INVOLUTIVE CO-DISTRIBUTIONS PRESERVED BY TRANSITIVE FAMILIES OF VECTOR FIELDS*

VÍCTOR AYALA BRAVO

Universidad Católica del Norte, Antofagasta-Chile.

Abstract

This paper deals with integrability conditions of involutive co-distributions defined on the co-tangent bundle of a differentiable manifold M . Via Frobenius's integrability theorem, the analysis is aimed at the search of conditions so that this type of co-distributions be preserved by transitive families of vector fields in M . We rely on the work of Lobry, Sussmann, Matsuda and Stefan. The type of situation studied comes up naturally in weak-observability problems and weakly-minimal realizations of arbitrary control systems.

Keywords

Co-distributions, transitivity, co-distributions preserved by families of vector fields, integrability.

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1. INTRODUCTION

Let M a differentiable manifold and θ a co-distribution on the co-tangent bundle T^*M , that is, for each $x \in M$, $\theta(x)$ is a vector subspace of the dual of the tangent space to M in x . Frobenius's theorem ensures the integrability of θ if this co-distribution is involutive and regular, [10]. In this paper, we are interested in finding conditions that warrant regularity of the involutive co-distribution. As we shall see, the type of situation to be analysed appears in the study of weak-observability and weakly-minimal realization of control systems.

A control system Σ is determined by the following data:

$$\Sigma = (M, D, R^s, h)$$

where M is a differentiable manifold, $D \subset X(M)$ is a family of vector fields in M , (strategies) and $h : M \rightarrow R^s$ is a differentiable function.

Let's denote by X_t the flow associated to the vector field X . Then D induces a pseudo-group:

$$G_z = \left\{ X_{t_1}^1 \circ X_{t_2}^2 \circ \dots \circ X_{t_k}^k \mid X^j \in D, t_j \in \mathbb{R} \right\}$$

and a semi-pseudo-group

$$S_\Sigma = \left\{ X_{t_1}^1 \circ X_{t_2}^2 \circ \dots \circ X_{t_k}^k \mid X^j \in D, t_j \geq 0 \right\}$$

of local diffeomorphisms on M .

G_Σ acts over M on such a way that the orbits

$$G_\Sigma(x) = \{\rho(x) \mid \rho \in G_\Sigma\}, x \in M$$

form a foliation with singularities, [7]. Σ is said to be transitive if there is a single orbit and in this case, for each $x \in M$, $G_\Sigma(x) = M$.

Each orbit $G_\Sigma(x)$ has a structure of a differentiable manifold, however the topology can be strictly finer than the induced topology. It is possible to restrict Σ to each orbit $G_\Sigma(x)$, thus obtaining a natural environment of the system for the initial condition x , [9]. Particularly, it is always possible to consider Σ as being transitive (on its orbits).

The accessibility set of Σ from the point x is given by its positive orbit under Σ , that is:

$$S_\Sigma(x) = \{\rho(x) \mid \rho \in S_\Sigma\}.$$

If $S_\Sigma(x) = M$, we say that the system Σ is controllable from x . Two states $x, y \in M$ are said to be undistinguishable (by Σ) if

$$h \circ \rho(x) = h \circ \rho(y), \quad \forall \rho \in S_\Sigma.$$

Σ is said to be observable if there are no two undistinguishable states in M and weakly-observable in $x \in M$ if there is a neighborhood U of x such that for each

$y \in U$, x and y are not undistinguishable, i.e., the family of local diffeomorphisms S_Σ restricted to U , separates x from any point $y \in U$.

A minimal realization for Σ is a transitive and observable system

$$\Sigma_1 = (M_1, D_1, R^s, h_1)$$

and a submersion $\pi : M \longrightarrow M_1$ such that : each vector field $X \in D$ is projectable on M_1 via the differential $d\pi = \pi_*$, $\pi_*(D) = D_1$, and $h = h_1 \circ \pi$.

