PARAMETER-BASED ALGORITHMS FOR APPROXIMATING LOCAL SOLUTION OF NONLINEAR COMPLEX EQUATIONS

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Abstract

We study the Ostrowski-Kantorovich convergence for a family of Halley-Werner type iteration methods in the complex plane. We provide an upper error bound for all parameter $\alpha \in [1, 2)$. We show that the error bound is a decreasing function of α . We prove also that the Halley method has the largest error bound.

Key words: *Nonlinear equations, Halley-Werner type methods, Ostrowski-Kantorovich analysis, Upper error bound.*

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1. Introduction

The purpose of this paper is the study of Halley-Werner type methods with a parameter $\alpha \in [1, 2)$ for solving nonlinear complex equations. This family of methods was proposed by W. Werner in 1980 [5] in the following form:

$$0 = F(x_n) + F'(x_n)(y_n - x_n),$$

$$0 = F(x_n) + [F'(x_n) + \frac{\alpha}{2}F''(x_n)(x_{n+1} - x_n)]$$

$$+ \frac{1-\alpha}{2}F''(x_n)(y_n - x_n)^2$$

for some fixed $\alpha \in \mathbb{R}$, $n = 1, 2, \dots$. For $\alpha = 1$ the above method reduces to the famous Halley method or the method of tangent hyperbolas which has a long history [1]. In this paper Ostrowski-Kantorovich type assumptions (similar to the ones already used for Newton's method in [3,4]) are applied to this family of iteration methods. Under this type of assumptions, we establish the existence-uniqueness theorem and give an error bound for $\alpha \in [1, 2)$. Finally we show that the error bound of the Halley iteration method is not the best one in this family.

2. Basic Iteration Characteristics

First we define an equivalent form for this family:

$$y_n = x_n - \frac{F(x_n)}{F'(x_n)}, \quad (2.1)$$

$$x_{n+1} = -\frac{1}{2} \frac{\frac{F''(x_n)}{F'(x_n)}(y_n - x_n)^2}{1 + \frac{\alpha}{2} \frac{F''(x_n)}{F'(x_n)}(y_n - x_n)}. \quad (2.2)$$

Then we can obtain the following Ostrowski-Kantorovich representation for $F(x_{n+1})$.

Lemma 2.1. Let F be the function in an open convex domain D in C to C . Assume that F has second order continuous derivative on D . The the following approximation is true for all $n \geq 0$:

$$\begin{aligned}
 F(x_{n+1}) &= \int_0^1 F''(y_n + t(x_{n+1} - y_n))(1-t)dt(x_{n+1} - y_n)^2 \\
 &- \frac{\frac{F''(x_n)(y_n - x_n)^3}{2F'(x_n)}[(1-\alpha) \int_0^1 F''(x_n + t(y_n - x_n))(1-t)dt]}{1 + \frac{\alpha}{2} \frac{F''(x_n)(y_n - x_n)}{F'(x_n)}} \\
 &- \frac{\frac{F''(x_n)}{2F'(x_n)}(y_n - x_n)^3 \int_0^1 F''(x_n + t(y_n - x_n))t dt}{1 + \frac{\alpha}{2} \frac{F''(x_n)}{F'(x_n)}(y_n - x_n)} \\
 &+ \frac{\frac{1}{2}(y_n - x_n)^2 \int_0^1 [2F''(x_n + t(y_n - x_n))(1-t) - F''(x_n)]dt}{1 + \frac{\alpha}{2} \frac{F''(x_n)}{F'(x_n)}(y_n - x_n)}. \quad (2.3)
 \end{aligned}$$

Proof: To see this fact, we observe by usual routine that

$$\begin{aligned}
 F(x_{n+1}) &= F(x_{n+1}) - F(y_n) - F'(y_n)(x_{n+1} - y_n) \\
 &+ F(y_n) + F'(y_n)(x_{n+1} - y_n)
 \end{aligned}$$

and

$$\begin{aligned}
 F(y_n) + F'(y_n)(x_{n+1} - y_n) &= F(y_n) - F'(y_n) \frac{1}{2} \frac{\frac{F''(x_n)}{F'(x_n)}(y_n - x_n)^2}{1 + \frac{\alpha}{2} \frac{F''(x_n)}{F'(x_n)}(y_n - x_n)} \\
 &= \frac{F(y_n) + \frac{\alpha}{2} \frac{F''(x_n)}{F'(x_n)}(y_n - x_n)F(y_n) - \frac{1}{2} \frac{F''(x_n)}{F'(x_n)}(y_n - x_n)^2 F'(y_n)}{1 + \frac{\alpha}{2} \frac{F''(x_n)}{F'(x_n)}(y_n - x_n)}
 \end{aligned}$$

