A SHIL’NIKOV’S THEOREM IN $\mathbb{R}^4$ IN AN EXTENDED NEIGHBORHOOD OF A SADDLE POINT OF FOCUS - FOCUS TYPE

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Abstract
In this article we study the existence of chaotic solutions in a neighborhood of a homoclinic orbit of a saddle point of focus-focus type. We will prove that the solutions have an exponential expansion, this fact implies the existence of a subsystem of solutions, which is in one-to-one correspondence with the set of doubly infinite sequences.

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1. Introduction.

In [8] Shil'nikov proved that in a neighborhood of a trajectory which is doubly asymptotic to a hyperbolic equilibrium point of a saddle–focus type, under certain conditions for the spectrum of the linear part, there exists a subsystem of solution where the trajectories are in a one-to-one correspondence with a countable set of doubly infinite sequences of positive integers (Symbolic Dynamic). That is, linear part of the system has three eigenvalues, one real and the other two complex such that their real parts are closer to the origin than the real parts of the other eigenvalues, then there exists a subsystem of solutions in one-to-one correspondence with sequences of the type \((\ldots, k_n, k_{n+1}, \ldots)\) with \(k_n < \rho k_{n+1}, \rho > 1\), such that when it passes by a neighborhood of the fixed point, the orbit has \(k_n\) semi-turns around the point; it continues by a tubular neighborhood of a homoclinic orbit and when it returns to the neighborhood of the equilibrium point it has \(k_{n+1}\) semi-turns and so on. This result is known as the Shil'nikov Theorem.

Furthermore Tresser [9] proved in a more geometric manner the existence of chaotic solutions in a neighborhood of a homoclinic solution, where the presence of an infinite number of Smale horseshoes is illustrated.

Here, we prove the existence of chaotic solutions in a tubular neighborhood of the homoclinic orbit for a 4-dimensional dynamical system, such that the system has two pairs of complex conjugate eigenvalues. In this case, Shil'nikov [7] proved the existence of a denumerable set of periodic motions. The proof of our theorem is similar to that of Shil'nikov [8]. We construct a Poincaré map \(T\) whose domain of definition is representable as a countable union of mutually disjoint domains on which \(T\) is a composition of two maps \(T_0\) and \(T_1\) where \(T_0\) is defined in a neighborhood of the origin and \(T_1\) in a tubular neighborhood of the homoclinic orbit.

2. Presentation of the Problem.

We consider a system of four differential equations

\[
\dot{z} = Z(z)
\]

where \(Z \in C^k(\mathbb{R}^4, \mathbb{R}^4), k \geq 4\). We will assume that the system (1) has an equilibrium point 0; i.e., \(Z(0) = 0\), at which the characteristic equation

\[
\left| \frac{\partial Z}{\partial t}(0) - \lambda I_{4} d_4 \right| = 0
\]

satisfies the following conditions:
(H1) The system has two pairs of complex conjugate eigenvalues $\lambda \pm i\omega$ and $\gamma \pm i\mu$, where $\lambda < 0$ and $\gamma > 0$.

(H2) There exists a homoclinic orbit $\Gamma_0$, issuing from 0 and returning to it, as $t \longrightarrow +\infty$, i.e., $\Gamma_0 \subset (M^+ \cap M^-)$, where $M^+$ and $M^-$ are the stable and unstable manifolds, respectively. Furthermore, it satisfies:

\begin{equation}
\dim(TM_p^+ \cap TM_p^-) = 1
\end{equation}

where $TM_p^+$ and $TM_p^-$ are the tangent spaces at $p \in \Gamma_0$ to the stable and unstable manifolds, respectively.

(H3) $\gamma + \lambda \neq 0$.

**Definition 1.** A focus-focus point is a saddle point which satisfies condition (H1). A small enough tubular neighborhood of the homoclinic orbit is called an extended neighborhood of the focus-focus point.

We consider the set of doubly infinite sequences

$$(\ldots, j_i, j_{i+1}, \ldots)$$

consisting of the symbols 0, 1, 2, ... satisfying that for arbitrary adjacent indices $j_{i+1} < \rho j_i$ for some $\rho > 1$. We denote this set by $\Omega(\rho)$; i.e.:

$$\Omega(\rho) = \{ (\ldots, j_i, j_{i+1}, \ldots) / j_i \in \mathbb{N} \cup \{0\}, j_{i+1} < \rho j_i, \text{ for some } \rho > 1 \}.$$ 

Under assumptions (H1)–(H3) we prove the following theorem:

**Theorem 1.** For $\lambda + \gamma < 0$ (resp. for $\lambda + \gamma > 0$) there exists a subsystem of solutions of (1) in one-to-one correspondence with the set $\Omega(\rho)$ with $\rho > -\frac{\lambda}{\gamma} > 1$ (resp. $\rho > \frac{-\gamma}{\lambda} > 1$).

Without loss of generality we may assume that $\lambda + \gamma < 0$. For the other case we change $t$ for $-t$. 


3. **Construction of the map $T$.**

3.1. **Construction of $T_0$.** We can rewrite system (1) in a neighborhood of the origin by a nondegenerate analytic nonlinear change of variables as:

\[
\begin{align*}
\dot{x}_1 &= \lambda x_1 - \omega x_2 + P_1(x_1, x_2, y_1, y_2) \\
\dot{x}_2 &= \omega x_1 + \lambda x_2 + P_2(x_1, x_2, y_1, y_2) \\
\dot{y}_1 &= \gamma y_1 - \mu y_2 + Q_1(x_1, x_2, y_1, y_2) \\
\dot{y}_2 &= \mu y_1 + \gamma y_2 + Q_2(x_1, x_2, y_1, y_2)
\end{align*}
\]

where the origin is the focus–focus point and the functions $P_1$, $P_2$, $Q_1$, and $Q_2$ vanish at the origin together with their first derivatives.