Two states $x, y \in M$ are said to be quasi-undistinguishable (by Σ) if :

$$h \circ \rho(x) = h \circ \rho(y), \quad \forall \rho \in G_\Sigma.$$

The concepts of quasi-observability and weak-observability for minimal realization are defined in an analogous way.

The study of the observability of Σ naturally leads to the analysis of the curves $h_i \circ \rho(x)$, $1 \leq i \leq s$, $\rho \in S_\Sigma$.

Hermann and Krener, with the purpose of studying sufficient conditions of weak-observability, introduced in [4] the co-distribution defined by :

$$\Lambda = \text{Span} \{ L_{x^1} \circ L_{x^2} \circ \dots \circ L_{x^r} (dh_j) \mid X^j \in D, 1 \leq i \leq s \}$$

where $L_x(h_i)(\cdot) = dh_{i(\cdot)}X(\cdot)$, $1 \leq i \leq s$, is Lie's derivative of h_i in the direction of the vector field X . Since this derivative commutes with the exterior derivative [10], that is $dL_x(f) = L_x(df)$ for any function $f : \text{dom}(f) \subset M \longrightarrow R$, Λ is, to a certain extent, an infinitesimal measure of h with respect to all possible concatenations of the system strategies.

In [4] it is proved the so-called weak-observability rank condition:

$$\dim \Lambda(x) = \dim M \implies \Sigma \text{ is weakly observable in } x.$$

In [3] Basto Gonçalves studies the problem of the quasi-observability of systems and defines ∇ as the smallest Σ invariant co-distribution containing the dh_i , $1 \leq i \leq s$, and builds quasi-minimal realization of Σ , through the integral manifolds of the regular co-distribution ∇ .

Ayala and San Martin introduce in [1] the co-distribution

$$\nabla^+ = \text{Span} \{ \rho^*(dh_i) \mid \rho \in S_\Sigma, 1 \leq i \leq s \}$$

where the pull-back is defined by $\rho^*(dh_i) = d(h_i \circ \rho)$, obtaining weakly-minimal realization of controllable systems in the process of building minimal realization of this type of systems.

As we shall see for transitive systems, ∇ will always be integrable, however this may not true for Λ and ∇^+ . On the other hand, $\Lambda \subset \nabla^+ \subset \nabla$ and since Λ is in general less difficult to build, it is convenient to specify the situations such that $\Lambda = \nabla^+$, $\Lambda = \nabla^+ = \nabla$. Under these circumstances, the study of the observability of control systems leads to the search of integrability conditions of involutive co-distributions in the sense of differential ideals, in fact, ∇^+ and ∇ are generated by families of exact 1-forms. Consequently, the condition of regularity (i.e., that the dimension of this type of co-distributions remains constant on M) is sufficient to determine integrability in the classical sense by a direct application of Frobenius theorem [10].

For this purpose, we rely on the techniques used by Lobry [5], Sussmann [9], Matsuda [6] and Stefan [8] in relation to the study of the integrability of distributions on the tangent bundle TM .

2. CO-DISTRIBUTIONS

Let us begin with some basic concepts. For each differentiable manifold M and for each x in M , T_x^*M will denote the dual of the vector space T_xM , that is

$$T_x^*M = \{\lambda : T_xM \longrightarrow \mathbb{R} \mid \lambda \text{ linear}\}.$$

The co-tangent bundle on M is by definition the manifold

$$T^*M = \bigcup_{x \in M} T_x^*M$$

A co-distribution θ in M is a map such that for each $x \in M$, $\theta(x)$ is a vector sub-space of T_x^*M . θ will be said to be regular if $\dim \theta(x)$ is independent from x in M .

An immersion (N, i) is said to be a (maximal) integral manifold of a co-distribution θ if for each $x \in N$:

- a) $\theta(i(x))((di_x(T_xN))) = 0$,
- b) $\dim(N) = \dim(M) - \dim \theta(i(x))$.

θ is integrable in x [inM], if there is an integral manifold of θ containing x , denoted $In_\theta(x)$ [for each $x \in M$].

Let us look at these concepts under the light of the co-distributions Λ , ∇^+ and ∇ .

