) \quad \dim(TM_p^+ \cap TM_p^-) = 1$$

where TM_p^+ and TM_p^- are the tangent spaces at $p \in \Gamma_0$ to the stable and unstable manifolds, respectively.

(H3) $\gamma + \lambda \neq 0$.

Definition 1. A focus-focus point is a saddle point which satisfies condition (H1). A small enough tubular neighborhood of the homoclinic orbit is called an extended neighborhood of the focus-focus point.

We consider the set of doubly infinite sequences

$$(\dots, j_i, j_{i+1}, \dots)$$

consisting of the symbols $0, 1, 2, \dots$ satisfying that for arbitrary adjacent indices $j_{i+1} < \rho j_i$ for some $\rho > 1$. We denote this set by $\Omega(\rho)$; i.e.:

$$\Omega(\rho) = \{ (\dots, j_i, j_{i+1}, \dots) / j_i \in \mathbb{N} \cup \{0\}, j_{i+1} < \rho j_i, \text{ for some } \rho > 1 \}.$$

Under assumptions (H1)–(H3) we prove the following theorem:

Theorem 1. For $\lambda + \gamma < 0$ (resp. for $\lambda + \gamma > 0$) there exists a subsystem of solutions of (1) in one-to-one correspondence with the set $\Omega(\rho)$ with $\rho > -\frac{\lambda}{\gamma} > 1$ (resp. $\rho > -\frac{\gamma}{\lambda} > 1$)

Without loss of generality we may assume that $\lambda + \gamma < 0$. For the other case we change t for $-t$.

3. Construction of the map T .

3.1. Construction of T_0 . We can rewrite system (1) in a neighborhood of the origin by a nondegenerate analytic nonlinear change of variables as:

$$(3) \quad \begin{aligned} \dot{x}_1 &= \lambda x_1 - \omega x_2 + P_1(x_1, x_2, y_1, y_2) \\ \dot{x}_2 &= \omega x_1 + \lambda x_2 + P_2(x_1, x_2, y_1, y_2) \\ \dot{y}_1 &= \gamma y_1 - \mu y_2 + Q_1(x_1, x_2, y_1, y_2) \\ \dot{y}_2 &= \mu y_1 + \gamma y_2 + Q_2(x_1, x_2, y_1, y_2) \end{aligned}$$

where the origin is the focus-focus point and the functions P_1 , P_2 , Q_1 , and Q_2 vanish at the origin together with their first derivatives.

By stable and unstable manifolds theorem, the equations in the new variables for the stable manifold M^+ are:

$$y_1 = \phi_1(x_1, x_2) \quad y_2 = \phi_2(x_1, x_2)$$

and the equations for the unstable manifold M^- are:

$$x_1 = \psi_1(y_1, y_2) \quad x_2 = \psi_2(y_1, y_2)$$

ϕ_1 , ϕ_2 , ψ_1 , and ψ_2 are C^{k-1} which vanish at the origin together with their first derivatives.

System (3) can be reduced by the normal form and the substitution:

$$\begin{aligned} \xi_1 &= x_1 - \psi_1(y_1, y_2) \quad , \quad \xi_2 = x_2 - \psi_2(y_1, y_2) \\ \eta_1 &= y_1 - \phi_1(x_1, x_2) \quad , \quad \eta_2 = y_2 - \phi_2(x_1, x_2) \end{aligned}$$

to the system:

$$(4) \quad \begin{aligned} \dot{\xi}_1 &= \lambda \xi_1 - \omega \xi_2 + P_{11}(\xi_1, \xi_2, \eta_1, \eta_2) \xi_1 + \\ &\quad + P_{12}(\xi_1, \xi_2, \eta_1, \eta_2) \xi_2 \\ \dot{\xi}_2 &= \omega \xi_1 + \lambda \xi_2 + P_{21}(\xi_1, \xi_2, \eta_1, \eta_2) \xi_1 + \\ &\quad + P_{22}(\xi_1, \xi_2, \eta_1, \eta_2) \xi_2 \\ \dot{\eta}_1 &= \gamma \eta_1 - \mu \eta_2 + Q_{11}(\xi_1, \xi_2, \eta_1, \eta_2) \eta_1 + \\ &\quad + Q_{12}(\xi_1, \xi_2, \eta_1, \eta_2) \eta_2 \\ \dot{\eta}_2 &= \mu \eta_1 + \gamma \eta_2 + Q_{21}(\xi_1, \xi_2, \eta_1, \eta_2) \eta_1 + \\ &\quad + Q_{22}(\xi_1, \xi_2, \eta_1, \eta_2) \eta_2 \end{aligned}$$

where P_{ij} and Q_{ij} vanish at the origin of the new variables whose equations for the stable manifold M^+ are $\eta_1 = 0$, $\eta_2 = 0$ and for the unstable manifold M^- are $\xi_1 = 0$ and $\xi_2 = 0$, with $i, j = 1, 2$.

