EXISTENCE AND MULTIPLICITY OF SOLUTIONS FOR SYSTEMS OF SEMILINEAR ELLIPTIC EQUATIONS*

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Abstract

In this work we study the existence and multiplicity of positive solutions of the quasimonotone system of semilinear elliptic equations

$$\begin{aligned}\Delta u_i + \lambda f_i(x, u_1, \dots, u_m) &= 0 && \text{in } \Omega \\ u_i|_{\partial\Omega} &= 0, \quad i = 1, \dots, m\end{aligned}$$

where f is restricted to a certain type of superlinear growth at infinity.

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1. INTRODUCTION

There are already a large number of established results [8] on the existence and multiplicity of non-negative solutions for a scalar semilinear elliptic equation

$$(1.1) \quad \Delta u + f(x, u) = 0$$

$$(1.2) \quad u|_{\partial\Omega} = 0$$

in a smooth bounded domain Ω in R^n , $n \geq 2$, where $f(x, 0) > 0$, f is monotone increasing in u , and satisfies some increasing rate in u uniformly in x . However, as pointed out in [8], the corresponding problem for a system of semilinear elliptic equations is not that well understood. In this work, we shall examine the system of equations

$$(1.3) \quad \Delta u_i + \lambda f_i(x, u_1, \dots, u_m) = 0 \quad \text{in } \Omega$$

$$(1.4) \quad u_i|_{\partial\Omega} = 0, \quad i = 1, \dots, m$$

where $f_i(x, 0, \dots, 0) > 0$ for all $x \in \Omega$, $i = 1, 2, \dots, m$. The functions f_i , $i = 1, \dots, m$, satisfy the quasimonotone condition and a certain blow up rate as to be made precise in the assumptions (H1) and (H2) below. Then results similar to those of the scalar equation case can be established. It should be noted that unless $\frac{\partial f_i}{\partial u_j} = \frac{\partial f_j}{\partial u_i}$ for all $1 \leq i, j \leq m$, the problem cannot be formulated in a variational form, hence techniques associated with variational structure are not applicable. The principal tools that we employ in this work are construction of upper and lower solutions, apriori estimate, and Leray-Schauder degree theory. The proofs are similar to that of [5] where we treat one scalar equation with non-variational structure. In what follows, solution means a classical solution in $(C^2(\bar{\Omega}))^m$.

We will now make our assumptions more precise. For ease of notation, we denote $\vec{u} = (u_1, \dots, u_m)^T$, $\|\vec{u}\|_s = \left(\sum_{i=1}^m u_i^s \right)^{\frac{1}{s}}$ and $\vec{f} = (f_1, \dots, f_m)^T$. For equations (3) and (4) defined in a smooth bounded domain Ω in R^n , $n \geq 2$, we assume the following two conditions:

(H1) For $\vec{x} \in \bar{\Omega}$, $1 \leq i, j \leq m$, $\vec{u} \geq \vec{0}$ (i.e. $u_i \geq 0$ componentwise), let $f_i \in C^3(\bar{\Omega} \times \mathbb{R}^m)$, $\frac{\partial f_i}{\partial u_i}(x, \vec{u}) > 0$ and $\vec{f}$ satisfies the quasimonotone condition $\frac{\partial f_i}{\partial u_j}(x, \vec{u}) \geq 0$ for $i \neq j$. For sufficiently large $M > 0$, there exist constants $c_1, c_2 > 0$, independent of $\vec{x}$, such that when $u_i \geq M$, $i = 1, \dots, m$, then

$$(1.5) \quad f_i(\vec{x}, \vec{u}) \leq c_2 \|\vec{u}\|_s^s$$

$$(1.6) \quad c_1 u_i^s \leq f_i(\vec{x}, 0, \dots, 0, u_i, 0, \dots, 0) \quad \text{for all } \vec{x} \in \Omega,$$

where $1 < s < \frac{n}{n-2}$ for $n > 2$, and any $s \in (1, \infty)$ for $n = 2$.

(H2) Ω is convex, $\partial\Omega$ has positive curvature everywhere, and there exist $r, \delta > 0$ such that for all $\vec{u} \geq 0$ and all $\vec{x} \in \Omega_r \equiv \{x \in \Omega \mid \text{dist}(\vec{x}, \partial\Omega) < r\}$, $i = 1, \dots, m$,

$$\nabla_x f_i(\cdot, \vec{u}) \cdot \vec{\mu} \leq 0,$$

where $\vec{\mu}$ is a unit vector satisfying

$$(1.7) \quad |\vec{\mu} - \vec{n}(\vec{x})| < \delta,$$

and $\vec{n}(\vec{x})$ is defined for $\vec{x} \in \Omega$ to be $\vec{n}(\vec{y})$, which is the unit outward normal vector at $\vec{y} \in \partial\Omega$ with $\vec{y}$ defined by $|\vec{y} - \vec{x}| = \text{dist}(\vec{x}, \partial\Omega)$ (the question of well definedness of $\vec{n}(\vec{x})$ is discussed in [4].)

