BANACH LATTICES WITH PROPERTIES OF DUNFORD - PETTIS TYPE

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Abstract

This paper is devoted to relationships between weakly compact operators and M - weakly compact (L -weakly compact) operators acting from (resp. into) Banach lattices. We define the M - Dunford - Pettis Property for Banach lattices and the L -Dunford-Pettis Property in Banach spaces. The paper contains some characterizations of this two properties.

KEY WORDS AND PHRASES : Banach lattices, weak compactness, weakly compact operators, Schur spaces.

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1. Introduction.

For notations and terminology concerning Banach lattices, we refer the reader to [1], [7] and [8].

For a Banach lattice E we denote its norm dual by E' . The absolute weak topology on E is the locally convex-solid topology generated by the family of Riesz seminorms $\{p_{x'} : x' \in E'\}$, where $p_{x'}(x) = |x'|(|x|)$ for each $x \in E$.

Let E be a Banach lattice and x' in E' , $x' \geq 0$. We consider the following equivalence relation in E : $x \sim 0$ if and only if $x'(|x|) = 0$. Then the completion of $E/\sim$, with respect to the norm $\|x\| = x'(|x|)$ is an abstract L-space denoted by (E, x') . The canonical mapping $E \longrightarrow (E, x')$ will be denoted by $j_{x'}$.

A subset S of a Banach lattice E is called a solid set if for every $x \in E$ such that $|x| \leq |y|$ for some $y \in S$, then $x \in S$. If A is a subset of E , the solid hull of A is denoted by $S(A)$.

Let E be a Banach lattice. Following [5] (see also [2]), we say that a set $A \subset E$ is L-weakly compact if A is norm bounded and $\|y_n\| \longrightarrow 0$ as $n \longrightarrow \infty$ whenever $\{y_n\}$ is a disjoint sequence in the positive part of the solid hull of A . It is well known that every L-weakly compact subset of a Banach lattice is relatively weakly compact (see, [5] or [9]), and it is also known that a norm bounded subset of a Banach lattice with the Positive Schur Property (i. e. positive weakly converging sequences in E are norm converging) is relatively weakly compact if and only if is L-weakly compact (see, e. g. [6] or [8]).

Recall the following notions introduced in [5] and [3] respectively : let E be a Banach lattice and X be a Banach space, a linear bounded operator T from E into X is called M-weakly compact if T maps norm bounded disjoint sequences into norm convergent sequences ; and T is called o-weakly compact if T maps order bounded disjoint sequences into norm convergent sequences. A linear bounded operator T from X into E is called L-weakly compact if T maps norm bounded sets of X into L-weakly compact sets in E . It is known that T is M-weakly compact (resp. L-weakly compact) if and only if its adjoint T' is L-weakly compact (resp. M-weakly compact). Clearly each M-weakly compact operator is o-weakly compact ; moreover M-weakly compact operators are weakly compact, see [5].

Some basic properties of M and L-weakly compact operators are given in the next two theorems.

2. Basic properties of M and L-weakly compact operators.

Theorem 1 : Let E be a Banach Lattice, X be a Banach space and T be a linear bounded operator from E into X . The following statements are equivalent :

- (a) T is M-weakly compact.
- (b) T is o-weakly compact and (*) "For each norm bounded set $A \subset E$ and every $\epsilon > 0$ there exists some y in E^+ such that $\|T(|x| - y)^+\| < \epsilon$, for all $x \in A$ ".

Proof :

(a) $\implies$ (b) Let $A \subset E$ be a norm bounded set and $\epsilon > 0$. If (*) does not hold, then for each $y \in E$, there exists some x in A with $\|T(|x| - y)^+\| \geq \epsilon$. In particular there exists a sequence $\{x_n\}$ in A with :

$$(1.1) \quad \left\| T \left(|x_{n+1}| - 4^n \sum_{i=1}^n |x_i| \right)^+ \right\| \geq \epsilon,$$

Let $x \in E$ such that $x = \sum_{n=1}^{\infty} 2^{-n} |x_n|$. Clearly $2^{-n} |x_n| \leq x$ holds for all n . Therefore the sequence $y_n = \left(|x_{n+1}| + 4^n \sum_{i=1}^n |x_i| - 2^{-n} x \right)^+$ is norm bounded and pairwise disjoint and the sequence $u_n = \left(|x_{n+1}| - 4^n \sum_{i=1}^n |x_i| \right)^+$ is such that $0 \leq y_n \leq u_n \leq y_n + 2^{-n} x$, for all n . Since T is M-weakly compact, we have $\|T(u_n)\| \rightarrow 0$ as $n \rightarrow \infty$ which contradicts (1.1).