Let Σ_0 and Σ_1 be the surfaces $\Sigma_0 : \xi_1^2 + \xi_2^2 = r_1^2$, $\eta_1^2 + \eta_2^2 \leq r_1^2$ and $\Sigma_1 : \xi_1^2 + \xi_2^2 \leq r_2^2$, $\eta_1^2 + \eta_2^2 = r_2^2$, with $r_1, r_2 > 0$, $r_1^2 \neq r_2^2$ sufficiently small and the orbit of system (4) passing through the point $M_0(\xi_1^0, \xi_2^0, \eta_1^0, \eta_2^0) \in \Sigma_0$ at $t = 0$, where $(\eta_1^0)^2 + (\eta_2^0)^2 \neq 0$, and by the point $M_1(\xi_1^1, \xi_2^1, \eta_1^1, \eta_2^1) \in \Sigma_1$ at $t = t_0$, where $(\xi_1^0)^2 + (\xi_2^0)^2 < r_2^2$.

We consider the following integral form for T_0 :

$$\begin{aligned}
 \xi_1(t) &= e^{\lambda t} [\xi_1^0 \cos \omega t - \xi_2^0 \sin \omega t] + \\
 &\quad + \int_0^t e^{\lambda(t-s)} [\bar{P}_1 \cos \omega(t-s) - \bar{P}_2 \sin \omega(t-s)] ds \\
 \xi_2(t) &= e^{\lambda t} [\xi_1^0 \sin \omega t + \xi_2^0 \cos \omega t] + \\
 &\quad + \int_0^t e^{\lambda(t-s)} [\bar{P}_1 \sin \omega(t-s) + \bar{P}_2 \cos \omega(t-s)] ds \\
 \eta_1(t) &= e^{\gamma(t-t_0)} [\eta_1^1 \cos \mu(t-t_0) - \eta_2^1 \sin \mu(t-t_0)] + \\
 &\quad + \int_{t_0}^t e^{\gamma(t-s)} [\bar{Q}_1 \cos \mu(t-s) - \bar{Q}_2 \sin \mu(t-s)] ds \\
 \eta_2(t) &= e^{\gamma(t-t_0)} [\eta_1^1 \sin \mu(t-t_0) + \eta_2^1 \cos \mu(t-t_0)] + \\
 &\quad + \int_{t_0}^t e^{\gamma(t-s)} [\bar{Q}_1 \sin \mu(t-s) + \bar{Q}_2 \cos \mu(t-s)] ds
 \end{aligned}
 \tag{5}$$

this form is equivalent to equation (4). Solutions of (5) are also solutions of system (4), passing through M_0 and M_1 and t_0 is the time of transition from Σ_0 to Σ_1 , $\bar{P}_1, \bar{P}_2, \bar{Q}_1$, and $\bar{Q}_2$ denote the nonlinear terms of the first, second, third, and forth equation of the system (4), respectively. This system is equivalent to:

$$\begin{aligned}
 \dot{\theta} &= A\theta + R_1(\theta, \sigma) \theta \\
 \dot{\sigma} &= B\sigma + S_1(\theta, \sigma) \sigma
 \end{aligned}
 \tag{6}$$

where $\theta(t) = (\xi_1(t), \xi_2(t))$, $\sigma(t) = (\eta_1(t), \eta_2(t))$, $A = \begin{pmatrix} \lambda & -\omega \\ \omega & \lambda \end{pmatrix}$, $B = \begin{pmatrix} \gamma & -\mu \\ \mu & \gamma \end{pmatrix}$ and $R_1(\theta(t), \sigma(t))$ and $S_1(\theta(t), \sigma(t))$ are the nonlinear part matrices of the integral representation of system (4). Furthermore,

this system satisfies:

$$\begin{aligned} S_1(0, \sigma) &= 0 \quad , \quad \left. \frac{\partial^k S_1}{\partial \sigma^k} \right|_{\theta=0} = 0 \\ S_1(\theta, 0) &= 0 \quad , \quad \left. \frac{\partial^k S_1}{\partial \theta^k} \right|_{\sigma=0} = 0 \end{aligned}$$

Remark: This result was proved by Deng in [2].

The parametric representation for T_0 is:

$$\begin{aligned} \theta(t) &= e^{At} \theta_0 + \int_0^t e^{A(t-s)} R_1(\theta(s), \sigma(s)) \theta(s) ds \\ \sigma(t) &= e^{B(t-t_0)} \sigma_1 + \int_{t_0}^t e^{B(t-s)} S_1(\theta(s), \sigma(s)) \sigma(s) ds \end{aligned} \quad (7)$$

A similar proof to the following theorem can be find in [1], which is reported without proof.

Theorem 2. *There is $\delta > 0$ such that there exists a unique solution of system (6) such that $\|(\theta(t), \sigma(t))\| < \delta$, for $\|R_1\| \leq M(\delta)(\|\theta\| + \|\sigma\|)$, $\|S_1\| \leq M(\delta)(\|\theta\| + \|\sigma\|)$, $\|\theta_0\|$ and $\|\sigma_1\| \leq \frac{\delta}{2K}$, with δ sufficiently small and $t_0 > 0$.*

Furthermore, we can choose δ small enough, to obtain the following theorem:

Theorem 3. *For $\|\theta_0\|, \|\sigma_1\| \leq \delta$ and $t_0 > 0$, we have:*

$$\begin{aligned} \|\theta(t)\| &\leq \overline{K} e^{\lambda t} \\ \|\sigma(t)\| &\leq \overline{K} e^{\gamma(t-t_0)} \end{aligned} \quad (8)$$

for $0 \leq t \leq t_0$, $\overline{K} > 0$.