By stable and unstable manifolds theorem, the equations in the new variables for the stable manifold $M^+$ are:

\[
y_1 = \phi_1(x_1, x_2) \quad y_2 = \phi_2(x_1, x_2)
\]

and the equations for the unstable manifold $M^-$ are:

\[
x_1 = \psi_1(y_1, y_2) \quad x_2 = \psi_2(y_1, y_2)
\]

where $\phi_1$, $\phi_2$, $\psi_1$, and $\psi_2$ are $C^{k-1}$ which vanish at the origin together with their first derivatives.

System (3) can be reduced by the normal form and the substitution:

\[
\begin{align*}
\xi_1 &= x_1 - \psi_1(y_1, y_2) \quad \xi_2 = x_2 - \psi_2(y_1, y_2) \\
\eta_1 &= y_1 - \phi_1(x_1, x_2) \quad \eta_2 = y_2 - \phi_2(x_1, x_2)
\end{align*}
\]

to the system:

\[
\begin{align*}
\dot{\xi}_1 &= \lambda \xi_1 - \omega \xi_2 + P_{11}(\xi_1, \xi_2, \eta_1, \eta_2)\xi_1 + P_{12}(\xi_1, \xi_2, \eta_1, \eta_2)\xi_2 \\
\dot{\xi}_2 &= \omega \xi_1 + \lambda \xi_2 + P_{21}(\xi_1, \xi_2, \eta_1, \eta_2)\xi_1 + P_{22}(\xi_1, \xi_2, \eta_1, \eta_2)\xi_2 \\
\dot{\eta}_1 &= \gamma \eta_1 - \mu \eta_2 + Q_{11}(\xi_1, \xi_2, \eta_1, \eta_2)\eta_1 + Q_{12}(\xi_1, \xi_2, \eta_1, \eta_2)\eta_2 \\
\dot{\eta}_2 &= \mu \eta_1 + \gamma \eta_2 + Q_{21}(\xi_1, \xi_2, \eta_1, \eta_2)\eta_1 + Q_{22}(\xi_1, \xi_2, \eta_1, \eta_2)\eta_2
\end{align*}
\]
where $P_{ij}$ and $Q_{ij}$ vanish at the origin of the new variables whose equations for the stable manifold $M^+$ are $\eta_1 = 0, \eta_2 = 0$ and for the unstable manifold $M^-$ are $\xi_1 = 0$ and $\xi_2 = 0$, with $i, j = 1, 2$.

Let $\Sigma_0$ and $\Sigma_1$ be the surfaces $\Sigma_0 : \xi_1^2 + \xi_2^2 = r_1^2$, $\eta_1^2 + \eta_2^2 \leq r_1^2$ and $\Sigma_1 : \xi_1^2 + \xi_2^2 \leq r_2^2$, $\eta_1^2 + \eta_2^2 = r_2^2$, with $r_1, r_2 > 0$, $r_1^2 \neq r_2^2$ sufficiently small and the orbit of system (4) passing through the point $M_0(\xi_1^0, \xi_2^0, \eta_1^0, \eta_2^0) \in \Sigma_0$ at $t = 0$, where $(\eta_1^0)^2 + (\eta_2^0)^2 \neq 0$, and by the point $M_1(\xi_1^1, \xi_2^1, \eta_1^1, \eta_2^1) \in \Sigma_1$ at $t = t_0$, where $(\xi_1^0)^2 + (\xi_2^0)^2 < r_2^2$.

We consider the following integral form for $T_0$:

\[
\xi_1(t) = e^{\lambda t}[\xi_1^0 \cos\omega t - \xi_2^0 \sin\omega t] + \int_0^t e^{\lambda(t-s)}[\overline{P}_1 \cos\omega(t-s) - \overline{P}_2 \sin\omega(t-s)]ds
\]

\[
\xi_2(t) = e^{\lambda t}[\xi_1^0 \sin\omega t + \xi_2^0 \cos\omega t] + \int_0^t e^{\lambda(t-s)}[\overline{P}_1 \sin\omega(t-s) + \overline{P}_2 \cos\omega(t-s)]ds
\]

\[
(5)
\]

\[
\eta_1(t) = e^{\gamma(t-t_0)}[\eta_1^1 \cos\mu(t-t_0) - \eta_2^1 \sin\mu(t-t_0)] + \int_{t_0}^t e^{\gamma(t-s)}[\overline{Q}_1 \cos\mu(t-s) - \overline{Q}_2 \sin\mu(t-s)]ds
\]

\[
\eta_2(t) = e^{\gamma(t-t_0)}[\eta_1^1 \sin\mu(t-t_0) + \eta_2^1 \cos\mu(t-t_0)] + \int_{t_0}^t e^{\gamma(t-s)}[\overline{Q}_1 \sin\mu(t-s) + \overline{Q}_2 \cos\mu(t-s)]ds
\]

this form is equivalent to equation (4). Solutions of (5) are also solutions of system (4), passing through $M_0$ and $M_1$ and $t_0$ is the time of transition from $\Sigma_0$ to $\Sigma_1$, $\overline{P}_1$, $\overline{P}_2$, $\overline{Q}_1$, and $\overline{Q}_2$ denote the nonlinear terms of the first, second, third, and forth equation of the system (4), respectively. This system is equivalent to:

\[
\dot{\theta} = A\theta + R_1(\theta, \sigma) \theta
\]

