ON COMMUTATIVITY OF RIGHT S-UNITAL RINGS WITH SOME POLYNOMIAL CONSTRAINTS

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Abstract

We study the commutativity of certain class of rings, namely rings with unity 1 and right s-unital under each of the following properties $[yx^m - x^n f(y) x^p, x] = 0$, $[yx^m + x^n f(y) x^p, x] = 0$, where $f(t)$ is a polynomial in $t^2\mathbb{Z}[t]$ varying with pair of ring elements x, y and m, n, p are fixed non-negative integers. Moreover, the results have been extended to the case when m and n depend on the choice of x and y and the ring satisfies the Chacron's Theorem.

1. Introduction.

There is a multitude of conditions each of which implies the commutativity of certain rings. The equivalence of few such conditions to that of commutativity of rings was established by Komatsu and Tominaga [10] and Ashraf et al. [6]. The list of these equivalent conditions was further enlarged by authors jointly with Samman in [4]. The major purpose of this paper is to extend the work of Abujabal [5] and Ashraf et al. [7]. In fact, several commutativity theorems can be obtained as corollaries to our results, for instance [1, Theorem], [2, Theorem], [3, Theorem], [4, Theorem], [10, Theorem], [11, Theorem] and [12, Theorem].

Throughout this paper, R will represent an associative ring (not necessarily with 1). Let $\mathbf{Z}[t]$ be the totality of polynomials in t with coefficient in $\mathbf{Z}$ the ring of integers. As usual for any x, y in R the symbol $[x, y]$ will denote the commutator $xy - yx$. In Section 2, we investigate the commutativity of rings with unity 1 under the ring properties :

(B_1) For every x, y in R , there exists $f(x) \in X^2\mathbf{Z}[X]$, such that $[yx^m - x^n f(y) x^p, x] = 0$, where m, n and p are fixed non-negative integers.

(S_1) For every x, y in R , there exists $f(X) \in X^2\mathbf{Z}[X]$, such that $[yx^m + x^n f(y) x^p, x] = 0$, where m, n and p are fixed non-negative integers.

(B_2) For every x, y in R , there exist non-negative integers m, n, p and $f(X) \in X^2\mathbf{Z}[X]$, such that $[yx^m + x^n f(y) x^p, x] = 0$.

(S_2) For every x, y in R , there exist non-negative integers m, n, p and $f(X) \in X^2\mathbf{Z}[X]$, such that $[yx^m + x^n f(y) x^p, x] = 0$.

(CA) For every x, y in R , there exist $f(X), g(X)$ in $X^2\mathbf{Z}[X]$, such that $[x - f(x), y - g(y)] = 0$.

The results of this paper are further extended to the right s-unital rings in the subsequent section 3.

2. Commutativity of rings with unity 1.

The following are the main results of this paper.

Theorem 1. Let R be a ring with unity 1. If R is satisfying any one of the conditions (B_1) and (S_1), then R is commutative (and conversely).

Theorem 2. Let R be a ring with unity 1 satisfying (CA). If R is satisfying any one of the conditions (B_2) and (S_2) , then R is commutative (and conversely).

In order to develop the proof of the above theorems first, we consider the following type of rings.

$$(1) \begin{pmatrix} GF(p) & GF(p) \\ 0 & GF(p) \end{pmatrix}, p \text{ a prime.}$$

$$(1) r \begin{pmatrix} 0 & GF(p) \\ 0 & GF(p) \end{pmatrix}, p \text{ a prime.}$$

$$(1) l \begin{pmatrix} GF(p) & GF(p) \\ 0 & 0 \end{pmatrix}, p \text{ a prime.}$$

(2) $M_\sigma(F) = \left\{ \begin{pmatrix} a & b \\ 0 & \sigma(a) \end{pmatrix} \mid a, b \text{ in } F \right\}$, where F is a finite field with a non-trivial automorphism σ .

(3) A non-commutative division ring.