Proof: Let us define

$$I = \{(\theta(t), \sigma(t)) / \theta(t), \sigma(t) \text{ satisfy (8)}\},$$

I is a complete metric space with the metric given by

$$d((\theta_1, \sigma_1); (\theta_2, \sigma_2)) = \|\theta_1 - \theta_2\| + \|\sigma_1 - \sigma_2\|$$

We define an operator Q in I by

$$Q : I \longrightarrow I$$

such that

$$Q(\theta, \sigma) = (Q_1(\theta, \sigma), Q_2(\theta, \sigma))$$

where Q_1, Q_2 are the right-hand sides of (7).

We prove that Q is well defined in I and is a contraction, so, there exists a fixed point in I .

We show the proof for Q_1 , because for Q_2 it is similar.

$$Q_1(\theta, \sigma) = e^{At}\theta_0 + \int_0^t e^{A(t-s)}R_1(\theta(s), \sigma(s))\theta(s)ds$$

$$\begin{aligned} \|Q_1(\theta, \sigma)\| &\leq \|e^{At}\theta_0\| + \int_0^t \|e^{A(t-s)}\| \|R_1(\theta(s), \sigma(s))\| \|\theta(s)\| ds \\ &\leq \overline{K}e^{\lambda t}\|\theta_0\| + \int_0^t \overline{K}e^{\lambda(t-s)}\|R_1(\theta(s), \sigma(s))\| \|\theta(s)\| ds \\ &\leq e^{\lambda t} \left(\overline{K}\delta + \overline{K}^2 \int_0^t e^{-\lambda s} M(\delta) (\|\theta(s)\| + \|\sigma(s)\|) e^{\lambda s} ds \right) \\ &\leq e^{\lambda t} \left(\overline{K}\delta + \overline{K}^3 M(\delta) \int_0^t (e^{\lambda s} + e^{\gamma(s-t_0)}) ds \right) \end{aligned}$$

then, the expression in parenthesis is less than a constant $\overline{K}$, i.e.,

$$\|Q_1(\theta, \sigma)\| \leq \overline{K}e^{\lambda t}$$

Now, we prove that Q_1 is a contraction

$$\begin{aligned} \|Q_1(\theta^1, \sigma^1) - Q_1(\theta^2, \sigma^2)\| &\leq \overline{K} M(\delta) \int_0^t e^{\lambda(t-s)} d((\theta^1, \sigma^1); (\theta^2, \sigma^2)) ds \\ &\leq q_3 d((\theta^1, \sigma^1); (\theta^2, \sigma^2)) \end{aligned}$$

with $0 < q_3 < 1$ for δ small enough.

Thus, there exists a fixed point for Q in I .

□

Definition 2. We say that $\theta(t)$ and $\sigma(t)$ admits an exponential expansion of regularity r if there exists C^r functions $\varphi_1 = \varphi_1(\theta_0, \sigma_1)$, $\varphi_2 = \varphi_2(t_0, \theta_0, \sigma_1)$, $U_1 = U_1(t, \theta_0, \sigma_1)$, and $U_2 = U_2(t, t_0, \theta_0, \sigma_1)$ with $r \geq 1$ such that

$$\begin{aligned}\theta(t) &= e^{-At}(\varphi_1(\theta_0, \sigma_1) + U_1(t, \theta_0, \sigma_1)) \\ \sigma(t) &= e^{-B(t-t_0)}(\varphi_2(t_0, \theta_0, \sigma_1) + U_2(t, t_0, \theta_0, \sigma_1))\end{aligned}$$

where

$$\begin{aligned}\frac{\partial \varphi_1}{\partial \sigma_1}(0, 0) &= 0, \quad \frac{\partial \varphi_1}{\partial \theta_0}(0, 0) = Id_2 \\ \frac{\partial \varphi_2}{\partial \theta_0}(t_0, 0, 0) &= 0, \quad \frac{\partial \varphi_2}{\partial \sigma_1}(t_0, 0, 0) = Id_2\end{aligned}$$

and U_1 and U_2 tend exponentially to zero together with their derivatives, as $t_0 \rightarrow +\infty$.

The variable $\sigma(t)$ has an exponential expansion in the unstable manifold (see [2]). We prove that $\theta(t)$ admits an exponential expansion.

Theorem 4. For a system in the form (6), $\theta(t)$ admits an exponential expansion of regularity $r - 2$.

Proof: We have the bounds (previous to the above theorem):

$$\|\theta(t)\| \leq \overline{K}e^{\lambda t} \quad \text{and} \quad \|R_1\| \leq M(\delta)(\|\theta(t)\| + \|\sigma(t)\|)$$

and if

$$\theta(t) = e^{At}\theta_0 + \int_0^t e^{A(t-s)}R_1(\theta(s), \sigma(s))\theta(s)ds$$

we obtain:

$$\|\theta(t)\|e^{-\lambda t} \leq \|\theta_0\| + M(\delta) \int_0^t (\|\theta(s)\| + \|\sigma(s)\|)ds$$

Let us define

$$\varphi_1(t_0, \theta_0, \sigma_1) = \lim_{t \rightarrow \infty} \theta(t)e^{-At}$$

$$\|\varphi_1(t_0, \theta_0, \sigma_1)\| \leq \|\theta_0\| + M(\delta) \int_0^\infty (e^{\lambda s} + e^{\gamma(t-s)})ds$$

and

$$U_1(t_0, \theta_0, \sigma_1) = e^{At}(\theta(t)e^{-At} - \varphi_1(t_0, \theta_0, \sigma_1))$$

So, $\theta(t)$ has an exponential expansion.