\[
(6)
\]

\[
\dot{\sigma} = B\sigma + S_1(\theta, \sigma) \sigma
\]

where $\theta(t) = (\xi_1(t), \xi_2(t))$, $\sigma(t) = (\eta_1(t), \eta_2(t))$, $A = \begin{pmatrix} \lambda & -\omega \\ \omega & \lambda \end{pmatrix}$, $B = \begin{pmatrix} \gamma & -\mu \\ \mu & \gamma \end{pmatrix}$ and $R_1(\theta(t), \sigma(t))$ and $S_1(\theta(t), \sigma(t))$ are the nonlinear part matrices of the integral representation of system (4). Furthermore,
this system satisfies:

\[ S_1(0, \sigma) = 0 , \quad \frac{\partial^k S_1}{\partial \sigma^k} \bigg|_{\sigma=0} = 0 \]

\[ S_1(\theta, 0) = 0 , \quad \frac{\partial^k S_1}{\partial \theta^k} \bigg|_{\theta=0} = 0 \]

**Remark:** This result was proved by Deng in [2].

The parametric representation for \( T_0 \) is:

\[ \theta(t) = e^{ At } \theta_0 + \int_0^t e^{ A(t-s) } R_1(\theta(s), \sigma(s)) \theta(s) \, ds \]

(7)

\[ \sigma(t) = e^{ B(t-t_0) } \sigma_1 + \int_{t_0}^t e^{ B(t-s) } S_1(\theta(s), \sigma(s)) \sigma(s) \, ds \]

A similar proof to the following theorem can be find in [1], which is reported without proof.

**Theorem 2.** There is \( \delta > 0 \) such that there exists a unique solution of system (6) such that \( \| (\theta(t), \sigma(t)) \| < \delta \), for \( \| R_1 \| \leq M(\delta)(\| \theta \| + \| \sigma \|) \), \( \| S_1 \| \leq M(\delta)(\| \theta \| + \| \sigma \|) \), \( \| \theta_0 \| \) and \( \| \sigma_1 \| \leq \frac{\delta}{2K} \), with \( \delta \) sufficiently small and \( t_0 > 0 \).

Furthermore, we can choose \( \delta \) small enough, to obtain the following theorem:

**Theorem 3.** For \( \| \theta_0 \| \), \( \| \sigma_1 \| \leq \delta \) and \( t_0 > 0 \), we have:

\[ \| \theta(t) \| \leq K e^{\lambda t} \]

(8)

\[ \| \sigma(t) \| \leq K e^{\gamma(t-t_0)} \]

for \( 0 \leq t \leq t_0 \), \( K > 0 \).

**Proof:** Let us define

\[ I = \{ (\theta(t), \sigma(t)) / \theta(t), \sigma(t) \text{ satisfy (8)} \} , \]
$I$ is a complete metric space with the metric given by
\[d((\theta_1, \sigma_1); (\theta_2, \sigma_2)) = \|\theta_1 - \theta_2\| + \|\sigma_1 - \sigma_2\|\]

We define an operator $Q$ in $I$ by
\[Q : I \rightarrow I\]
such that
\[Q(\theta, \sigma) = (Q_1(\theta, \sigma), Q_2(\theta, \sigma))\]
where $Q_1$, $Q_2$ are the right-hand sides of (7).

We prove that $Q$ is well defined in $I$ and is a contraction, so, there exists a fixed point in $I$.

We show the proof for $Q_1$, because for $Q_2$ it is similar.

\[
Q_1(\theta, \sigma) = e^{At} \theta_0 + \int_0^t e^{A(t-s)} R_1(\theta(s), \sigma(s)) \, ds
\]

\[
\|Q_1(\theta, \sigma)\| \leq \|e^{At} \theta_0\| + \int_0^t \|e^{A(t-s)} R_1(\theta(s), \sigma(s))\| \|\theta(s)\| ds
\]

\[
\leq \bar{K} e^{\lambda t} \|\theta_0\| + \int_0^t \bar{K} e^{\lambda(t-s)} \|R_1(\theta(s), \sigma(s))\| \|\theta(s)\| ds
\]

\[
\leq e^{\lambda t} \left( \bar{K} \delta + \bar{K}^2 \int_0^t e^{-\lambda s} M(\delta)(\|\theta(s)\| + \|\sigma(s)\|) e^{\lambda s} ds \right)
\]

\[
\leq e^{\lambda t} \left( \bar{K} \delta + \bar{K}^3 \int_0^t (e^{\lambda s} + e^{\gamma(s-t_0)}) ds \right)
\]

then, the expression in parenthesis is less than a constant $\bar{K}$, i.e.,
\[
\|Q_1(\theta, \sigma)\| \leq \bar{K} e^{\lambda t}
\]

Now, we prove that $Q_1$ is a contraction

\[
\|Q_1(\theta^1, \sigma^1) - Q_1(\theta^2, \sigma^2)\| \leq \bar{K} M(\delta) \int_0^t e^{\lambda(t-s)} d((\theta^1, \sigma^1); (\theta^2, \sigma^2)) ds
\]

\[
\leq q_3 d((\theta^1, \sigma^1); (\theta^2, \sigma^2))
\]

with $0 < q_3 < 1$ for $\delta$ small enough.