Under these hypothesis, we shall prove there exists a $\lambda^* > 0$ such that for $\lambda < \lambda^*$, there are at least two solutions, and for $\lambda = \lambda^*$, there exists at least one solution, while for $\lambda > \lambda^*$, there is no solution. In case that strict convexity of $\vec{f}$ is assumed, i.e., $\left(\frac{\partial^2 f_i}{\partial u_j \partial u_k}\right)_{j,k=1,\dots,m}$ is a positive definite matrix for each $i = 1, \dots, m$, then the statements can be made more precise for $\lambda = \lambda^*$: there exists exactly one solution for such λ , and it is a simple turning point.

Finally we remark that similar techniques can be applied to study the existence and multiplicity of the system of quasimonotone semilinear equations

$$(1.8) \quad \Delta u_i + \vec{b} \cdot \nabla u_i + \lambda f_i(x, u_1, \dots, u_m) = 0 \quad \text{in } \Omega$$

$$(1.9) \quad u_i|_{\partial\Omega} = 0, \quad i = 1, \dots, m$$

where $\vec{b} \in C^1(\bar{\Omega})$ satisfies the additional condition

$$\vec{b} \cdot \vec{\mu} \geq 0,$$

in assumption (H2).

2. Existence of Solutions

For simplicity we use the vectors $\vec{1} = (1, 1, \dots, 1)$ and $\vec{0} = (0, 0, \dots, 0)$. When $\lambda = 0$, $\vec{u} \equiv \vec{0}$ is the unique solution. Let S denote the set of nonnegative λ for which equations (3)-(4) have a nonnegative solution. We shall begin by proving that S is a bounded interval.

Lemma 2.1. *If λ is sufficiently small, then $\lambda \in S$.*

Proof: Let $\vec{u} = \tilde{\phi}\vec{1}$ where

$$(2.1) \quad \Delta\tilde{\phi} = -1,$$

$$(2.2) \quad \tilde{\phi}|_{\partial\Omega} = 0.$$

By the Maximum Principle, $\tilde{\phi} > 0$. Moreover for all $1 \leq i \leq m$, there exists a sufficiently small λ , independent of i , such that

$$\Delta\tilde{\phi} + \lambda f_i(x, \tilde{\phi}, \dots, \tilde{\phi}) = -1 + \lambda f_i(x, \tilde{\phi}, \dots, \tilde{\phi}) < 0,$$

hence u_i is an upper solution of (1)_i. Consequently $\vec{u}$ is an upper solution for the system of equations (3)-(4). Since $\vec{u} \equiv \vec{0}$ is always a lower solution and the system of equations is quasimonotone, solution exists by monotone iteration methods (Thm. 2 in [9]).

Lemma 2.2.

If a solution of the equations (3)–(4) exists for $\tilde{\lambda} > 0$, then the solution exists for all λ such that $0 \leq \lambda \leq \tilde{\lambda}$.

Proof:

Let $\tilde{u}_i$ satisfies (3)-(4) with $\lambda = \tilde{\lambda}$, then $(\tilde{u}_1, \tilde{u}_2, \dots, \tilde{u}_m)$ is an upper solution for

$$\Delta u_i + \lambda f_i(\vec{x}, u_1, \dots, u_m) = 0, \quad i = 1, \dots, m,$$

for $0 < \lambda < \tilde{\lambda}$. Indeed,

$$\Delta\tilde{u}_i + \lambda f_i(\vec{x}, \tilde{u}_1, \dots, \tilde{u}_m) = -\tilde{\lambda} f_i(\vec{x}, \tilde{u}_1, \dots, \tilde{u}_m) + \lambda f_i(\vec{x}, \tilde{u}_1, \dots, \tilde{u}_m) < 0$$

and $\vec{u} \equiv \vec{0}$ is a lower solution. So, by monotone iteration, there exists a solution for $0 < \lambda < \tilde{\lambda}$.

It should be remarked that such monotone iterations give minimal positive solution for each λ for which solution exists: There exists a minimal positive solution $\vec{u}$ of (3)-(4) that satisfies $0 < \vec{u} \leq \vec{v}$ in Ω (componentwise) for all positive solution $\vec{v}$ of (3)-(4). It is also clear from the monotone iteration technique that $\vec{u}_{\min}^{\lambda_1} < \vec{u}_{\min}^{\lambda_2}$ in Ω for $\lambda_1 < \lambda_2$, where $\vec{u}_{\min}^{\lambda_i}$ is the minimal positive solution for $\lambda = \lambda_j$, $j = 1, 2$.

We will establish there is a second solution besides this minimal positive solution.

Lemma 2.3.. *S is bounded.*

Proof:

Let ϕ_1 be the eigenfunction corresponding to the smallest eigenvalue $\mu_1 > 0$ of the problem

$$(2.3) \quad \Delta\phi + \mu\phi = 0,$$

$$(2.4) \quad \phi|_{\partial\Omega} = 0.$$

Now multiply equation (3) by ϕ_1 and integrate over Ω . It follows from the hypothesis in (H1) that we can choose $\alpha > 0$ such that for all i and for all $\vec{x} \in \Omega$

$$(2.5) \quad f(\vec{x}, u_1, \dots, u_m) \geq f(\vec{x}, 0, \dots, 0, u_i, 0, \dots, 0) \geq \alpha u_i.$$

Thus we get

$$(2.6) \quad \int_{\Omega} \mu_1 u_i \phi_1 d\vec{x} = \lambda \int_{\Omega} f(x, u_1, \dots, u_m) \phi_1 d\vec{x} \geq \lambda \alpha \int_{\Omega} u_i \phi_1 d\vec{x}.$$

Therefore, for any $\lambda \in S$ we have

$$(2.7) \quad \lambda \leq \frac{\mu_1}{\alpha} < \infty.$$

Hence the lemma.