(b) $\implies$ (a) Let $\{x_n\}$ be a norm bounded disjoint sequence in E , then by (*), for each $\epsilon > 0$, there exists some y in E^+ such that $\|T(x_n - y)^+\| < \frac{\epsilon}{2}$ for all n . Since T is o-weakly compact and the sequence $\{x_n \wedge y\}$ is order bounded and pairwise disjoint, then $\|T(x_n \wedge y)\| \rightarrow 0$ as $n \rightarrow \infty$. The desired conclusion follows from the relation : $x_n = x_n \wedge y + (x_n - y)^+$.

Theorem 2. Let E be a Banach lattice with order continuous norm, X be a Banach space and T be linear bounded operator from X into E . The following statements are equivalent :

- (a) T is L-weakly compact.

- (b) For each norm bounded set $B \subset X$ and every $\epsilon > 0$ there exists some w in E^+ such that $\|(|Tx| - w)^+\| < \epsilon$, for all $x \in B$.

Proof :

(a) $\implies$ (b) It follows from [2], Corollary 2.10.

(b) $\implies$ (a) If $\{y_n\}_n$ is any disjoint sequence in the positive part of the solid hull of $T(B(X))$, let $x_n \in B(X)$ such that $0 \leq y_n \leq |Tx_n|$. Let $\epsilon > 0$ be given. By (b), there exists w in E^+ such that $\|(Tx_n - w)^+\| \leq \frac{\epsilon}{2}$. The sequence $\{y_n \wedge w\}_n$ is norm null since it is order bounded and pairwise disjoint and E has order continuous norm, now the conclusion follows from the relation $y_n = y_n \wedge w + (y_n - w)^+$.

From the above Theorem, we conclude that if E has order continuous norm, then every compact operator T from a Banach space X to E is L-weakly compact. However the operator $T : l_1 \rightarrow l_\infty$ defined by $T(\{a_n\}) = (\sum a_n, \sum a_n, \dots, \sum a_n)$ is compact and is not L-weakly compact. We don't know if the requirement that E has order continuous norm is necessary in order to conclude that compact operators are L-weakly compact too.

3. Main results.

Theorem 3 [6]. Let E be a Banach lattice, then the following statements are equivalent :

- (a) E' has order-continuous norm.
- (b) Every Dunford-Pettis operator T from E into an arbitrary Banach space is M-weakly compact.
- (c) Every compact operator T from E into an arbitrary Banach space is M-weakly compact.
- (d) Every positive operator T from E into an abstract L-space is M-weakly compact.

Proof :

(a) $\implies$ (b) If E' has order-continuous norm, then every norm bounded disjoint sequence is weakly convergent to zero (see, e. g. [2, Corollary 2.9].

(b) $\implies$ (c) It is trivial.

(c) $\implies$ (a) If E' does not have an order-continuous norm, then there exists an order bounded disjoint sequence $\{x'_n\}$ in E' such that $\inf \|x'_n\| > 0$. Then there exists $\epsilon > 0$ and a norm bounded pairwise disjoint sequence $\{x_n\}$ in E with $x'_n(x_n) \geq \epsilon$ for all n . The operator $T : E \longrightarrow l_1$ defined by $T(x) = \left(\sum x'_n(x), \sum x'_n(x), \dots, \sum x'_n(n) \right)$ is compact (it is of rank one) and is not M-weakly compact since $\|Tx_n\| \geq \epsilon$ for all n .

(a) $\implies$ (d) This follows from fact that every abstract L-space has the Positive Schur Property.

(d) $\implies$ (a) This is similar to the proof of (c) $\implies$ (a) by considering the operator $Tx = \{x'_n(x)\}$ from E to l_1 .

The next result gives a simple characterization of Banach lattices for which weakly compact operators are M-weakly compact.

Theorem 4 [6]. Let E be a Banach lattice. Then the following statements are equivalent :

(a) E' is the Positive Schur Property.

(b) For each Banach space F , every weakly compact operator $T : E \longrightarrow F$ is M-weakly compact.

Proof :

(a) $\implies$ (b) From Gantmacher's Theorem, T' is weakly compact. Since E' has the Positive Schur Property, the set $T'(\{y'_n \in F' : \|y'_n\| \leq 1\})$ is L-weakly compact, therefore T is M-weakly compact.