Thus, there exists a fixed point for $Q$ in $I$.  

\[ \square \]
Definition 2. We say that $\theta(t)$ and $\sigma(t)$ admits an exponential expansion of regularity $r$ if there exists $C^r$ functions $\varphi_1 = \varphi_1(\theta_0, \sigma_1), \varphi_2 = \varphi_2(t_0, \theta_0, \sigma_1), U_1 = U_1(t, \theta_0, \sigma_1),$ and $U_2 = U_2(t, t_0, \theta_0, \sigma_1)$ with $r \geq 1$ such that

\[
\begin{align*}
\theta(t) &= e^{-At}(\varphi_1(\theta_0, \sigma_1) + U_1(t, \theta_0, \sigma_1)) \\
\sigma(t) &= e^{-B(t-t_0)}(\varphi_2(t_0, \theta_0, \sigma_1) + U_2(t, t_0, \theta_0, \sigma_1))
\end{align*}
\]

where

\[
\begin{align*}
\frac{\partial \varphi_1}{\partial \sigma_1}(0, 0) &= 0, \quad \frac{\partial \varphi_1}{\partial \theta_0}(0, 0) = 1d_2 \\
\frac{\partial \varphi_2}{\partial \theta_0}(t_0, 0, 0) &= 0, \quad \frac{\partial \varphi_2}{\partial \sigma_1}(t_0, 0, 0) = 1d_2
\end{align*}
\]

and $U_1$ and $U_2$ tend exponentially to zero together with their derivatives, as $t_0 \to +\infty$.

The variable $\sigma(t)$ has an exponential expansion in the unstable manifold (see [2]). We prove that $\theta(t)$ admits an exponential expansion.

Theorem 4. For a system in the form (6), $\theta(t)$ admits an exponential expansion of regularity $r - 2$.

Proof: We have the bounds (previous to the above theorem):

\[
\|\theta(t)\| \leq K e^{\lambda t} \quad \text{and} \quad \|R_1\| \leq M(\delta)(\|\theta(t)\| + \|\sigma(t)\|)
\]

and if

\[
\theta(t) = e^{At}\theta_0 + \int_0^t e^{A(t-s)} R_1(\theta(s), \sigma(s))\theta(s) ds
\]

we obtain:

\[
\|\theta(t)\|e^{-\lambda t} \leq \|\theta_0\| + M(\delta) \int_0^t (\|\theta(s)\| + \|\sigma(s)\|) ds
\]

Let us define

\[
\begin{align*}
\varphi_1(t_0, \theta_0, \sigma_1) &= \lim_{t \to -\infty} \theta(t)e^{-At} \\
\|\varphi_1(t_0, \theta_0, \sigma_1)\| &\leq \|\theta_0\| + M(\delta) \int_0^\infty (e^{\lambda s} + e^{\gamma(t-s)}) ds
\end{align*}
\]
and

\[ U_1(t_0, \theta_0, \sigma_1) = e^{At}(\theta(t)e^{-At} - \varphi_1(t_0, \theta_0, \sigma_1)) \]

So, \( \theta(t) \) has an exponential expansion.

Due to the fact that we have this exponential expansion, the solution of system (4):

\[
\begin{align*}
\xi_1(t) &= \xi_1^0(t, t_0, \xi_1^0, \xi_2^0, \eta_1^0, \eta_2^0) \\
\xi_2(t) &= \xi_2^0(t, t_0, \xi_1^0, \xi_2^0, \eta_1^0, \eta_2^0) \\
\eta_1(t) &= \eta_1^0(t, t_0, \xi_1^0, \xi_2^0, \eta_1^0, \eta_2^0) \\
\eta_2(t) &= \eta_2^0(t, t_0, \xi_1^0, \xi_2^0, \eta_1^0, \eta_2^0)
\end{align*}
\]

can be represented in the following form:

\[
\begin{align*}
\xi_1(t) &= e^{\lambda t}[\xi_1^0(1 + \alpha_{11}^0 + \beta_{11}^0) \cos \omega t - \xi_2^0(1 + \alpha_{12}^0 + \beta_{12}^0) \sin \omega t] \\
\xi_2(t) &= e^{\lambda t}[\xi_1^0(1 + \alpha_{21}^0 + \beta_{21}^0) \sin \omega t + \xi_2^0(1 + \alpha_{22}^0 + \beta_{22}^0) \cos \omega t] \\
\eta_1(t) &= e^{\gamma (t-t_0)}[\eta_1^1(1 + \alpha_{11}^1 + \beta_{11}^1) \cos \mu (t-t_0) - \\
&\quad - \eta_2^1(1 + \alpha_{12}^1 + \beta_{12}^1) \sin \mu (t-t_0)] \\
\eta_2(t) &= e^{\gamma (t-t_0)}[\eta_1^1(1 + \alpha_{21}^1 + \beta_{21}^1) \sin \mu (t-t_0) + \\
&\quad + \eta_2^1(1 + \alpha_{22}^1 + \beta_{22}^1) \cos \mu (t-t_0)]
\end{align*}
\]

where \( \alpha_{ij}^0(t-t_0, \xi_1^0, \xi_2^0, \eta_1^0, \eta_2^0) \), \( \alpha_{ij}^1(t_0, \xi_1^0, \xi_2^0, \eta_1^0, \eta_2^0) \), \( \beta_{ij}^0(t, t_0, \xi_1^0, \xi_2^0, \eta_1^0, \eta_2^0) \), \( \beta_{ij}^1(t, t_0, \xi_1^0, \xi_2^0, \eta_1^0, \eta_2^0) \) are analytic functions satisfying conditions \( \alpha_{ij}^k(0, \xi_1^0, \xi_2^0, \eta_1^0, \eta_2^0) = \alpha_{ij}^k(t_0, \xi_1^0, \xi_2^0, \eta_1^0, \eta_2^0) = \beta_{ij}^k(0, \xi_1^0, \xi_2^0, \eta_1^0, \eta_2^0) = \beta_{ij}^k(t, t_0, \xi_1^0, \xi_2^0, \eta_1^0, \eta_2^0) = 0 \) for \( \xi_1^0 = \xi_2^0 = \eta_1^0 = \eta_2^0 = 0 \) and \( \beta_{ij}^0(t_0, t_0, \xi_1^0, \xi_2^0, \eta_1^0, \eta_2^0) \), \( \beta_{ij}^1(t_0, t_0, \xi_1^0, \xi_2^0, \eta_1^0, \eta_2^0) \) converge to zero, together with their first derivatives, as \( t \rightarrow +\infty \) with \( i, j, k = 1, 2 \).