From the previous lemmas it follows that there exists a $\lambda^* = \sup S$ such that a solution exists for $\lambda \in [0, \lambda^*)$ or $\lambda \in [0, \lambda^*]$ and there is no solution for $\lambda > \lambda^*$.

3. *A-priori* Bounds for Solutions

In this section we will establish that for any solution $\vec{u}$ of (3)-(4) with $\lambda \geq \varepsilon > 0$, its $C^{2+\alpha}$ -norm,

$$\|\vec{u}\|_{C^{2+\alpha}} = \sum_{i=1}^m \|u_i\|_{C^{2+\alpha}}$$

is uniformly bounded in Ω . The bound might depend on ε .

Lemma 3.1.. *For $\lambda \geq \varepsilon > 0$ and any given compact set $K \subset \Omega$ there exists $\tilde{C} > 0$, which depends on K , such that for all solutions of (3)-(4) (subject to the assumptions (H1) and (H2)),*

$$(3.1) \quad \|\vec{u}\|_{L^1(K)} = \sum_{i=1}^m \|u_i\|_{L^1(K)} \leq \tilde{C}$$

Proof:

For fixed λ and i , choose $\alpha = \frac{1+\mu_1}{\lambda}$. By the superlinearity of f_i there is $p > 0$, which depends on α , such that for all u_i , $i = 1, \dots, m$, $f_i(\vec{x}, u_1, \dots, u_m) > \alpha u_i - p$ on $\bar{\Omega}$.

As in equation (6), here we get

$$(3.2) \quad \int_{\Omega} \mu_1 \phi_1 u_i d\vec{x} = \lambda \int_{\Omega} f(\vec{x}, \vec{u}) \phi_1 d\vec{x} \geq \lambda \alpha \int_{\Omega} u_i \phi_1 d\vec{x} - \lambda p \int_{\Omega} \phi_1 d\vec{x}.$$

Thus by the choice of α

$$\int_{\Omega} u_i \phi_1 d\vec{x} \leq \lambda p \int_{\Omega} \phi_1 d\vec{x}.$$

Since $\lambda \leq \lambda^*$ and in any compact set $K \subset \Omega$, ϕ_1 is bounded below by a constant $c > 0$ depending only on K , this leads to the uniform boundedness of $\|u_i\|_{L^1(K)}$ and so of $\|\vec{u}\|_{L^1(K)}$ for any solution $\vec{u}$ of (3)-(4). The value of $\tilde{C}$ in (1) depends on the compact set K .

Lemma 3.2..

There exists $\delta > 0$, that depends only on the geometry of Ω and $C > 0$ (which can depend on δ , but independent of $\vec{u}$) such that if we define

$$\Omega_{\delta} = \{\vec{x} \in \Omega | \text{dist}(\vec{x}, \partial\Omega) \leq \delta\}$$

then any solution $\vec{u}$ of (3)-(4) has the a-priori bound

$$(3.3) \quad \|\vec{u}\|_{L^{\infty}(\Omega_{\delta})} \leq C$$

Proof:

We shall use r and δ as defined in the assumption (H2). Due to the assumed strict convexity of the domain Ω , if all the f_i 's are independent of $\vec{x}$ explicitly, i.e. $f_i = f_i(\vec{u})$, then we can apply the main theorem in [11] to conclude that any positive solution $\vec{u}$ of the system of equations (3)-(4) is nondecreasing along the inward normal direction for at least some distance $r_1 < r$ from $\partial\Omega$. With the assumption $\nabla_x f_i(\cdot, \vec{u}) \cdot \vec{n} \leq 0$, $i = 1, \dots, m$, a similar proof as in [11] can be carried out to yield the same conclusion for the case $\vec{f} = \vec{f}(\vec{x}, \vec{u})$.

Our additional assumptions (H2) on $\vec{f}$ will then allow us to conclude that there exist $0 < r_0 \leq r$ and $0 < \delta_0 \leq \delta$, which depend on the geometry of Ω but independent of the solution $\vec{u}$, satisfying:

- * For any $\vec{x} \in \partial\Omega$, $u(x - t\vec{\mu})$ is nondecreasing for any $t \in [0, r_0]$, and $\vec{\mu}$ such that $\delta_0 > |\vec{\mu} - \vec{n}(x)|$.

The proof proceeds exactly as in Step 2 in section 1 of [4]: The statement (*) implies that for $\vec{x} \in \Omega_{r_0}^\bullet$, there exists a truncated cone $I_{\vec{x}}$ as indicated in fig. 1 such that $u(\vec{y}) \geq u(\vec{x})$ for all $\vec{y} \in I_{\vec{x}}$. Moreover there exists a γ , independent of $\vec{x} \in \Omega_{r_0}^\bullet$, such that $meas(I_{\vec{x}}) \geq \gamma > 0$ for all $\vec{x} \in \Omega_{r_0}^\bullet$. By the lemma 3.2, for $1 \leq i \leq m$,

$$\tilde{C} \geq \int_{I_{\vec{x}}} u_i(\vec{y}) \phi_1(\vec{y}) d\vec{y} \geq \gamma u(\vec{x}) \min_{\vec{y} \in I_{\vec{x}}} \{\phi_1(\vec{y})\}.$$

Since ϕ_1 is positive in a compact set in Ω and r_0 is independent of $\vec{u}$, so the last term on the right hand side has a positive lower bound independent of $\vec{u}$, hence the proof.