Let us denote the above numerator by L . We obtain in turn

$$F(y_n) = \int_0^1 F''(x_n + t(y_n - x_n))(1-t)dt(y_n - x_n)^2,$$

we now obtain

$$\begin{aligned}
 L &= -\frac{1}{2F'(x_n)}[F''(x_n)(y_n - x_n)^2 \int_0^1 F''(x_n + t(y_n - x_n))dt(y_n - x_n) \\
 &- \alpha F''(x_n)(y_n - x_n) \int_0^1 F''(x_n + t(y_n - x_n))(1-t)dt(y_n - x_n)^2 \\
 &- F'(x_n) \int_0^1 [2F''(x_n + t(y_n - x_n))(1-t) - F''(x_n)]dt(y_n - x_n)^2]
 \end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{2F'(x_n)}[(1-\alpha)F''(x_n) \int_0^1 F''(x_n + t(y_n - x_n))(1-t)dt(y_n - x_n)^3 \\
&\quad + F''(x_n) \int_0^1 F''(x_n + t(y_n - x_n))tdt(y_n - x_n)^3 \\
&\quad + \frac{1}{2} \int_0^1 [2F''(x_n + t(y_n - x_n))(1-t) - F''(x_n)]dt(y_n - x_n)^2].
\end{aligned}$$

Substituting it in the above, we obtain (2.3).

3. The Ostrowski-Kantorovich Convergence

Theorem 3.1. *Let $F : D \subset C \rightarrow C$, where C is the complex plan and D is an open convex domain. Assume that F has 2nd order continuous derivatives on D , and for given an initial value $x_0 \in D$ the following Ostrowski-Kantorovich conditions are satisfied:*

$$|F''(x)| \leq M, \quad |F''(x) - F''(y)| \leq N |x - y|, \quad \text{for all } x, y \in D, \quad (3.1)$$

$$|F'(x_0)^{-1}| \leq \beta, \quad |F'(x_0)^{-1}F(x_0)| \leq \eta. \quad (3.2)$$

Set $h = K\beta\eta$ and assume that

$$h \leq \frac{1}{2}, \quad M[1 + \frac{2N}{3M^2\beta(2-\alpha)}]^{\frac{1}{2}} \leq K, \quad 1 \leq \alpha < 2, \quad (3.3)$$

$$\overline{S(x_0, r_1)} \subset D, \quad (3.4)$$

where r_1 is the roots of the equation $g(t) = 0$, where

$$g(t) = \frac{K}{2}t^2 - \frac{1}{\beta}t + \frac{\eta}{\beta}, \quad (3.5)$$

and

$$r_1 = \frac{1 - \sqrt{1 - 2h}}{h}\eta, \quad r_2 = \frac{1 + \sqrt{1 - 2h}}{h}\eta. \quad (3.6)$$

The scalar sequences are defined as

$$s_n = t_n - \frac{g(t_n)}{g'(t_n)}, \quad t_0 = 0, \quad (3.7)$$

$$t_{n+1} = s_n - \frac{\frac{g''(t_n)}{g'(t_n)}(s_n - t_n)^2}{2 + \frac{\alpha}{2} \frac{g''(t_n)}{g'(t_n)}(s_n - t_n)}, \quad \text{for } n \geq 0. \quad (3.8)$$

Then the sequences are well-defined and convergent with $\{x_n\}, \{y_n\} \in \overline{S(x_0, r_1)}$. The limit is the unique solution of the equation $F(x) = 0$ in $S(x_0, r_3)$, $r_3 \in [r_1, r_2]$. We also have the following error estimates for all $n \geq 0$:

$$|x_n - x^*| \leq r_n - t_n, \quad (3.9)$$

$$|y_n - x^*| \leq r_1 - s_n, \quad (3.10)$$

$$r_1 - t_n \leq \frac{(1 - \theta^2)\eta}{1 - \theta^3} \theta^{3^n - 1}, \text{ for all } \alpha \in [1, 2], \quad (3.11)$$

$$\text{where } \theta = \frac{1 - \sqrt{1 - 2h}}{1 + \sqrt{1 - 2h}}.$$

Proof: By using induction on n , it is enough to show that the following statements are true for all $n \geq 0$:

$$\begin{aligned} (a_n) : & \quad x_n \in \overline{S(x_0, r_1)}, \\ (b_n) : & \quad |F'(x_n)^{-1}| \leq -\frac{1}{g'(t_n)}, \\ (c_n) : & \quad |y_n - x_n| \leq s_n - t_n, \\ (d_n) : & \quad y_n \in \overline{S(x_0, r_1)}, \\ (e_n) : & \quad |x_{n+1} - y_n| \leq t_{n+1} - s_n. \end{aligned}$$