3.2. Construction of \( T_1 \). The theorem of continuous dependence on initial conditions guarantees that the map \( T_1 \) can be considered in the
neighborhood of the orbit of system (4), and it can be represented in the form:

\begin{equation}
\bar{z}_2^0 = f(\xi_1^0, \xi_2^0, \eta_1^0) = A_1\xi_1^0 + A_2\xi_2^0 + B\eta_1^0 + \ldots
\end{equation}

\begin{equation}
\bar{\eta}_i^0 = g_i(\xi_1^0, \xi_2^0, \eta_1^0) = A_{i1}\xi_1^0 + A_{i2}\xi_2^0 + B_i\eta_1^0 + \ldots \ i = 1, 2
\end{equation}

Let consider the mapping \( T = T_1 \circ T_0 \) as:

\begin{equation}
\bar{z}_2^0 = f(\xi_1^0(t_0, t_0, \xi_1^0, \xi_2^0, \eta_1^0, \eta_2^0), \xi_2^0(t_0, t_0, \xi_1^0, \xi_2^0, \eta_1^0, \eta_2^0))
\end{equation}

\begin{equation}
\bar{\eta}_i^0 = g_i(\xi_1^0(t_0, t_0, \xi_1^0, \xi_2^0, \eta_1^0, \eta_2^0), \xi_2^0(t_0, t_0, \xi_1^0, \xi_2^0, \eta_1^0, \eta_2^0))
\end{equation}

where \( \xi_1^0 = \sqrt{r_1^2 - (\xi_2^0)^2} \) and \( \eta_1^0 = \sqrt{r_2^2 - (\eta_2^0)^2} \) with \( i = 1, 2 \).

Let \( M_0^+(\xi_1^0, \xi_2^0, 0, 0) \) be the point of intersection \( \Gamma_0 \) with \( \Sigma_0 \). We define in \( \Sigma_0 \) the neighborhood

\[ U_0 = \left\{ (\xi_1^0, \xi_2^0, \eta_1^0, \eta_2^0) / (\xi_1^0 - \xi_1^0)^2 + (\xi_2^0)^2 \leq \epsilon^2, (\eta_1^0)^2 + (\eta_2^0)^2 \leq \epsilon^2 \right\} \cap \Sigma_0 \]

and let \( M_1(0, 0, \eta_1^-, \eta_2^-) \) be the point of intersection \( \Gamma_0 \) with \( \Sigma_1 \). We define in \( \Sigma_1 \) the neighborhood

\[ U_1 = \left\{ (\xi_1^1, \xi_2^1, \eta_1^1, \eta_2^1) / (\xi_1^1)^2 + (\xi_2^1)^2 \leq \epsilon^2, (\eta_1^1)^2 + (\eta_2^1)^2 \leq \epsilon^2 \right\} \cap \Sigma_1 \]

If

\[ U_\xi = \left\{ (\xi_1^0, \xi_2^0) / (\exists(\eta_1^0, \eta_2^0)) (\eta_1^0)^2 + (\eta_2^0)^2 \leq r_1^2, (\eta_1^0, \xi_2^0, \eta_1^0, \eta_2^0) \in U_0 \right\} \]

\[ U_\eta = \left\{ (\eta_1^0, \eta_2^0) / (\exists(\xi_1^0, \xi_2^0)) (\xi_1^0)^2 + (\xi_2^0)^2 \leq r_2^2, (\eta_1^0, \xi_2^0, \eta_1^0, \eta_2^0) \in U_1 \right\} \]

then, for given \( \epsilon \), we can find \( t(\epsilon) \), such that for \( (\xi_1^0, \xi_2^0) \in U_\xi, (\eta_1^0, \eta_2^0) \in U_\eta \), and Theorem 3, we obtain:

\[ (\xi_1) = (\xi_1^0)^2 + (\xi_2^0)^2 \leq \epsilon^2 \quad \text{and} \quad (\eta_1)^2 + (\eta_2)^2 = (\eta_1^0)^2 + (\eta_2^0)^2 \leq \epsilon^2 \]

for \( t_0 > t(\epsilon) \), i.e.,

\[ T_0(U_0) \cap U_1 \neq \emptyset \]