From the previous two lemmas since $\vec{u}$ is uniformly bounded near the boundary and locally in L^1 it follows immediately that

$$(3.4) \quad \|\vec{u}\|_{L^1} \leq C_1.$$

Next we have

Lemma 3.3.. For $\lambda \geq \varepsilon > 0$, there exists a $C'_s > 0$ such that for all solutions of (3)-(4) ,

$$(3.5) \quad \|\vec{u}\|_{L^s} \leq C'_s$$

where s is the blow up rate of $\vec{f}$ as defined in the assumption (H1).

Proof: Multiply equation $(3)_i$ by $\tilde{\phi}$ defined in (1)-(2) and integrate to obtain

$$\begin{aligned} \int_{\Omega} u_i d\vec{x} &= \lambda \int_{\Omega} f_i(\vec{x}, u_1, \dots, u_m) \tilde{\phi} d\vec{x} \\ &\geq \lambda \int_{\Omega} f_i(\vec{x}, 0, \dots, 0, u_i, 0, \dots, 0) \tilde{\phi} \geq \varepsilon c_0 \int_{\Omega} u_i^s \tilde{\phi} \end{aligned}$$

where by our assumptions on f_i there exists a sufficiently small $c_1 > c_0 > 0$ such that $f_i(\vec{x}, 0, \dots, 0, u_i, 0, \dots, 0) \geq c_0 u_i^s$. Hence using the result in (4) , $u_i \in L_{\text{loc}}^s$. Together with lemma 3.3 we conclude that there exists a constant C'_s such that

$$\|\vec{u}\|_{L^s} \leq C'_s$$

for $\lambda \geq \varepsilon > 0$.

Lemma 3.4.. For $\lambda \geq \varepsilon > 0$, there exists a $C > 0$ such that for all solutions of (3)-(4)

$$(3.6) \quad \|\vec{u}\|_{\mathcal{H}^1} = \sum_{i=1}^m \|u_i\|_{\mathcal{H}^1} \leq C$$

Proof: From

$$\Delta u_i + \lambda f_i(\vec{x}, u_1, \dots, u_n) = 0,$$

we multiply by u_i and integrate. By assumption (H1) with $c_3 = \max\{c_2, mM^s c_2\}$ we obtain

$$\begin{aligned} \int_{\Omega} \frac{1}{2} |\nabla u_i|^2 d\vec{x} &= \lambda \int_{\Omega} f_i(\vec{x}, u_1, \dots, u_n) u_i d\vec{x} \\ (3.7) \quad &\leq \lambda^* c_3 \int_{\Omega} ((u_1^s + \dots + u_n^s) + 1) u_i d\vec{x} \end{aligned}$$

For $i \neq j$ we have

$$\int_{\Omega} u_j^s u_i \, d\bar{x} \leq \left(\int_{\Omega} u_j^{s+1} \right)^{\frac{s}{s+1}} \left(\int_{\Omega} u_i^{s+1} \right)^{\frac{1}{s+1}} \, d\bar{x} \leq \frac{s}{s+1} \|u_j\|_{L^{s+1}}^{s+1} + \frac{1}{s+1} \|u_i\|_{L^{s+1}}^{s+1}$$

using Hölder's inequality. Adding over all i we arrive at

$$\sum_{i=1}^m \|u_i\|_{\mathcal{H}^1} \leq c_4 \left(\sum_{j=1}^m \|u_j\|_{L^{s+1}}^{s+1} + 1 \right)$$

for some $c_4 > 0$.

From the interpolation inequality

$$\|u_i\|_{L^p} \leq \|u_i\|_{L^q}^{\eta} \|u_i\|_{L^r}^{1-\eta}$$

with $\frac{1}{p} = \frac{\eta}{q} + \frac{1-\eta}{r}$, $0 \leq \eta \leq 1$, it follows that

$$\sum_{i=1}^m \|u_i\|_{\mathcal{H}^1} \leq c_4 \left(\sum_{i=1}^m (\|u_i\|_{L^{n_c}}^{\eta} \|u_i\|_{L^s}^{1-\eta})^{s+1} + 1 \right)$$

where $n_c \equiv \frac{2n}{n-2}$, $\eta = \frac{n_c}{(1+s)(n_c-s)}$

Since $\|u_i\|_{L^s}$ is bounded we have

$$\sum_{i=1}^m \|u_i\|_{\mathcal{H}^1} \leq c_4 + c_5 \sum_{i=1}^m \|u_i\|_{L^{n_c}}^{\beta}$$

with $\beta = \eta(s+1) < 2$ by assumption (H2).