All the above statements are true for $n = 0$ by initial conditions. Then we assume that they are true for a fixed n and all smaller integer values. From (a_n)

$$\begin{aligned} |x_{n+1} - x_0| & \leq |x_{n+1} - y_n| + |y_n - x_n| + |x_n - x_0| \\ & \leq (t_{n+1} - s_n) + (s_n - t_n) + (t_n - t_0) \\ & = t_{n+1} - t_0 \\ & = t_{n+1}. \end{aligned}$$

Hence (a_{n+1}) is true. Since

$$\begin{aligned} |F'(x_{n+1}) - F'(x_0)| & \leq M |x_{n+1} - x_0| \\ & \leq K(t_{n+1} - t_0) \\ & = Kt_{n+1}. \end{aligned}$$

then

$$|F'(x_{n+1})^{-1}| \leq \frac{1}{\frac{1}{\beta} - Kt_{n+1}} = -\frac{1}{g'(t_{n+1})}.$$

Thus (b_n) is true. Using (2.3) and taking norms we obtain

$$\begin{aligned} |F(x_{n+1})| & \leq \frac{M}{2} |x_{n+1} - y_n|^2 + \frac{\frac{|F''(x_n)|}{2|F'(x_n)|} [(1 - \alpha)\frac{M}{2} + \frac{M}{2}] |y_n - x_n|^3}{1 - \frac{\alpha}{2} \frac{|F''(x_n)|}{|F'(x_n)|} |y_n - x_n|} \\ & \quad + \frac{\frac{N}{6} |y_n - x_n|^3}{1 - \frac{\alpha}{2} \frac{|F''(x_n)|}{|F'(x_n)|} |y_n - x_n|} \\ & \leq \frac{K}{2} (t_{n+1} - s_n)^2 + \frac{\frac{K}{-g'(t_n)} (s_n - t_n)^3 (2 - \alpha) \frac{K}{4}}{1 + \frac{\alpha}{2} \frac{K}{g'(t_n)} (s_n - t_n)} \\ & = g(t_{n+1}). \end{aligned}$$

It follows immediately that

$$\begin{aligned} |y_{n+1} - x_{n+1}| &\leq \frac{|F(x_{n+1})|}{|F'(x_{n+1})|} \\ &\leq -\frac{g(t_{n+1})}{g'(t_{n+1})} \\ &= s_{n+1} - t_{n+1}. \end{aligned}$$

and

$$\begin{aligned} |y_{n+1} - x_0| &\leq |y_{n+1} - x_{n+1}| + |x_{n+1} - y_n| + |y_n - x_0| \\ &\leq (s_{n+1} - t_{n+1}) + (t_{n+1} - s_n) + (s_n - t_0) \\ &\leq s_{n+1} - t_0 \\ &= s_{n+1}. \end{aligned}$$

Thus (c_{n+1}) and (d_{n+1}) are also true. Moreover from (2.2),

$$x_{n+2} = y_{n+1} - \frac{1}{2} \frac{\frac{F''(x_{n+1})}{F'(x_{n+1})}(y_{n+1} - x_{n+1})^2}{1 + \frac{\alpha}{2} \frac{F''(x_{n+1})}{F'(x_{n+1})}(y_{n+1} - x_{n+1})},$$

and by taking norms we shall obtain

$$\begin{aligned} |x_{n+2} - y_{n+1}| &\leq \frac{\frac{1}{2} \frac{|F''(x_{n+1})|}{|F'(x_{n+1})|} |y_{n+1} - x_{n+1}|^2}{1 - \frac{\alpha}{2} \frac{|F''(x_{n+1})|}{|F'(x_{n+1})|} |y_{n+1} - x_{n+1}|} \\ &\leq -\frac{1}{2} \frac{\frac{K}{g'(t_{n+1})}(s_{n+1} - t_{n+1})^2}{1 + \frac{\alpha}{2} \frac{K}{g'(t_{n+1})}(s_{n+1} - t_{n+1})} \\ &= -\frac{1}{2} \frac{\frac{g''(t_{n+1})}{g'(t_{n+1})}(s_{n+1} - t_{n+1})^2}{1 + \frac{\alpha}{2} \frac{g''(t_{n+1})}{g'(t_{n+1})}(s_{n+1} - t_{n+1})} \\ &= t_{n+2} - s_{n+1}. \end{aligned}$$