Let \( \bar{t} = N \pi + s^* \), \( s^* \in [0, \pi) \). If we write \( [\bar{t}, \infty) = \bigcup_{k=-N}^{\infty} I_k \) with

\[ I_k = \left[ \frac{k\pi + s^*}{\mu}, \frac{(k + 1)\pi + s^*}{\mu} \right] \]

we obtain that \( D(T_0) \) and \( R(T_0) \) are the images of \( U_\xi \times U_\eta \times [\bar{t}, \infty) \) under the maps \( \tau_0 \) and \( \tau_1 \), which are given by:

\[ \tau_0(t_0, \xi_1^0, \xi_2^0, \eta_1^0, \eta_2^0) = (\xi_1^0, \xi_2^0, \eta_1^0, \eta_2^0) \]

\[ \tau_1(t_0, \xi_1^0, \xi_2^0, \eta_1^0, \eta_2^0) = (\xi_1^0, \xi_2^0, \eta_1^0, \eta_2^0) \]
so, the domain and the range of $T_0$ are given by:

$$D(T_0) = \bigcup_{k=N}^{\infty} \sigma_k^0 \quad \text{and} \quad R(T_0) = \bigcup_{k=N}^{\infty} \sigma_k^1$$

where

$$\sigma_k^0 = \tau_0(U_\xi, U_\eta, I_k) \quad \sigma_k^1 = \tau_1(U_\xi, U_\eta, I_k)$$

so, $\sigma_k^0$ is stratified into a collection of surfaces given by:

$$\eta_i^{0} = \eta_i^p(\pi k + S_0 + s^*, s_1^0, s_2^0, \eta_1, \eta_2) \equiv (\eta_i^k, \eta_i^1) \in U_\eta \quad \text{and} \quad \sigma_k^1$$

which we denote by $P_{(\eta_1^1, \eta_1^2)}(\eta_1^1, \eta_2^1) \in U_\eta$ and $\sigma_k^1$ is stratified into a collection of surfaces given by:

$$\xi_i^{1} = \xi_i^p(\pi k + S_1 + s^*, s_1^0, s_2^0, \eta_1, \eta_2) \equiv (\xi_i^k, \xi_i^1, \xi_i^0) \in U_\xi$$

which we denote by $Q_{(\xi_1^1, \xi_1^0, \xi_2^0)}(\xi_1^1, \xi_2^0) \in U_\xi$.

**Lemma 1.** There exists $N_1 \geq N$, such that for $k > N_1$ and for each $(\eta_1^1, \eta_1^2) \in U_\eta$ the surface $P_{(\eta_1^1, \eta_1^2)}(\eta_1^1, \eta_2^1)$ intersects $D^+$ in a unique point, where $D^+ = T_1(D)$ with $D = \{(0,0)\} \times U_\eta$.

**Corollary 1.** $\overline{T_0(D^+ \cap D(T_0))}$ is locally disconnected.

**Remark:** The following lemma shows the condition to obtain an invariant set.

**Lemma 2.** For $m, k > N$, such that $\lambda k + \gamma m < -N_0$, with $N_0 > N$ we have

$$\sigma_{m,k} = T\sigma_k^0 \cap \sigma_m^0 \neq \emptyset$$

**Proof:** We prove that for $k$ and $m$ large enough

$$T_1(Q_{(\xi_1^1, \xi_1^0, \xi_2^0)^m}) \cap P_{(\eta_1^1, \eta_1^2)}^m \neq \emptyset.$$

The equations of $T_1$ are:

$$\xi_2^0(m) = A_1 \xi_1^p(k) + A_2 \xi_2^p(k) + B \eta_2^1(k) + \ldots$$

$$\eta_i^0(m) = A_{i1} \xi_1^p(k) + A_{i2} \xi_2^p(k) + B_i \eta_2^1(k) + \ldots, \quad i = 1, 2.$$
If $B_1 \neq 0$, from the first equation of (11) we can obtain:

$$
\eta_2^{(k)} = \frac{1}{B_1} \left[ \frac{\eta_1^0(m)}{B_1} - A_{11} \xi_1^{(k)} - A_{12} \xi_2^{(k)} \right]
$$

So, if we apply the Implicit Function Theorem, we obtain $\eta_2^{(k)}$ and replacing it in (10) and in (11) for $i = 2$, we have:

$$
\begin{align*}
\bar{\xi}_2^0(m) & = A_1 \xi_1^{(k)} + A_2 \xi_2^{(k)} + \frac{B}{B_1} \left[ \eta_1^0(m) - A_{11} \xi_1^{(k)} - A_{12} \xi_2^{(k)} \right] \\
B_1 \bar{\xi}_2^0(m) & = (A_1 B_1 - BA_{11}) \xi_1^{(k)} + (A_2 B_1 - BA_{12}) \xi_2^{(k)} + B \eta_1^0(m)
\end{align*}
$$

and

$$
\begin{align*}
\bar{\eta}_2^0(m) & = A_{21} \xi_1^{(k)} + A_{22} \xi_2^{(k)} + \frac{B_2}{B_1} \left[ \eta_1^0(m) - A_{11} \xi_1^{(k)} - A_{12} \xi_2^{(k)} \right] \\
B_1 \bar{\eta}_2^0(m) & = (A_{21} B_1 - A_{11} B_2) \xi_1^{(k)} + (A_{22} B_1 - A_{12} B_2) \xi_2^{(k)} + B_2 \eta_1^0(m)
\end{align*}
$$

Replacing the expression given in (9), we get:

$$
\begin{align*}
\epsilon^\lambda(k \pi + S_1 + s^*) \left\{ \xi_1^0 \left[ \Delta_1 (1 + \alpha_{11}^0 + \beta_{11}^0) \cos \frac{\omega(k \pi + S_1 + s^*)}{\mu} + \Delta_2 (1 + \alpha_{21}^0 + \beta_{21}^0) \sin \frac{\omega(k \pi + S_1 + s^*)}{\mu} \right] \\
- \xi_2^0 \left[ \Delta_1 (1 + \alpha_{12}^0 + \beta_{12}^0) \sin \frac{\omega(k \pi + S_1 + s^*)}{\mu} - \Delta_2 (1 + \alpha_{22}^0 + \beta_{22}^0) \cos \frac{\omega(k \pi + S_1 + s^*)}{\mu} \right] \right\}
\end{align*}
$$