Using Sobolev's estimates for $\|u_i\|_{L^{n_c}}$ we arrive at

$$\|\vec{u}\|_{\mathcal{H}^1}^2 \leq c_4 + c_6 \|\vec{u}\|_{\mathcal{H}^1}^{\beta}$$

Hence the lemma.

Lemma 3.5.. *For $\lambda \geq \varepsilon > 0$, there exists a $C_0 > 0$ such that for all solutions of (3)-(4)*

$$(3.8) \quad \|\vec{u}\|_{C^{2+\alpha}} = \sum_{i=1}^m \|u_i\|_{C^{2+\alpha}} \leq C_0$$

Proof: Since $\|u_i\|_{L^{n_c}} \leq \|u_i\|_{\mathcal{H}^1} \leq C$, the growth restriction on f_i implies (Prop. B.1, Appendix B, [10]) that $\|f_i(\cdot, \vec{u}(\cdot))\|_{L^\gamma} \leq k_1$, with $\gamma > 2$.

By the regularity estimate for linear elliptic equations [1], for each i , $i = 1, \dots, m$, $\|u_i\|_{W^{2,2}} \leq K_2$. We can now boot-strap, so that the function u_i is in $C^{2+\alpha}$. Finally, using Schauder estimate, $\|u_i\|_{C^{2+\alpha}} \leq C$ for $\lambda \geq \varepsilon > 0$.

An immediate consequence of (8) is the existence of solution for $\lambda = \lambda^*$.

Lemma 3.6.. *There exists a minimal positive solution for $\lambda = \lambda^*$. Thus $S = [0, \lambda^*]$.*

Proof:

Because of lemma 3.5, we have a $C^{2+\alpha}$ bound on all solutions. Thus the minimal solution for $\lambda < \lambda^*$ converges uniformly in $C(\bar{\Omega})$ to a function $\vec{u}^*$ as $\lambda \rightarrow \lambda^*$, a standard argument using the regularity of the elliptic equations shows that $\vec{u}^*$ is indeed a solution for $\lambda = \lambda^*$.

4. Multiplicity of solutions

For $0 < \alpha < 1$, let $T : (C^{1+\alpha}(\Omega))^m \rightarrow (C^{1+\alpha}(\bar{\Omega}))^m$ be the inverse operator of $-\Delta$ subject to zero Dirichlet boundary conditions. Choose $\delta > 0$ such that $\delta\phi_1\vec{1}$ is less than the minimal solution at $\lambda = \frac{\varepsilon}{2}$ (recall that ϕ_1 is defined as in the equations (3)-(4)), and

$$G = \{\vec{u} \in (C^{1+\alpha}(\bar{\Omega}))^m \mid \vec{u} \geq \delta\phi_1\vec{1}, \|\vec{u}\|_{C^{1-\alpha}} \leq C_0 + 1\}.$$

Lemma 4.1.. *Let $\varepsilon < \lambda \leq \lambda^* < \mu$. Then $\deg(I - \lambda T(\vec{f}(\vec{x}, \cdot)), G, 0) = 0$.*

Proof:

From the previous section we know that uniform *a-priori* bound can be constructed for λ in the range of $\varepsilon \leq \lambda \leq \lambda^* < \mu$. As we homotopy from ε to μ , there cannot be a solution which touches $\delta\phi_1$, since the minimal positive solution can only get larger with λ . Also from the *a-priori* bounds established in section 3, there is no solution that touches the boundary of the ball $\|\vec{u}\|_{C^{1+\alpha}} = C_0 + 1$, hence the Leray-Schauder degree of the map $I - \lambda T(f(\vec{x}, \cdot))$ for $\varepsilon \leq \lambda \leq \mu$ is well defined.

Since in particular, there is no solution beyond λ^* , we have

$$0 = \deg(I - \mu T(\vec{f}(\vec{x}, \cdot)), G, 0) = \deg(I - \lambda T(\vec{f}(\vec{x}, \cdot)), G, 0).$$

Lemma 4.2.. *There exists at least 2 solutions of the equation (3)-(4), for λ in the range (ε, λ^*) .*

Proof:

Let $\lambda \in [\varepsilon, \lambda^*)$ be fixed and pick λ_1, λ_2 such that $\frac{\varepsilon}{2} < \lambda_1 < \lambda < \lambda_2 < \lambda^*$. Let $\underline{u}$ be the minimal positive solution for λ_1 and $\bar{u}$ be the minimal positive solution for λ_2 . Then $\underline{u}$ and $\bar{u}$ are lower and upper solutions respectively for λ . Define

$$D = \{\vec{u} \in (C^{1+\alpha}(\bar{\Omega}))^m \mid \underline{u} < \vec{u} < \bar{u}, \|\vec{u}\|_{C^{1+\alpha}} < C_0 + 1\}.$$

Because the system of equations is quasimonotone, Theorem 1 in [9] gives

$$(4.1) \quad \deg(I - \lambda T(\vec{f}(\vec{x}, \cdot)), D, 0) = 1.$$

Thus by lemma 4.1 and the excision property of the degree we conclude that

$$\deg(I - \lambda T(\vec{f}(\vec{x}, \cdot)), G \setminus \bar{D}, 0) = -1.$$

Hence there exists at least one more solution besides the minimal positive solution for $\varepsilon < \lambda < \lambda^*$.