Theorefore (e_{n+1}) is true. Now we are ready to find the error bound for $r_1 - t_n$. First we set $a_n = r_n - t_n$ and $b_n = r_2 - t_n$. Moreover, we notice the facts that

$$\begin{aligned} g(t_n) &= \frac{K}{2} a_n b_n, \\ g'(t_n) &= -\frac{K}{2} (a_n + b_n), \\ b_n &= a_n + \frac{(1 - \theta^2)\eta}{\theta}. \end{aligned}$$

Now by (3.8), we have

$$a_n = \frac{a_{n-1}^4 + (2 - \alpha)a_{n-1}^3 b_{n-1}}{(a_{n-1} + b_{n-1})^3 - \alpha a_{n-1} b_{n-1} (a_{n-1} + b_{n-1})}.$$

We obtain similarly that

$$b_n = \frac{b_{n-1}^4 + (2 - \alpha)b_{n-1}^3 a_{n-1}}{(a_{n-1} + b_{n-1})^3 - \alpha a_{n-1} b_{n-1} (a_{n-1} + b_{n-1})}.$$

Taking the ratio of a_n and b_n , we have

$$\begin{aligned} \frac{a_n}{b_n} &= \left[\frac{a_{n-1}}{b_{n-1}} \right]^3 \frac{a_{n-1} + (2 - \alpha)b_{n-1}}{b_{n-1} + (2 - \alpha)a_{n-1}} \\ &= \left[\frac{a_{n-1}}{b_{n-1}} \right]^3 \frac{\frac{a_{n-1}}{b_{n-1}} + (2 - \alpha)}{1 + (2 - \alpha)\frac{a_{n-1}}{b_{n-1}}} \end{aligned}$$

Use the facts that

$$a_{n-1} \leq b_{n-1}, 1 \leq \alpha < 2,$$

we can easily see that

$$\frac{a_n}{b_n} \leq \left[\frac{a_{n-1}}{b_{n-1}} \right]^3 \leq \dots \leq \left[\frac{a_0}{b_0} \right]^{3^n} = \theta^{3^n}.$$

By solving the above inequality for a_n , we obtain the error bound

$$r_1 - t_n = a_n \leq \frac{(1 - \theta^2)\eta}{1 - \theta^{3^n}} \theta^{3^{n-1}}.$$

To show uniqueness. let us assume that there exists a second solution y^* of equation $F(x) = 0$ in $\overline{S(x_0, r_3)}$ for $r_3 \in [r_1, r_2)$. Using the approximation

$$F'(x^* + t(y^* - x^*)) - F'(x^*) = \int_{x_0}^{x^* + t(y^* - x^*)} F''(z) dz,$$

and the estimate

$$\begin{aligned} &|F'(x_0)^{-1}| \left| \int |F'(x^* + t(y^* - x^*)) - F'(x_0)| dt \right. \\ &\leq \beta M \int_0^1 |x^* + t(y^* - x^*) - x_0| dt \end{aligned}$$

$$\begin{aligned}
&\leq \beta M \int_0^1 [(1-t) |x^* - x_0| + t |y^* - x_0|] dt \\
&\leq \frac{\beta M}{2} (r_1 + r_3) \\
&\leq \frac{\beta M}{2} (r_1 + r_2) \\
&< 1.
\end{aligned}$$

We deduce that the inverse of

$$\int_0^1 F'(x^* + t(y^* - x^*)) dt$$

exists. From the approximation

$$F(y^*) - F(x^*) = \int_0^1 F'(x^* + t(y^* - x^*)) dt (y^* - x^*)$$

we conclude that $x^* = y^*$. That completes the proof of theorem 3.1.

Some remarks: (a) Under the Ostrowski-Kantorovich assumptions, Gragg and Tapia provided the best error bound for Newton method which can not be improved [2]. That is showed that

$$r_1 - t_n = \frac{(1 - \theta^2)\eta}{1 - \theta^{2^n}} \theta^{2^n - 1}, \text{ for all } n \geq 0.$$

(b) Similarly, for the Halley method we have the best error bound

$$r_1 - t_n = \frac{(1 - \theta^2)\eta}{1 - \theta^{3^n}} \theta^{3^n - 1}, \text{ for all } n \geq 0.$$

which can also not be improved. Note that we have showed the existence of infinitely many methods for $\alpha \in [1, 2)$ where the upper bounds are less than that of the Halley method.

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