$$
\begin{align*}
= e^{-\gamma(m \pi + S_0 + s^*)} \{ \eta_1^1 \left[ B_1 (1 + \alpha_{21}^1 + \beta_{21}^1) \sin(S_0 + s^*) + B_2 (1 + \alpha_{11}^1 + \beta_{11}^1) \cos(S_0 + s^*) \right] \\
- \eta_2^1 \left[ B_1 (1 + \alpha_{22}^1 + \beta_{22}^1) \cos(S_0 + s^*) - B_2 (1 + \alpha_{12}^1 + \beta_{12}^1) \sin(S_0 + s^*) \right] \}
\end{align*}
$$
with $A_{11}B_2 - A_{21}B_1 = \Delta_1$ and $A_{12}B_2 - A_{22}B_1 = \Delta_2$, i.e.:

$$
\epsilon(\lambda k + \gamma m) \pi \epsilon(\lambda s_1 + \gamma s_0 + (\lambda + \gamma)s^*) \{ [\xi_1^0 \Delta_1(1 + \alpha_{11}^0 + \beta_{11}^0) + \\
+ \xi_2^0 \Delta_2(1 + \alpha_{22}^0 + \beta_{22}^0)] \cos \frac{\omega(k \pi + S_1 + s^*)}{\mu} + \\
+ [\xi_1^0 \Delta_2(1 + \alpha_{21}^0 + \beta_{21}^0) - \xi_2^0 \Delta_1(1 + \alpha_{12}^0 + \beta_{12}^0)] \sin \frac{\omega(k \pi + S_1 + s^*)}{\mu} \} = \\
= [\eta_1^1 B_1(1 + \alpha_{11}^1 + \beta_{11}^1) + \eta_2^1 B_2(1 + \alpha_{12}^1 + \beta_{12}^1)] \sin(S_0 + s^*) + \\
+ [\eta_1^1 B_2(1 + \alpha_{11}^1 + \beta_{11}^1) - \eta_2^1 B_1(1 + \alpha_{22}^1 + \beta_{22}^1)] \cos(S_0 + s^*)
$$

since $\lambda + \gamma < 0$ and $\lambda k + \gamma m < -N_0$, with $N_0 > N$, then the right-hand side of the last equation tends to zero. The solution is of the type

$$
\overline{A}_1 \sin(S_0 + s^*) + \overline{A}_2 \cos(S_0 + s^*) = \overline{A} \sin(S_0 + s^* + \theta) = 0.
$$

If we choose $s^*$ as $\theta - \frac{\pi}{2}$ and apply the Implicit Function Theorem, we obtain:

$$
\overline{S}_0 = \frac{\pi}{2} + \alpha(S_1, \xi_1^0, \xi_2^0, \eta_1^0, \eta_2^0, m, k).
$$

Replacing it in the remainder equations, we get the other variables, i.e., $\overline{\xi}_2^0$ and $\eta_i^1$ with $i = 1, 2$. For the other case we change $t$ for $-t$.

\[\square\]

**Proof of theorem 1** We define

$$
T_{m,k} : \sigma_k \rightarrow \sigma_m
$$

by

$$
T_{m,k}(\xi_1^0, \xi_2^0, \eta_1^1, \eta_2^1, S) = (\overline{\xi}_1^0, \overline{\xi}_2^0, \overline{\eta}_1^1, \overline{\eta}_2^1, \overline{S})
$$

where:

$$
\overline{\xi}_2^0 = f(\xi_1^p(k \pi + S + s^*, \xi_1^0, \xi_2^0, \eta_1^1, \eta_2^1), \xi_2^p(k \pi + S + s^*, \xi_1^0, \xi_2^0, \eta_1^1, \eta_2^1))
$$

$$
\overline{\xi}_1^0 = \sqrt{\overline{\eta}_1^0 - (\overline{\xi}_2^0)^2}
$$

$$
\overline{\eta}_i^p(m \pi + \overline{S} + s^*, \overline{\xi}_1^0, \overline{\xi}_2^0, \eta_1^1, \eta_2^1) =
$$

$$
= g_i(\xi_1^p(k \pi + S + s^*, \xi_1^0, \xi_2^0, \eta_1^1, \eta_2^1), \xi_2^p(k \pi + S + s^*, \xi_1^0, \xi_2^0, \eta_1^1, \eta_2^1))
$$

$$
\overline{S} = \frac{\pi}{2} + \overline{\alpha}(S, \xi_1^0, \xi_2^0, \eta_1^1, \eta_2^1, m, k) \quad \text{for } i = 1, 2.
$$
Using lemma 2, this system can be reduced to:

\[
\begin{align*}
\xi_2^0 &= \overline{F}(\xi_1^0, \xi_2^0, \eta_1^1, \eta_2^1, S, k, m) \\
\eta_i^1 &= \overline{G}_i(\xi_1^0, \xi_2^0, \eta_1^1, \eta_2^1, S, k, m), \quad i = 1, 2. \\
\eta &= \frac{\pi}{2} + \overline{\alpha}(\xi_1^0, \xi_2^0, \eta_1^1, \eta_2^1, S, k, m)
\end{align*}
\]

where \(\overline{F}, \overline{G}_1, \overline{G}_2\) and \(\overline{\alpha}\) are contraction mappings for \(\lambda k + \gamma m < -N_0\) with \(N_0\) large enough.