(4) $S = \langle 1 \rangle + T$, T as non-commutative radical subring of S .

(5) $S = \langle 1 \rangle + T$, T is a non-commutative subring of S such that

$$T[T, T] = [T, T]T = 0.$$

In [14], Streb gave a nice classification for non-commutative rings, which yields a powerful tool in obtaining a number of commutativity theorems (cf. [1], [2], [3], [10] and [11]). It follows from the proof of [13, Corollary (1)], that if R is a non-commutative ring with unity 1, then there exists a factor subring of R which is of type (1), (2), (3), (4) and (5). This observation gives the following result which plays the key role in our subsequent study.

Lemma 1. Let P be a ring property which is inherited by factor subrings. If no rings of type (1), (2), (3), (4) or (5) satisfy P , then every ring with unity 1 satisfying P is commutative.

For the sake of convenience, we state the following well-known result due to Herstein [8].

Lemma 2. Let R be a ring in which for every x, y in R there exists $f(x) \in X^2\mathbb{Z}[X]$, such that $[x - f(x), y] = 0$. Then R is commutative.

Proof of Theorem 1. Let R be a ring satisfying (B_1) . First, we shall show that no rings of type (1), (2), (3), (4) or (5) satisfies (B_1) . Let us begin with the ring of type (1). Then in $M_2(GF(p))$, p a prime, we observe that $[e_{12}e_{12+22}^m - e_{12+22}^n f(e_{12})e_{12+22}^p, e_{12+22}] \neq 0$ for all integers $m \geq 0, n \geq 0, p \geq 0$ and $f(X) \in X^2\mathbf{Z}[X]$. Hence the ring of type (1) does not satisfy (B_1) .

Suppose that B is the ring of type (2). Taking

$$x = \begin{pmatrix} a & 0 \\ 0 & \sigma(a) \end{pmatrix} (\sigma(a) \neq a), y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

in $(B = M_\sigma(F))$.

One observes that

$$[yx^m - x^m f(y)x^p, x] = -[x, y]x^m = (\sigma(a) - a)(\sigma(a))^m y \neq 0$$

for all integers $m \geq 0, n \geq 0, p \geq 0$ and $f(X) \in X^2\mathbf{Z}[X]$. Hence no rings of type (2) satisfy condition (B_1) .

Now, consider the ring R of type (3). If x is a unit of R , then for every y in R choose $f(x) \in X^2\mathbf{Z}[X]$, such that $[yx^{-m} - x^{-n}f(y)x^{-p}, x^{-1}] = 0$. This implies that

$$[yx^{-m} - x^{-n}f(y)x^{-p}, x] = 0.$$

Thus it follows that

$$(i) \quad x^n[x, y]x^p = [x, f(y)]x.$$

Further, we consider $g(x) \in x^2\mathbf{Z}[x]$ such that $[f(y)x^m - x^n g(f(y))x^p, x] = 0$. This becomes

$$(ii) \quad x^n[x, g(f(y))]x^p = [x, f(y)]x^m.$$

Combining (i) and (ii), we get $x^n[x, g(f(y))]x^p = x^n[x, y]x^p$.

But since x is a unit, $[x, y - h(y)] = 0$ for $g(f(y)) = h(y) \in Y^2\mathbf{Z}[Y]$. Hence by Lemma 2, R is commutative which gives a contradiction. Thus a ring R of type (3) does not satisfy (B_1) .

Suppose that R has a factor subring of type (4). Let $a, b \in T$. Since $1 - a$ is unit, using above paragraph there exists $f(X) \in X^2\mathbf{Z}[X]$ such that $[a, b - f(b)] = -[1 - a, b - f(b)] = 0$. Thus T is commutative by Lemma 2. This is a contradiction, so R does not satisfy a factor subring of type (4) which satisfy (B_1) .