5. Behavior of the Solution for $\lambda = \lambda^*$ for Strictly Convex $\vec{f}$

In this section we shall further assume that for each $i = 1, \dots, m$, the matrix $\left(\frac{\partial^2 f_i}{\partial u_j \partial u_k}\right)_{j,k=1,\dots,m}$ is positive definite and that $\frac{\partial f_i}{\partial u_j} > 0$ for all $1 \leq i, j \leq m$. Then as in the case of a scalar equation we can prove the following properties of the solution set,

(i) The solution for $\lambda = \lambda^*$ is unique.

(ii) Around a neighborhood of $\lambda = \lambda^*$, the solution set can be parametrized by $\lambda = \lambda(s)$ and $u = u(s)$ for $-\delta < s < \delta$ for some $\delta > 0$, with $\lambda(0) = \lambda^*$. Further, $\lambda(s) < \lambda^*$ for $s \neq 0$ in that neighborhood. Hence $\lambda = \lambda^*$ corresponds to a simple turning point.

To prove these claims, we define

$$(5.1) \quad \vec{F}(\vec{x}, \vec{u}, \lambda) \equiv \Delta \vec{u} + \lambda \vec{f}(\vec{x}, \vec{u})$$

for $\vec{u}$ in $(C^{2+\alpha}(\bar{\Omega}))^m$ with zero Dirichlet boundary condition.

Thus solutions of the equations (3)-(4) correspond to

$$(5.2) \quad \vec{F}(\vec{x}, \vec{u}, \lambda) = 0.$$

Denote the minimal positive solution for λ by $\vec{u}_{\min}^\lambda$. The Frechet derivative of $\vec{F}$ evaluated at the minimal solution is given by

$$(5.3) \quad D_{\vec{u}} \vec{F}(\vec{x}, \vec{u}_{\min}^\lambda, \lambda) \vec{v} \equiv \Delta \vec{v} + \lambda D_{\vec{u}} \vec{f}(\vec{x}, \vec{u}_{\min}^\lambda) \vec{v}$$

for any $\vec{v}$ in $(C^{2+\alpha}(\bar{\Omega}))^m$, where

$$D_{\vec{u}} \vec{f}(\vec{x}, \vec{u}_{\min}^\lambda) = \left(\frac{\partial f_i}{\partial u_j}(\vec{x}, \vec{u}_{\min}^\lambda) \right)_{i,j=1,\dots,m}$$

First we establish a lemma:

Lemma 5.1. *Let $A(\vec{x}) = (a_{ij})_{i,j=1,\dots,m} > 0$, i.e. $a_{ij}(\vec{x}) > 0$ for all $1 \leq i, j \leq m$. Then there exists a positive eigenvalue η_1 and a positive vector eigenfunction $\vec{\psi}$ such that*

$$(5.4) \quad \Delta \vec{\psi} + \eta_1 A(\vec{x}) \vec{\psi} = 0,$$

$$(5.5) \quad \vec{\psi}|_{\partial\Omega} = 0.$$

Proof: Define $\tilde{T} : (C(\bar{\Omega}))^m \rightarrow (C(\bar{\Omega}))^m$ by $\tilde{T}\vec{\varphi} \equiv -\Delta^{-1}(A\vec{\varphi})$ subject to zero boundary conditions. Then $\tilde{T}$ is a positive operator: If $\vec{\varphi} \geq 0$ with at least one component not identically zero in Ω , then if $\vec{\phi} = \tilde{T}\vec{\varphi}$ we have $-\Delta\vec{\phi} = A\vec{\varphi} \geq \vec{0}$, with no component being identically zero in Ω . Hence by the Maximum Principle $\vec{\phi} > \vec{0}$, i.e. each component of ϕ is positive on Ω .

Since $\tilde{T}$ is also compact, the Krein-Rutman Theorem implies the existence of a positive eigenvalue μ_1 and a positive vector eigenfunction $\vec{\psi}$ such that

$$\tilde{T}\vec{\psi} = \mu_1 \vec{\psi}$$

so

$$A\vec{\psi} = -\mu_1 \Delta \vec{\psi}.$$

The lemma follows with $\eta_1 = \frac{1}{\mu_1}$.

For a fixed $\lambda_0 < \lambda^*$, by the previous lemma, there exists a positive eigenvalue κ_1 and a positive eigenfunction vector $\vec{\varphi}_1$ such that

$$\Delta \vec{\varphi}_1 + \kappa_1 \lambda_0 D_{\vec{u}} \vec{f}(\vec{x}, \vec{u}_{\min}^{\lambda_0}) \varphi_1 = 0.$$

hence for $\lambda < \min\{\kappa_1 \lambda_0, \lambda_0\}$ there cannot be a nontrivial solution for the problem

$$\Delta \phi + \lambda D_{\vec{u}} \vec{f}(\vec{x}, \vec{u}_{\min}^{\lambda}) \phi = 0$$

because of the comparison theorem 1.13 in [2] and $D_{\vec{u}} \vec{f}(\vec{x}, \vec{u}_{\min}^{\lambda})$ being an increasing function in λ for each entry in the matrix. Thus for sufficiently small λ the Frechet derivative in (3) is non-singular. On the other hand there should be a first $\lambda = \bar{\lambda} \leq \lambda^*$ at which $D_{\vec{u}} \vec{F}$ becomes singular. Otherwise we can continue the minimal solution branch to $\lambda = \lambda^*$ since the solution can never blow up due to the *a-priori* bound that we have established. Using Implicit Function theorem at $\lambda = \lambda^*$, we can obtain a solution of (2) with $\lambda > \lambda^*$, which is a contradiction.