So, if \(\Gamma\) is an orbit contained in a neighborhood of \(\Gamma_0, \Gamma_0 \neq \Gamma\), then \(\Gamma\) intersects \(\Sigma_0\) in points of \(D(T_0)\), which defines a sequence of points \(\{P_n\}_{n=0}^{\infty}\) with \(P_n \in \sigma_{k_n}\), for some \(k_n, k_n > N\).

In section 2, we define the set \(\Omega(\rho)\) as the set of doubly infinite sequences consisting of the symbols \(j_n\). Let us define \(j_n = k_n - N\), then

\[\{\ldots, j_n, j_{n+1}, \ldots\} \in \Omega(\rho).\]

On the other hand, for a given sequence

\[\beta = \{\ldots, j_n, j_{n+1}, \ldots\} \in \Omega(\rho)\]

we define \(k_n = j_n + N\), then for \(k_n > N\), we have that \(\gamma k_{n+1} + \lambda k_n < -N\) thus \(\rho k_n > k_{n+1}\) with \(\rho > -\frac{\lambda}{\gamma} > 1\).

Let \((\xi_1^0 n, \xi_2^0 n, \eta_1^0 n, \eta_2^0 n)\) be the coordinates of \(M_n^0 \in \sigma_{k_n}(\xi_1^1 n, \xi_2^1 n, \eta_1^1 n, \eta_2^1 n)\) the coordinates of \(M_n^1 = T_0 M_n^0\) and \(t_0 = k_n \pi + S_n + s^*\) the transition time from \(M_n^0\) to \(M_n^1\). So, the relations between the coordinates of the points \(M_n^0, M_{n+1}^0, \ldots\) can be written as:

\[
\begin{align*}
\xi_2^{n+1} &= \overline{F}(\xi_1^n, \xi_2^n, \eta_1^{n+1}, \eta_2^{n+1}, S_n, k_n, k_{n+1}) \\
\xi_1^{n+1} &= \sqrt{r_1^2 - (\xi_2^{n+1})^2} \\
\eta_i^n &= \overline{G}_i(\xi_1^n, \xi_2^n, \eta_1^{n+1}, \eta_2^{n+1}, S_n, k_n, k_{n+1}), \quad i = 1, 2. \\
S_{n+1} &= \frac{\pi}{2} + \overline{\alpha}(\xi_1^n, \xi_2^n, \eta_1^{n+1}, \eta_2^{n+1}, S_n, k_n, k_{n+1})
\end{align*}
\]

If we use the following fixed point lemma for product spaces given by Shil'nikov in [8], there exists a unique sequence which is in one-to-one correspondence with the solution \(\Gamma\) in a neighborhood of \(\Gamma_0\).
Lemma 3. Suppose that the complete metric spaces $X_i$ and $Y_i$ and the operators $A_i$ and $B_i$, $i = 0, \pm 1, \ldots$ satisfy the following conditions:

(1) $\sup_{x_i', x_i'' \in X_i} \rho_{X_i}(x_i', x_i'') < d$, $\sup_{y_i', y_i'' \in Y_i} \rho_{Y_i}(y_i', y_i'') < d$;

(2) $A_i(X_i \times Y_{i+1}) \subset X_{i+1}$, $B_{i+1}(X_i \times Y_{i+1}) \subset Y_i$,

\[ \rho_{X_{i+1}}(\bar{x}_i', \bar{x}_i'') < \frac{q}{2} (\rho_{X_i}(x_i', x_i'') + \rho_{Y_{i+1}}(y_i', y_i'')) \]

(3) $\rho_{Y_i}(\bar{y}_i', \bar{y}_i'') < \frac{q}{2} (\rho_{X_i}(x_i', x_i'') + \rho_{Y_{i+1}}(y_i', y_i''))$

where $q < 1$ and $\bar{x}_i = A_i(x_i, y_{i+1})$, $\bar{y}_i = B_{i+1}(x_i, y_{i+1})$. Then there exists a unique sequence

\[ (\ldots, (x_{i-1}^*, y_{i-1}^*), (x_i^*, y_i^*), (x_{i+1}^*, y_{i+1}^*), \ldots), \]

satisfying conditions

\[ x_{i+1}^* = A_i(x_i^*, y_{i+1}^*), \quad y_i^* = B_i(x_i^*, y_{i+1}^*). \]

We define

\[ X_n = U_\xi \times I, \quad Y_n = U_\eta \]

where $I = [0, \pi]$

\[ A_n : \begin{cases} \xi_{2}^{n+1} = \overline{F}(\xi_1^n, \xi_2^n, \eta_1^{n+1}, \eta_2^{n+1}, S_n, k_n, k_{n+1}) \\ \xi_1^{n+1} = \sqrt{r_1^2 - (\xi_2^{n+1})^2} \\ S_{n+1} = \frac{\pi}{2} + c(\xi_1^n, \xi_2^n, \eta_1^{n+1}, \eta_2^{n+1}, S_n, k_n, k_{n+1}) \end{cases} \]

and

\[ B_{n+1} : \begin{cases} \eta_i^n = \overline{G}_i(\xi_1^n, \xi_2^n, \eta_1^{n+1}, \eta_2^{n+1}, S_n, k_n, k_{n+1}), \quad i = 1, 2. \end{cases} \]

Because of the properties of $T_m, k$, this sequence of spaces and operators satisfies the conditions of the lemma. Thus, there exists a unique sequence

\[ (\ldots, (\xi_1^n, \xi_2^n, \eta_1^n, \eta_2^n, S_n), (\xi_1^{n+1}, \xi_2^{n+1}, \eta_1^{n+1}, \eta_2^{n+1}, S_{n+1}), \ldots) \]
which corresponds to a unique solution $\Gamma$ in a neighborhood of $\Gamma_0$. 

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References


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