Examples 2.0.

1. Let $\Sigma = (\mathbb{R}, D = \{u \frac{d}{dx} \mid u \in \mathbb{R}\}, \mathbb{R}^2, h = (\cos x, \sin x))$. Then,

$$\Lambda(x) = \text{Span} \{ \sin x \, dx, \cos x \, dx \} = T_x^* \mathbb{R}.$$

But $T_x^* \mathbb{R}$ can be nullified only at the origin x . Thus, the integral manifold of Λ at $x \in \mathbb{R}$ is $In_\Lambda(x) = \{x\}$.

Evidently, i is the inclusion. Now, since $\Lambda \subset \nabla^+ \subset \nabla$ we have that $\Lambda = \nabla^+ = \nabla$

Let $\Sigma = (\mathbb{R}^2, D = \{u \frac{\partial}{\partial x} \mid u \in \mathbb{R}\}, \mathbb{R}, h)$ be a control system such that

$$h(x, y) = \int_{-\infty}^x f(t) dt$$

where f is a function of class C^∞ that is zero in $(-\infty, 0]$ and strictly increasing on $(0, +\infty)$.

For each $w = (x, y) \in \mathbb{R}^2$ we have

$$\Lambda(w) = \text{Span} \{ f^{(k)}(x) \, dx \mid K \geq 0 \}.$$

Then,

$$\Lambda(w) = \begin{cases} \text{Span} \{ dx \} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

Thus, if π_1 denotes the projection on the x axis we have:

- a) $J = \{w \in \mathbb{R}^2 \mid \pi_1(w) < 0\}$ is the integral manifold of Λ through each $w \in J$.
- b) $\text{Span} \{ \frac{\partial}{\partial y} \mid_w \}$ is the integral manifold of Λ containing w for each w satisfying $\pi_1(w) > 0$.
- c) There are no integral manifolds of Λ containing elements with null first coordinate. In fact, since all the manifolds considered are boundaryless, each integral manifolds of Λ "passing" over these points will have to "go through" the y axis. This situation is not possible because of the difference of the dimension of Λ in any neighborhood of this type of state.

The integrability of ∇^+ comes from the equality

$$\nabla^+(x, y) = \text{Span} \{ f^{(k)}(x+t) \, dx \mid t > 0 \} = \text{Span} \{ dx \}$$

In particular, ∇^+ is regular and for each $w = (x, y) \in \mathbb{R}^2$ we have $In_{\nabla^+}(x, y) = Span \left\{ \frac{\partial}{\partial y} \mid_w \right\}$. Besides, we can observe that in this case $\Lambda \subset \nabla^+ = \nabla$.

Definition 2.1. Let $X \in X(M)$ and θ a co-distribution in M . We say that X preserves θ if for each $x \in dom(X)$:

$$X_t^* \theta(X_t(x)) \subset \theta(x), \text{ for each possible } t \in \mathbb{R}.$$

$F \subset X(M)$ preserves θ if each $X \in F$ preserves θ . $\square$

Remarks 2.2.

1. If X preserves θ then, for each $x \in dom(X)$

$$X_t^* \theta(X_t(x)) = \theta(x) \text{ for each possible } t \in \mathbb{R}.$$

2. Let F be a transitive family of vector fields on M , that is, the pseudo-group G_F built analogously to pseudo-group G_Σ for control systems, is such that:

$$\exists x \in M \text{ satisfying } G_F(x) = M.$$

Let us suppose that F preserves θ , then, according to the remark above, the dimension of θ is independent from $x \in M$ and thus, θ is regular. It is important to note here the relevance of the orbit theorem [9]. Indeed, given a control system Σ on M , the field family $F = D_\Sigma$ is transitive over each orbit of Σ . Particularly, each co-distribution θ preserved by D_Σ will be regular.