□

Due to the fact that we have this exponential expansion, the solution of system (4):

$$\begin{aligned}\xi_1(t) &= \xi_1^p(t, t_0, \xi_1^0, \xi_2^0, \eta_1^1, \eta_2^1) \\ \xi_2(t) &= \xi_2^p(t, t_0, \xi_1^0, \xi_2^0, \eta_1^1, \eta_2^1) \\ \eta_1(t) &= \eta_1^p(t, t_0, \xi_1^0, \xi_2^0, \eta_1^1, \eta_2^1) \\ \eta_2(t) &= \eta_2^p(t, t_0, \xi_1^0, \xi_2^0, \eta_1^1, \eta_2^1)\end{aligned}$$

can be represented in the following form:

$$\begin{aligned}\xi_1(t) &= e^{\lambda t}[\xi_1^0(1 + \alpha_{11}^0 + \beta_{11}^0) \cos \omega t - \xi_2^0(1 + \alpha_{12}^0 + \beta_{12}^0) \sin \omega t] \\ \xi_2(t) &= e^{\lambda t}[\xi_1^0(1 + \alpha_{21}^0 + \beta_{21}^0) \sin \omega t + \xi_2^0(1 + \alpha_{22}^0 + \beta_{22}^0) \cos \omega t] \\ (9) \quad \eta_1(t) &= e^{\gamma(t-t_0)}[\eta_1^1(1 + \alpha_{11}^1 + \beta_{11}^1) \cos \mu(t - t_0) - \\ &\quad - \eta_2^1(1 + \alpha_{12}^1 + \beta_{12}^1) \sin \mu(t - t_0)] \\ \eta_2(t) &= e^{\gamma(t-t_0)}[\eta_1^1(1 + \alpha_{21}^1 + \beta_{21}^1) \sin \mu(t - t_0) + \\ &\quad + \eta_2^1(1 + \alpha_{22}^1 + \beta_{22}^1) \cos \mu(t - t_0)]\end{aligned}$$

where $\alpha_{ij}^0(t - t_0, \xi_1^0, \xi_2^0, \eta_1^1, \eta_2^1)$, $\alpha_{ij}^1(t_0, \xi_1^0, \xi_2^0, \eta_1^1, \eta_2^1)$, $\beta_{ij}^k(t, t_0, \xi_1^0, \xi_2^0, \eta_1^1, \eta_2^1)$ are analytic functions satisfying conditions $\alpha_{ij}^k(0, \xi_1^0, \xi_2^0, \eta_1^1, \eta_2^1) = 0$ for $\xi_1^0 = \xi_2^0 = \eta_1^1 = \eta_2^1 = 0$ and $\beta_{ij}^0(t_0, t_0, \xi_1^0, \xi_2^0, \eta_1^1, \eta_2^1)$, $\beta_{ij}^0(0, t_0, \xi_1^0, \xi_2^0, \eta_1^1, \eta_2^1)$ converge to zero, together with their first derivatives, as $t \rightarrow +\infty$ with $i, j, k = 1, 2$.

3.2. Construction of T_1 . The theorem of continuous dependence on initial conditions guarantees that the map T_1 can be considered in the

neighborhood of the orbit of system (4), and it can be represented in the form:

$$(10) \quad \bar{\xi}_2^0 = f(\xi_1^1, \xi_2^1, \eta_2^1) = A_1 \xi_1^1 + A_2 \xi_2^1 + B \eta_2^1 + \dots$$

$$(11) \quad \bar{\eta}_i^0 = g_i(\xi_1^1, \xi_2^1, \eta_2^1) = A_{i1} \xi_1^1 + A_{i2} \xi_2^1 + B_i \eta_2^1 + \dots \quad i = 1, 2$$

Let consider the mapping $T = T_1 \circ T_0$ as:

$$\begin{aligned} \bar{\xi}_2^0 &= f(\xi_1^p(t_0, t_0, \xi_1^0, \xi_2^0, \eta_1^1, \eta_2^1), \xi_2^p(t_0, t_0, \xi_1^0, \xi_2^0, \eta_1^1, \eta_2^1), \eta_2^1) \\ \eta_i^0(0, \bar{t}_0, \bar{\xi}_1^0, \bar{\xi}_2^0, \bar{\eta}_1^1, \bar{\eta}_2^1) &= g_i(\xi_1^p(t_0, t_0, \xi_1^0, \xi_2^0, \eta_1^1, \eta_2^1), \xi_2^p(t_0, t_0, \xi_1^0, \xi_2^0, \eta_1^1, \eta_2^1), \eta_2^1) \end{aligned}$$

where $\xi_1^0 = \sqrt{r_1^2 - (\xi_2^0)^2}$ and $\eta_1^1 = \sqrt{r_2^2 - (\eta_2^1)^2}$ with $i = 1, 2$.