(b) $\implies$ (c) If $\{x'_n\}$ is a positive weakly convergent to zero sequence in E' , the operator $T : E \longrightarrow c_0$ defined by $T(x) = \{x'_n(x)\}$ is weakly compact, then by our hypothesis, T is M-weakly compact. Therefore if $\{x_n\}$ is a norm bounded disjoint sequence in E^+ , we have that $\lim_{n \rightarrow \infty} Tx_n = 0$, thus $\lim_{n \rightarrow \infty} x'_n(x_n) = 0$ for each norm bounded disjoint sequence $\{x_n\}$ on E , which implies by [2, Corollary 2.6], $\lim_{n \rightarrow \infty} x'_n = 0$.

In his monumental paper devoted to the study of weakly compact operators on $C(K)$ -spaces Grothendieck [4], introduced the "Dunford-Pettis Property". Following this idea, we can define the next property.

Let E be a Banach lattice, we shall say that E has the M-Dunford-Pettis Property if for each Banach space F , every M-weakly compact operator T from E into F is a Dunford-Pettis operator. Clearly, if a Banach lattice E has the Dunford-Pettis Property, E has the M-Dunford-Pettis Property. The converse does not hold as we can see later..

Theorem 5. Let E be a Banach lattice. Then the following statements are equivalent :

- (a) E has the M-Dunford-Pettis property.
- (b) If $\{x_n\}$ is a weakly null sequence in E and $\{x'_n : n \in N\}$ is a L-weakly compact set in E' , then $\lim_{n \rightarrow \infty} x'_n(x_n) = 0$.

Proof :

(a) $\implies$ (b) If $\{x_n\}$ is a weakly null sequence in E and $\{x'_n : n \in N\}$ is a L-weakly compact set in E' , the operator $Tx = \{x'_n(x)\}$ is M-weakly compact since $\lim_{n \rightarrow \infty} x'_n(u_n) = 0$ for each norm bounded disjoint sequence $\{u_n\}$ in E , therefore T is Dunford-Pettis operator and thus $\lim_{n \rightarrow \infty} x'_n(x_n) = 0$.

(b) $\implies$ (a) Let $T : E \longrightarrow F$ be a M-weakly compact operator. If T is not a Dunford-Pettis operator, then by passing to a subsequence if necessary we can assume that there exists $\epsilon > 0$ and a weakly null sequence $\{x_n\}$ in E such that $\|Tx_n\| \geq \epsilon$ for all n . Let us choose a sequence $\{y'_n\}$ in F' with $\|y'_n\| \leq 1$ such that $y'_n(Tx_n) \geq \epsilon$ for all n . Since T is M-weakly compact, the set $\{T(y'_n) : n \in N\}$ is L-weakly compact and $T'y'_n(x_n) \geq \epsilon$, for all n , which is a contradiction to (b).

Corollary 1. If E' has the Positive Schur Property. Then E has the Dunford-Pettis property if and only if E has the M-Dunford-Pettis property.

Corollary 2. Let E be a Banach lattice. If the lattice operations are weakly sequentially continuous, then E has the M-Dunford-Pettis Property. Moreover if E' has order-continuous norm, the converse holds.

Proof :

Let $\{x_n\}$ be a weakly null sequence in E and $\{x'_n : n \in N\}$ be a L-weakly compact subset of E' . Then, if $\epsilon > 0$, by [7, Corollary 2.6], there exists some y' in E' , $y' \geq 0$ such that $\|(|x'| - y')^+\| < \frac{\epsilon}{2}$, for all n . By our hypothesis the sequence $\{x'_n\}$ is weakly null, so $|x'_n(x_n)| \leq y'(|x_n|) + \frac{\epsilon}{2}$, for all n , thus $\lim_{n \rightarrow \infty} x'_n(x_n) = 0$ and the conclusion follows from Theorem 5.

Suppose that E' has order -continuous norm and E has the M-Dunford-Pettis property. Let $\{x_n\}$ be a weakly null sequence in E and $x' \in E'$. Consider the abstract L-space (E, x') and the canonical surjection $j_{x'}$. Since E' has order continuous norm and j is positive, by Theorem 3, $j_{x'}$ is M-weakly compact, so $j_{x'}$ is a Dunford-Pettis operator and thus $\lim_{n \rightarrow \infty} j_{x'}(x_n) = 0$, but $|j_{x'}(x_n)| = x'(|x_n|)$ which implies that $\{x_n\}$ is also a weakly null sequence.

Corollary 3. Let E be a Banach Lattice. Then :

- (a) Every discrete Banach lattice with order-continuous norm has the M-Dunford-Pettis property.
- (b) If E is a reflexive Banach lattice, then E has the M-Dunford-Pettis Property if and only if E is discrete.