We claim this $\bar{\lambda}$ is the first eigenvalue for

$$(5.6) \quad \Delta \vec{\psi} + \lambda D_{\vec{u}} \vec{f}(\vec{x}, \vec{u}_{\min}^{\bar{\lambda}}) \vec{\psi} = 0$$

subject to Dirichlet boundary condition, and therefore its corresponding vector eigenfunction $\vec{\psi}$ is positive. If not, there exists a first eigenvalue $\nu_1 < \bar{\lambda}$ and a corresponding eigenfunction vector $\vec{\psi}_1 > 0$ to the problem

$$(5.7) \quad \Delta \vec{\psi} + \nu D_{\vec{u}} \vec{f}(\vec{x}, \vec{u}_{\min}^{\bar{\lambda}}) \vec{\psi} = 0.$$

However

$$\nu_1 D_{\bar{u}} \vec{f}(\vec{x}, \vec{u}_{\min}^{\bar{\lambda}}) < \beta D_{\bar{u}} \vec{f}(\vec{x}, \vec{u}_{\min}^{\beta}) < \bar{\lambda} D_{\bar{u}} \vec{f}(\vec{x}, \vec{u}_{\min}^{\bar{\lambda}})$$

for some $\beta < \lambda$ and close to $\bar{\lambda}$ by simple continuity.

Since

$$\nu_1 D_{\bar{u}} \vec{f}(\vec{x}, \vec{u}_{\min}^{\nu_1}) < \nu_1 D_{\bar{u}} \vec{f}(\vec{x}, \vec{u}_{\min}^{\bar{\lambda}}) < \beta D_{\bar{u}} \vec{f}(\vec{x}, \vec{u}_{\min}^{\beta})$$

there exist positive first eigenvalues ρ_1, ρ_2, ρ_3 to the problems

$$\Delta Z_1 + \rho_1 \nu_1 D_{\bar{u}} \vec{f}(\vec{x}, \vec{u}_{\min}^{\nu_1}) Z_1 = 0$$

$$\Delta Z_2 + \rho_2 \nu_1 D_{\bar{u}} \vec{f}(\vec{x}, \vec{u}_{\min}^{\bar{\lambda}}) Z_2 = 0$$

$$\Delta Z_3 + \rho_3 \beta D_{\bar{u}} \vec{f}(\vec{x}, \vec{u}_{\min}^{\beta}) Z_3 = 0$$

with $\rho_2 = 1$ because of equation (7) and $\rho_1 \geq \rho_2 \geq \rho_3$ by the comparison theorem 1.13 in [2].

Since $\nu D_{\bar{u}} \vec{f}(\vec{x}, \vec{u}_{\min}^{\nu})$ is a continuous function of ν , and eigenvalues depend continuously on the coefficients, hence ρ is a continuous function of ν . By the Intermediate Value Theorem, there exist a $\rho = 1$ and a $\nu_1 < \nu < \beta$ such that

$$\Delta \vec{Z} + \rho \nu D_{\bar{u}} \vec{f}(\vec{x}, \vec{u}_{\min}^{\nu}) \vec{Z} = 0.$$

But this fact contradicts our assumption that $\lambda = \lambda^*$ is the first value where the Frechet derivative (3) becomes singular. Hence we have proved that there exists a first eigenvalue $\bar{\lambda} \leq \lambda^*$ and a positive vector eigenfunction $\vec{\psi}$ such that

$$(5.8) \quad \Delta \vec{\psi} + \bar{\lambda} D_{\bar{u}} \vec{f}(\vec{x}, \vec{u}_{\min}^{\bar{\lambda}}) \vec{\psi} = 0.$$

By lemma 5.1, we have the existence of a first eigenvalue $\bar{\lambda}$ and a positive vector eigenfunction $\vec{\psi}^* > 0$ such that

$$(5.9) \quad \Delta \vec{\psi}^* + \bar{\lambda} \left(D_{\bar{u}} \vec{f}(\vec{x}, \vec{u}_{\min}^{\bar{\lambda}}) \right)^T \vec{\psi}^* = 0.$$

Let $\langle \vec{f}, \vec{g} \rangle = \sum_{i=1}^m f_i g_i$ denote the usual inner product in $\mathbb{R}^m$. Take inner product of (8) with $\vec{\psi}^*$ and (9) with $\vec{\psi}$. Integrating by parts and subtracting, since $\int_{\Omega} \langle D_{\bar{u}} \vec{f}, \vec{\psi} \rangle d\vec{x}$ is positive it follows that $\bar{\lambda} = \bar{\bar{\lambda}}$.