Let $M_0^+(\xi_1^{0+}, \xi_2^{0+}, 0, 0)$ be the point of intersection Γ_0 with Σ_0 . We define in Σ_0 the neighborhood

$$U_0 = \left\{ (\xi_1^0, \xi_2^0, \eta_1^0, \eta_2^0) / (\xi_1^0 - \xi_1^{0+})^2 + (\xi_2^0)^2 \leq \epsilon^2, (\eta_1^0)^2 + (\eta_2^0)^2 \leq \epsilon^2 \right\} \cap \Sigma_0$$

and let $M_1^-(0, 0, \eta_1^{1-}, \eta_2^{1-})$ be the point of intersection Γ_0 with Σ_1 . We define in Σ_1 the neighborhood

$$U_1 = \left\{ (\xi_1^1, \xi_2^1, \eta_1^1, \eta_2^1) / (\xi_1^1)^2 + (\xi_2^1)^2 \leq \epsilon^2, (\eta_1^1 - \eta_1^{1-})^2 + (\eta_2^1)^2 \leq \epsilon^2 \right\} \cap \Sigma_1$$

If

$$U_\xi = \left\{ (\xi_1^0, \xi_2^0) / (\exists (\eta_1^0, \eta_2^0)) (\eta_1^0)^2 + (\eta_2^0)^2 \leq r_1^2, (\xi_1^0, \xi_2^0, \eta_1^0, \eta_2^0) \in U_0 \right\}$$

$$U_\eta = \left\{ (\eta_1^1, \eta_2^1) / (\exists (\xi_1^1, \xi_2^1)) (\xi_1^1)^2 + (\xi_2^1)^2 \leq r_2^2, (\xi_1^1, \xi_2^1, \eta_1^1, \eta_2^1) \in U_1 \right\}$$

then, for given ϵ , we can find $t(\epsilon)$, such that for $(\xi_1^0, \xi_2^0) \in U_\xi$, $(\eta_1^1, \eta_2^1) \in U_\eta$, and Theorem 3, we obtain:

$$(\xi_1^1)^2 + (\xi_2^1)^2 = (\xi_1^p)^2 + (\xi_2^p)^2 \leq \epsilon^2 \quad \text{and} \quad (\eta_1^0)^2 + (\eta_2^0)^2 = (\eta_1^p)^2 + (\eta_2^p)^2 \leq \epsilon^2$$

for $t_0 > t(\epsilon)$, i.e.,

$$T_0(U_0) \cap U_1 \neq \emptyset$$

Let $\bar{t} = N\pi + s^*$, $s^* \in [0, \pi)$. If we write $[\bar{t}, \infty) = \cup_{k=N}^\infty I_k$ with $I_k = \left[\frac{k\pi + s^*}{\mu}, \frac{(k+1)\pi + s^*}{\mu} \right]$, we obtain that $D(T_0)$ and $R(T_0)$ are the images of $U_\xi \times U_\eta \times [\bar{t}, \infty)$ under the maps τ_0 and τ_1 , which are given by:

$$(12) \quad \begin{aligned} \tau_0(t_0, \xi_1^0, \xi_2^0, \eta_1^1, \eta_2^1) &= (\xi_1^0, \xi_2^0, \eta_1^p, \eta_2^p) \\ \tau_1(t_0, \xi_1^0, \xi_2^0, \eta_1^1, \eta_2^1) &= (\xi_1^p, \xi_2^p, \eta_1^1, \eta_2^1) \end{aligned}$$

so, the domain and the range of T_0 are given by:

$$D(T_0) = \cup_{k=N}^{\infty} \sigma_k^0 \quad \text{and} \quad R(T_0) = \cup_{k=N}^{\infty} \sigma_k^1$$

where

$$\begin{aligned} \sigma_k^0 &= \tau_0(U_\xi, U_\eta, I_k) \\ \sigma_k^1 &= \tau_1(U_\xi, U_\eta, I_k) \end{aligned}$$

so, σ_k^0 is stratified into a collection of surfaces given by:

$$\eta_i^0 = \eta_i^p(\pi k + S_0 + s^*, \xi_1^0, \xi_2^0, \eta_1^1, \eta_2^1) \equiv (\eta_i^k)_{(\eta_1^1, \eta_2^1)}(S_0, \xi_1^0, \xi_2^0), \quad 0 \leq S_0 < \pi$$

which we denote by $P_{(\eta_1^1, \eta_2^1)}^k$, $(\eta_1^1, \eta_2^1) \in U_\eta$ and σ_k^1 is stratified into a collection of surfaces given by:

$$\xi_i^1 = \xi_i^p(\pi k + S_1 + s^*, \xi_1^0, \xi_2^0, \eta_1^1, \eta_2^1) \equiv (\xi_i^k)_{(S_1, \xi_1^0, \xi_2^0)}(\eta_1^1, \eta_2^1), \quad 0 \leq S_1 < \pi$$

which we denote by $Q_{(S_1, \xi_1^0, \xi_2^0)}^k$, $(\xi_1^0, \xi_2^0) \in U_\xi$.