Proof :

- (a) It is follows inmediately from the above proposition.
- (b) Let E be a reflexive Banach lattice with the M-Dunford-Pettis property. If E is not discrete then by [1, Theorem 21.12], there exists some x in E such that the order interval $[0, x]$ is not compact, therefore there exists an $\epsilon > 0$ and a sequence $\{x_n\}$ in $[0, x]$ such that $\|x_n - x_m\| \geq 4\epsilon$ whenever $n \neq m$. Take x'_n in E' , $\|x'\| \leq 1$ such that $x'_n(|x_n - x_m|) \geq 3\epsilon$ for all n . Since E' has order continuous norm, there exists some z in E' , with $(|x'| - z')^+(x) \leq \epsilon$ whenever x' in E' satisfies $\|x'\| \leq 1$ (see [2, Corollary 2.8]). Then, for all n :

$$\begin{aligned} x'_n(|x_n - x_{n+1}|) &= (x'_n \wedge z')(|x_n - x_{n+1}|) + (x' - z')^+(|x_n - x_{n+1}|) \\ &\leq z'(|x_n - x_{n+1}|) + (x' - z')^+(2x) \end{aligned}$$

Thus $z'(|x_n - x_{n+1}|) \geq \epsilon$, for all n . Therefore the operator $j_{z'} : E \rightarrow (E, z')$ is M-weakly compact and is not a Dunford-Pettis operator.

It's well known that L-weakly compact operators are weakly compact. The next theorem shows that the converse is also true for Banach spaces that X' is a Schur space.

Theorem 6 [6]. Let X be a Banach space the following statements are equivalent :

- (a) X' is a Schur space.
- (b) For each Banach lattice E with order continuous norm, every weakly compact operator $T : X \rightarrow E$ is L-weakly compact.

Proof :

Let E be a Banach lattice with order continuous norm and T be a weakly compact operator from X to E . We shall proof that T' is M-weakly compact.

Let $\{x'_n\}$ be a norm bounded disjoint sequence in E' . Since E has order continuous norm, $x'_n \rightarrow 0$ $\sigma(E', E)$ and T is continuous with respect the topologies $\sigma(E', E)$ in E' and $\sigma(X', X'')$ in X' , we have $T'x'_n \rightarrow 0$ $\sigma(X', X'')$. Thus by our hypothesis $\{T'x'_n\}$ is a norm null sequence in X' .

Conversely, let $\{x'_n\}$ be a sequence which is weakly convergent to zero in X' . The operator $T : X \rightarrow c_0$ defined by $Tx = \{x'_n(x)\}$ is weakly compact and, by the condition (b), T is L-weakly compact and so T' is M-weakly compact. Thus if $\{e_n\}$ denotes the usual basis of l_1 , which is a disjoint sequence, then $\lim_{n \rightarrow \infty} T'e_n = 0$ and the conclusion follows from the fact that $T'e_n = x'_n$ for all n .

Let X be a Banach space, we say that X has the L-Dunford-Pettis Property if for each Banach lattice E with order continuous norm, continuous norm, every L-weakly compact operator T from X into E is a Dunford-Pettis operator.

Clearly, every Banach space with the Dunford-Pettis Property always has the L-Dunford-Pettis Property ; the converse does not hold since for each $2 < p < \infty$, l_p has the L-Dunford-Pettis Property.

Theorem 7. A Banach space X has the L-Dunford-Pettis Property if and only if each weakly compact operator T from X into an Abstract L-space is a Dunford-Pettis operator.

Proof :

Let E be a Banach lattice with order continuous norm and $T : E \longrightarrow X$ be a L-weakly compact operator. Take $\{x_n\}$ a weakly convergent to zero sequence in X . Since T' is M-weakly compact, $\lim y'_n(Tx_n) = 0$ for each norm bounded disjoint sequence $\{y'_n\}$ in E' . By [2, Corollary 2.6], it is enough to show that $\lim y'(|Tx_n|) = 0$ for each $y' \in E'$, $y' \geq 0$. Take $y' \geq 0$ in E' , consider the abstract L-space (E, y') and its canonical mapping $j_{x'}$. The operator $j_{x'} \circ T$ is weakly compact, thus, by our hypothesis $j_{x'} \circ T$ is a Dunford-Pettis operator, so $\lim_{n \rightarrow \infty} j_{x'} \circ T(x_n) = 0$. Therefore T is a Dunford-Pettis operator since $\|(j_{x'} \circ T)x\| = y'(|Tx|)$.

The converse holds from the very well known result : norm bounded subsets of an abstract L-space is relatively weakly compact if and only if is L-weakly compact.

If E is a Banach lattice with the L-Dunford-Pettis Property and such that E' has order continuous norm, then the lattice operations are weakly sequentially continuous, thus by Corollary 2 of Theorem 5, we conclude that E also has the M-Dunford-Pettis Property.

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