Finally, we consider $S = \langle 1 \rangle + T$, where T is a non-commutative subring of S such that $T[T, T] = [T, T]T = 0$. Let R has a factor subring S . Choose $a, b \in T$ such that

$$[a, b] = (a + 1)^n [a, b] (a + 1)^p = [a, f(b)] (a + 1)^m = 0.$$

This is a contradiction.

Hence, we have seen that no rings of Type (1), (2), (3), (4) or (5) satisfy (B_1) . By Lemma 1, R is commutative.

Remark 1. If R satisfies property (S_1) , then the proof will follow on the same lines as above.

Proof of Theorem 2. Suppose that R satisfies property (B_2) . First, we assume that the rings of type (1). Then $[e_{12}e_{22}^m + e_{22}^n f(e_{12})e_{22}^p, e_{22}] \neq 0$ for any integer $m \geq 0, n \geq 0, p \geq 0$, and $f(X) \in X^2\mathbf{Z}[X]$. This shows that no rings of type (1) satisfying (B_2) .

Using similar arguments as above one can prove that no rings of type (2) satisfy condition (B_2) . From this fact and Corollary 1 of [10], we obtain the required result.

Remark 2. The proof for the case R satisfies (S_2) , will follow on the same lines as above.

Since there are non-commutative rings with R^2 being central, neither of these conditions guarantee the commutativity of arbitrary rings. However, we extend the study to the class of rings, which are called right *s*-unital. The results obtained here generalize [1], [2], [3], [6], [9], [11].

3. Extensions to right *s*-unital rings.

An associative ring R is called right (resp. left) *s*-unital if $x \in xR$ (resp. $x \in Rx$) for each $x \in R$. Further R is called *s*-unital if it is both right as well as left *s*-unital, i. e. $x \in Rx \cap xR$ for each $x \in R$.

If R is an s-unital (resp. right or left s-unital) ring, then for any finite subset A of R , there exists an element $e \in R$ such that $xe = ex = x$ (resp. $xe = x$ or $ex = x$) for all $x \in A$. Such an element e is called the pseudo (left or pseudo right) identity of A in R . Before we go ahead with our task, we state the following lemma due to Komatsu, Nishinaka and Tominaga [11] in order to make our paper self-contained as far as possible.

Lemma 3. Let R be a right s-unital ring and not left s-unital. Then R has a factor sybring of type $(1)_r$.

If R is a right s-unital ring satisfying (B_1) , then a careful scrutiny of the first paragraph of the proof of Theorem 1 shows that no rings of type $(1)_r$ satisfy (B_1) . By Lemma 3, R is s-unital and in view of Proposition 1 of [9], R has unity 1. Thus R is commutative by Theorem 1, which gives the following.

Theorem 3. Let R be a right s-unital ring satisfying any one of the conditions (B_1) and (S_1) . Then R is commutative (and conversely).

Corollary 1. Let $m \geq 0, n \geq 0$ be fixed integers. Suppose that if R is a right s-unital ring in which for every x, y in R , there exists an integer $r = r(x, y) > 1$ such that either $[yx^m - x^n y^r x^p, x] = 0$ or $[yx^m + x^n y^r x^p, x] = 0$. Then R is commutative (and conversely).

From [10, Corollary 1], similarly we can prove the following result as follows :

Theorem 4. Let R be a right s-unital ring satisfying (CA) . Suppose further that R satisfies any one of the conditions (B_2) and (S_2) . Then R is commutative (and conversely).

Remark 3. One might conjecture a possible generalization of Theorem 1 and 2 when R is left s-unital ring. The following example shows that there is a non-commutative left s-unital ring satisfying (B_1) or (S_1) .

Example 1.

Let $R = \left\{ \alpha = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \beta = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \gamma = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \delta = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}$ be a subring of 2×2 matrices over $GF(2)$. Then for any fixed positive integers m, n, p and $f(x)$ in $X^2\mathbf{Z}[X]$, R satisfies (B_1) or (S_1) . However, R is not commutative, although R is left s -unital.

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