Lemma 1. *There exists $N_1 \geq N$, such that for $k > N_1$ and for each $(\eta_1^1, \eta_2^1) \in U_\eta$ the surface $P_{(\eta_1^1, \eta_2^1)}^k$ intersects D^+ in a unique point, where $D^+ = T_1(D)$ with $D = \{(0, 0)\} \times U_\eta$.*

Corollary 1. *$\overline{T_0(D^+ \cap D(T_0))}$ is locally disconnected.*

Remark: The following lemma shows the condition to obtain an invariant set.

Lemma 2. *For $m, k > N$, such that $\lambda k + \gamma m < -N_0$, with $N_0 > N$ we have*

$$\sigma_{m,k} = T\sigma_k^0 \cap \sigma_m^0 \neq \emptyset$$

Proof: We prove that for k and m large enough

$$T_1(Q_{(S_1, \xi_1^0, \xi_2^0)}^k) \cap P_{(\eta_1^1, \eta_2^1)}^m \neq \emptyset.$$

The equations of T_1 are:

$$\bar{\xi}_2^{0(m)} = A_1 \xi_1^{p(k)} + A_2 \xi_2^{p(k)} + B \eta_2^{1(k)} + \dots$$

$$\bar{\eta}_i^{0(m)} = A_{i1} \xi_1^{p(k)} + A_{i2} \xi_2^{p(k)} + B_i \eta_2^{1(k)} + \dots, \quad i = 1, 2.$$

If $B_1 \neq 0$, from the first equation of (11) we can obtain:

$$\eta_2^{1(k)} = \frac{1}{B_1} \left(\bar{\eta}_1^0(m) - A_{11}\xi_1^p(k) - A_{12}\xi_2^p(k) \right)$$

So, if we apply the Implicit Function Theorem, we obtain $\eta_2^{1(k)}$ and replacing it in (10) and in (11) for $i = 2$, we have:

$$\begin{aligned} \bar{\xi}_2^0(m) &= A_1\xi_1^p(k) + A_2\xi_2^p(k) + \frac{B}{B_1} \left(\bar{\eta}_1^0(m) - A_{11}\xi_1^p(k) - A_{12}\xi_2^p(k) \right) \\ B_1\bar{\xi}_2^0(m) &= (A_1B_1 - BA_{11})\xi_1^p(k) + (A_2B_1 - BA_{12})\xi_2^p(k) + B\bar{\eta}_1^0(m) \end{aligned}$$

and

$$\begin{aligned} \bar{\eta}_2^0(m) &= A_{21}\xi_1^p(k) + A_{22}\xi_2^p(k) + \frac{B_2}{B_1} \left(\bar{\eta}_1^0(m) - A_{11}\xi_1^p(k) - A_{12}\xi_2^p(k) \right) \\ B_1\bar{\eta}_2^0(m) &= (A_{21}B_1 - A_{11}B_2)\xi_1^p(k) + (A_{22}B_1 - A_{12}B_2)\xi_2^p(k) + \\ &\quad + B_2\bar{\eta}_1^0(m) \end{aligned}$$

Replacing the expression given in (9), we get:

$$\begin{aligned} &\epsilon^{\lambda(k\pi+S_1+s^*)} \left\{ \xi_1^0 \left[\Delta_1(1 + \alpha_{11}^0 + \beta_{11}^0) \cos \frac{\omega(k\pi + S_1 + s^*)}{\mu} + \right. \right. \\ &\quad \left. \left. + \Delta_2(1 + \alpha_{21}^0 + \beta_{21}^0) \sin \frac{\omega(k\pi + S_1 + s^*)}{\mu} \right] \right. \\ &\quad \left. - \xi_2^0 \left[\Delta_1(1 + \alpha_{12}^0 + \beta_{12}^0) \sin \frac{\omega(k\pi + S_1 + s^*)}{\mu} - \right. \right. \\ &\quad \left. \left. - \Delta_2(1 + \alpha_{22}^0 + \beta_{22}^0) \cos \frac{\omega(k\pi + S_1 + s^*)}{\mu} \right] \right\} \\ &= \epsilon^{-\gamma(m\pi+S_0+s^*)} \{ \eta_1^1 [B_1(1 + \alpha_{21}^1 + \beta_{21}^1) \sin(S_0 + s^*) + \\ &\quad + B_2(1 + \alpha_{11}^1 + \beta_{11}^1) \cos(S_0 + s^*)] \\ &\quad - \eta_2^1 [B_1(1 + \alpha_{22}^1 + \beta_{22}^1) \cos(S_0 + s^*) - B_2(1 + \alpha_{12}^1 + \beta_{12}^1) \sin(S_0 + s^*)] \} \end{aligned}$$

with $A_{11}B_2 - A_{21}B_1 = \Delta_1$ and $A_{12}B_2 - A_{22}B_1 = \Delta_2$, i.e.:

$$\begin{aligned} & e^{(\lambda k + \gamma m)\pi} e^{(\lambda S_1 + \gamma S_0 + (\lambda + \gamma)s^*)} \left\{ [\xi_1^0 \Delta_1 (1 + \alpha_{11}^0 + \beta_{11}^0) + \right. \\ & \quad \left. + \xi_2^0 \Delta_2 (1 + \alpha_{22}^0 + \beta_{22}^0)] \cos \frac{\omega(k\pi + S_1 + s^*)}{\mu} + \right. \\ & \quad \left. + [\xi_1^0 \Delta_2 (1 + \alpha_{21}^0 + \beta_{21}^0) - \xi_2^0 \Delta_1 (1 + \alpha_{12}^0 + \beta_{12}^0)] \sin \frac{\omega(k\pi + S_1 + s^*)}{\mu} \right\} = \\ & = [\eta_1^1 B_1 (1 + \alpha_{21}^1 + \beta_{21}^1) + \eta_2^1 B_2 (1 + \alpha_{12}^1 + \beta_{12}^1)] \sin(S_0 + s^*) + \\ & \quad + [\eta_1^1 B_2 (1 + \alpha_{11}^1 + \beta_{11}^1) - \eta_2^1 B_1 (1 + \alpha_{22}^1 + \beta_{22}^1)] \cos(S_0 + s^*) \end{aligned}$$

since $\lambda + \gamma < 0$ and $\lambda k + \gamma m < -N_0$, with $N_0 > N$, then the right-hand side of the last equation tends to zero. The solution is of the type

$$\bar{A}_1 \sin(S_0 + s^*) + \bar{A}_2 \cos(S_0 + s^*) = \bar{A} \sin(S_0 + s^* + \theta) = 0.$$

If we choose s^* as $\theta - \frac{\pi}{2}$ and apply the Implicit Function Theorem, we obtain:

$$\bar{S}_0 = \frac{\pi}{2} + \alpha(S_1, \xi_1^0, \xi_2^0, \eta_1^1, \eta_2^1, m, k).$$

Replacing it in the remainder equations, we get the other variables, i.e., $\bar{\xi}_2^0$ and η_i^1 with $i = 1, 2$. For the other case we change t for $-t$.

□

Proof of theorem 1 We define

$$T_{m,k} : \sigma_k \longrightarrow \sigma_m$$

by

$$T_{m,k}(\xi_1^0, \xi_2^0, \bar{\eta}_1^1, \bar{\eta}_2^1, S) = (\bar{\xi}_1^0, \bar{\xi}_2^0, \eta_1^1, \eta_2^1, \bar{S})$$

where:

$$\bar{\xi}_2^0 = f(\xi_1^p(k\pi + S + s^*, \xi_1^0, \xi_2^0, \bar{\eta}_1^1, \bar{\eta}_2^1), \xi_2^p(k\pi + S + s^*, \xi_1^0, \xi_2^0, \bar{\eta}_1^1, \bar{\eta}_2^1), \bar{\eta}_2^1)$$

$$\bar{\xi}_1^0 = \sqrt{r_1^2 - (\bar{\xi}_2^0)^2}$$

$$\eta_i^p(m\pi + \bar{S} + s^*, \bar{\xi}_1^0, \bar{\xi}_2^0, \eta_1^1, \eta_2^1) =$$

$$= g_i(\xi_1^p(k\pi + S + s^*, \xi_1^0, \xi_2^0, \bar{\eta}_1^1, \bar{\eta}_2^1), \xi_2^p(k\pi + S + s^*, \xi_1^0, \xi_2^0, \bar{\eta}_1^1, \bar{\eta}_2^1), \bar{\eta}_2^1)$$

$$\bar{S} = \frac{\pi}{2} + \bar{\alpha}(S, \xi_1^0, \xi_2^0, \bar{\eta}_1^1, \bar{\eta}_2^1, m, k) \quad \text{for } i = 1, 2.$$

Using lemma 2, this system can be reduced to:

$$\bar{\xi}_2^0 = \bar{F}(\xi_1^0, \xi_2^0, \bar{\eta}_1^1, \bar{\eta}_2^1, S, k, m)$$

$$\eta_i^1 = \bar{G}_i(\xi_1^0, \xi_2^0, \bar{\eta}_1^1, \bar{\eta}_2^1, S, k, m), \quad i = 1, 2.$$

$$\bar{S} = \frac{\pi}{2} + \bar{\alpha}(\xi_1^0, \xi_2^0, \bar{\eta}_1^1, \bar{\eta}_2^1, S, k, m)$$

where $\bar{F}$, $\bar{G}_1$, $\bar{G}_2$ and $\bar{\alpha}$ are contraction mappings for $\lambda k + \gamma m < -N_0$ with N_0 large enough.

So, if $\bar{\Gamma}$ is an orbit contained in a neighborhood of Γ_0 , $\Gamma_0 \neq \bar{\Gamma}$, then, $\bar{\Gamma}$ intersects Σ_0 in points of $D(T_0)$, which defines a sequence of points $\{P_n\}_{-\infty}^{\infty}$ with $P_n \in \sigma_{k_n}^0$, for some k_n , $k_n > \bar{N}$.

In section 2, we define the set $\Omega(\rho)$ as the set of doubly infinite sequences consisting of the symbols j_n . Let us define $j_n = k_n - \bar{N}$, then

$$\{\dots, j_n, j_{n+1}, \dots\} \in \Omega(\rho).$$

On the other hand, for a given sequence

$$\beta = \{\dots, j_n, j_{n+1}, \dots\} \in \Omega(\rho)$$

we define $k_n = j_n + \bar{N}$, then for $k_n > \bar{N}$, we have that $\gamma k_{n+1} + \lambda k_n < -\bar{N}$ thus $\rho k_n > k_{n+1}$ with $\rho > -\frac{\lambda}{\gamma} > 1$.