Now it can be checked that

$$\vec{F}_{\bar{\lambda}}(\vec{x}, \vec{u}_{\min}^{\bar{\lambda}}, \bar{\lambda}) = \vec{f}(\vec{x}, \vec{u}_{\min}^{\bar{\lambda}}) > \vec{0}.$$

Since $\int_{\Omega} \langle \vec{\psi}^*, \vec{f}(\vec{x}, \vec{u}_{\min}^{\lambda}) \rangle d\vec{x}$ is positive, it follows that

$$\vec{F}_{\lambda} \notin \text{Range} \left(D_{\vec{u}} \vec{f}(\vec{x}, \vec{u}_{\min}^{\lambda}) \right).$$

So $\lambda = \bar{\lambda}$ is not a bifurcation point [3].

We can therefore parametrize the solution set in a neighborhood around λ by: $\vec{u} = \vec{u}(s)$, $\lambda = \lambda(s)$, for some sufficient small $\delta > 0$ and $-\delta < s < \delta$ with $\lambda(0) = \bar{\lambda}$ as a consequence of implicit function theorem.

With the assumed smoothness in f , we can differentiate the equation (2) with respect to s , which gives

$$(5.10) \quad \Delta \vec{v} + \lambda(s) D_{\vec{u}} \vec{f}(\vec{x}, \vec{u}_{\min}^{\lambda}) \vec{v} + \lambda'(s) \vec{f}(\vec{x}, \vec{u}_{\min}^{\lambda}) = 0$$

where $\vec{v} \equiv \frac{d\vec{u}}{ds}(s)$. We evaluate the equation at $s = 0$, take inner product with $\vec{\psi}^*$ and integrate over Ω , which results in

$$(5.11) \quad \lambda'(0) = 0.$$

Differentiate equation (10) once more. With $\vec{w} = \frac{d^2 \vec{u}}{ds^2}(s)$ we have, after evaluating at $s = 0$ and using the equation (11),

$$\Delta w_i + \bar{\lambda} \sum_{j=1}^m \frac{\partial f_i}{\partial u_j} w_j + \bar{\lambda} \sum_{j,k=1}^m \frac{\partial^2 f_i}{\partial u_j \partial u_k} v_j v_k + \lambda''(0) f_i = 0$$

for $i = 1, \dots, m$. Again take inner product with $\vec{\psi}^*$ and integrate to get $\lambda''(0) < 0$ after employing the assumption that $\left(\frac{\partial^2 f_i}{\partial u_j \partial u_k} \right)_{j,k=1,\dots,m}$ are positive definite for $i = 1, \dots, m$.

Thus around a neighborhood of λ^* ,

$$(5.12) \quad \lambda = \bar{\lambda} + \lambda''(0)s^2 + O(s^3).$$

So $\bar{\lambda}$ is a simple turning point.

Finally we show that $\bar{\lambda} = \lambda^*$, and there is only one solution for $\lambda = \lambda^*$. This will finish the proof of our claims.

Corresponding to $\lambda = \bar{\lambda}$ and $\lambda = \lambda^*$ we have

$$(5.13) \quad \Delta \vec{u}_{\min}^{\bar{\lambda}} + \bar{\lambda} \vec{f}(\vec{x}, \vec{u}_{\min}^{\bar{\lambda}}) = 0$$

$$(5.14) \quad \Delta \vec{u}_{\min}^{\lambda^*} + \lambda^* \vec{f}(\vec{x}, \vec{u}_{\min}^{\lambda^*}) = 0$$

Subtract equation (14) from (13), take inner product with $\vec{\psi}^*$ which is the first eigenfunction for $\lambda = \bar{\lambda}$, and integrate to obtain

$$\begin{aligned} \int_{\Omega} \langle -\bar{\lambda} D_{\bar{u}} \bar{f}(\bar{x}, \bar{u}_{\min}^{\bar{\lambda}})(\bar{u}_{\min}^{\bar{\lambda}} - \bar{u}_{\min}^{\lambda^*}) + \bar{\lambda} \bar{f}(\bar{x}, \bar{u}_{\min}^{\bar{\lambda}}) - \lambda^* \bar{f}(\bar{x}, \bar{u}_{\min}^{\lambda^*}), \bar{\psi}^* \rangle d\bar{x} \\ = 0 \end{aligned}$$

which can be written as

$$\begin{aligned} \bar{\lambda} \int_{\Omega} \langle -\bar{f}(\bar{x}, \bar{u}_{\min}^{\lambda^*}) + \bar{f}(\bar{x}, \bar{u}_{\min}^{\bar{\lambda}}) + D_{\bar{u}} \bar{f}(\bar{x}, \bar{u}_{\min}^{\bar{\lambda}})(\bar{u}_{\min}^{\lambda^*} - \bar{u}_{\min}^{\bar{\lambda}}), \bar{\psi}^* \rangle d\bar{x} \\ = \int_{\Omega} \langle (\lambda^* - \bar{\lambda}) \bar{f}(\bar{x}, \bar{u}_{\min}^{\lambda^*}), \bar{\psi}^* \rangle d\bar{x} \end{aligned}$$

By the convexness assumption on f , the left hand side is negative unless $u_* = u_0$ when it is zero. The right hand side is non-negative since $\lambda^* \geq \bar{\lambda}$, and can only be zero when $\lambda^* = \bar{\lambda}$. Hence the above equation holds only when $\lambda^* = \bar{\lambda}$, and $\bar{u}_{\min}^{\bar{\lambda}} = \bar{u}_{\min}^{\lambda^*}$.

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