Let $(\xi_1^{0n}, \xi_2^{0n}, \eta_1^{0n}, \eta_2^{0n})$ be the coordinates of

$$M_n^0 \in \sigma_k^0, (\xi_1^{1n}, \xi_2^{1n}, \eta_1^{1n}, \eta_2^{1n})$$

the coordinates of $M_n^1 = T_0 M_n^0$ and $t_0 = k_n \pi + S_n + s^*$ the transition time from M_n^0 to M_n^1 . So, the relations between the coordinates of the points $M_n^0, M_{n+1}^0, \dots$ can be written as:

$$\xi_2^{n+1} = \bar{F}(\xi_1^n, \xi_2^n, \eta_1^{n+1}, \eta_2^{n+1}, S_n, k_n, k_{n+1})$$

$$\xi_1^{n+1} = \sqrt{r_1^2 - (\xi_2^{n+1})^2}$$

$$\eta_i^n = \bar{G}_i(\xi_1^n, \xi_2^n, \eta_1^{n+1}, \eta_2^{n+1}, S_n, k_n, k_{n+1}), \quad i = 1, 2.$$

$$S_{n+1} = \frac{\pi}{2} + \bar{\alpha}(\xi_1^n, \xi_2^n, \eta_1^{n+1}, \eta_2^{n+1}, S_n, k_n, k_{n+1})$$

If we use the following fixed point lemma for product spaces given by Shil'nikov in [8], there exists a unique sequence which is in one-to-one correspondence with the solution $\bar{\Gamma}$ in a neighborhood of Γ_0 .

Lemma 3. Suppose that the complete metric spaces X_i and Y_i and the operators A_i and B_i , $i = 0, \pm 1, \dots$, satisfy the following conditions:

- (1) $\sup_{x'_i, x''_i \in X_i} \rho_{X_i}(x'_i, x''_i) < d, \quad \sup_{y'_i, y''_i \in Y_i} \rho_{Y_i}(y'_i, y''_i) < d;$
- (2) $A_i(X_i \times Y_{i+1}) \subset X_{i+1}, \quad B_{i+1}(X_i \times Y_{i+1}) \subset Y_i,$
- (3)
$$\begin{aligned} \rho_{X_{i+1}}(\bar{x}'_{i+1}, \bar{x}''_{i+1}) &< \frac{q}{2}(\rho_{X_i}(x'_i, x''_i) + \rho_{Y_{i+1}}(y'_{i+1}, y''_{i+1})) \\ \rho_{Y_i}(\bar{y}'_i, \bar{y}''_i) &< \frac{q}{2}(\rho_{X_i}(x'_i, x''_i) + \rho_{Y_{i+1}}(y'_{i+1}, y''_{i+1})) \end{aligned}$$

where $q < 1$ and $\bar{x}_{i+1} = A_i(x_i, y_{i+1})$, $\bar{y}_i = B_{i+1}(x_i, y_{i+1})$. Then there exists a unique sequence

$$(\dots, (x_{i-1}^*, y_{i-1}^*), (x_i^*, y_i^*), (x_{i+1}^*, y_{i+1}^*), \dots),$$

satisfying conditions

$$x_{i+1}^* = A_i(x_i^*, y_{i+1}^*), \quad y_i^* = B_{i+1}(x_i^*, y_{i+1}^*).$$

We define

$$\begin{aligned} X_n &= U_\xi \times \bar{I} \\ Y_n &= U_\eta \end{aligned}$$

where $\bar{I} = [0, \pi]$

$$A_n : \begin{cases} \xi_2^{n+1} &= \bar{F}(\xi_1^n, \xi_2^n, \eta_1^{n+1}, \eta_2^{n+1}, S_n, k_n, k_{n+1}) \\ \xi_1^{n+1} &= \sqrt{r_1^2 - (\xi_2^{n+1})^2} \\ S_{n+1} &= \frac{\pi}{2} + \bar{\alpha}(\xi_1^n, \xi_2^n, \eta_1^{n+1}, \eta_2^{n+1}, S_n, k_n, k_{n+1}) \end{cases}$$

and

$$B_{n+1} : \left\{ \begin{aligned} \eta_i^n &= \bar{G}_i(\xi_1^n, \xi_2^n, \eta_1^{n+1}, \eta_2^{n+1}, S_n, k_n, k_{n+1}), \quad i = 1, 2. \end{aligned} \right.$$

Because of the properties of $T_{m,k}$, this sequence of spaces and operators satisfies the conditions of the lemma. Thus, there exists a unique sequence

$$(\dots, (\xi_1^n, \xi_2^n, \eta_1^n, \eta_2^n, S_n), (\xi_1^{n+1}, \xi_2^{n+1}, \eta_1^{n+1}, \eta_2^{n+1}, S_{n+1}), \dots)$$

which corresponds to a unique solution $\bar{\Gamma}$ in a neighborhood of Γ_0 .

□

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