3. Note that $-\frac{\partial}{\partial x}$ does not preserve $\Lambda = Span\{f^{(k)}dx\}$ in example 2.0 (2).
4. It is well-known that the operator of derivative exterior d satisfies $d^2 = 0$. In particular, the exterior derivative of an exact form df is null. Remember that the co-distributions introduced in [section 1] were defined by

5.

$$\Lambda = Span. \left\{ d(L_{X_1} \circ \dots \circ L_{X_r}(h_i) \mid X^j \in D, 1 \leq i \leq s \right\}$$

$$\nabla^+ = Span \{ d(\rho \circ h_i) \mid \rho \in S_\Sigma, 1 \leq i \leq s \}$$

$$\nabla = Span \{ d(\rho \circ h_i) \mid \rho \in G_\Sigma, 1 \leq i \leq s \}$$

consequently,

$$d(\Lambda) = d(\nabla^+) = d(\nabla) = 0.$$

In particular, these co-distributions are involutive. So, we must try to find conditions under which this type of co-distributions is preserved by transitive families of vector fields.

The introduction of the concept that follows is thoroughly justified. Let w a 1-form in M , that is:

$$w : x \in \text{dom}(w) \subset M \longrightarrow w_x \in T_x^*M$$

and let $X \in X(M)$. Lie's derivative of w in the direction of the vector field X in $x \in \text{dom}(X) \cap \text{dom}(w)$ is by definition:

$$L_X(w) |_x (\cdot) = \frac{d}{dt} \Big|_{t=0} w_{X_t(x)}(X_t^*(x)(\cdot)).$$

Now, let θ a co-distribution in M parametrized by $\delta = (w^1, \dots, w^r)$ over a "piece of trajectory of X in x ", that is,

$$\exists \varepsilon > 0 : \theta(X_t(x)) = \text{Span}\{w_{X_t}^1(x), \dots, w_{X_t}^r(x)\}$$

for each $t \in (-\varepsilon, \varepsilon)$. Then, by definition, a vector field X preserves θ if:

$$X_t^*(\theta(X_t(x))) \subset \theta(x)$$

equivalently, if the curves

$$\Psi_j(t) = X_t^* w_{X_t}^j(x) \in \theta(x), 1 \leq j \leq r.$$

By derivation, we obtain for each j :

$$\frac{d}{dt} \Psi_j(t) = X_t^* L_x(w^j)(X_t(x)).$$

In particular, and since for each $j = 1, \dots, r$, $w_x^j \in \theta(x)$ we can conclude that a necessary condition for X to preserve θ is:

$$\frac{d}{dt} \Psi_j(t) \in \theta(x)$$

and then with $t = 0$, we obtain that Lie's derivative of w^j in the direction of the vector field X at x must belong to θ , that is

$$L_X(w^j) |_x \in \theta(x)$$

for each $x \in \text{dom}(X) \cap \text{dom}(w^j)$, $j = 1, \dots, r$.

However, as shown in example 2.0 (2), this condition is not sufficient and therefore, if we wish to study integrability via the above notion, we need some additional hypothesis on these derivatives.

Definition 2.3. Let θ be a co-distribution,

1. θ is said to be differentiable at x , if there are 1-forms $w^1, \dots, w^r$ differentiable in a neighborhood U in x satisfying

i) $w \in \theta$ on U

ii) $\theta(x) = \text{Span}\{w_x^1, \dots, w_x^r\}$.

2. Let θ be differentiable at x , the map

$$y \in U \longrightarrow \delta(y) : R^r \longrightarrow \theta(y) \subset T_y^* M$$

defined by

$$\delta(y)(t_1, \dots, t_r) = \sum_{j=1}^r t_j w_y^j$$

is called a parametrization of θ (centered in x). We denote $\delta = (w^1, \dots, w^r)$.

3. Let $\delta = (w^1, \dots, w^r)$ a parametrization of θ centered in x . We say that a vector field X in M preserves θ infinitesimally at x along δ if :

i) there is an interval J containing the origin such that

$$t \in J \implies \theta(X_t(x)) = \text{Span} \left\{ w_{X_t(x)}^1, \dots, w_{X_t(x)}^r \right\}.$$

ii) there are Lebesgue-integrable functions $a_{ij} : J \longrightarrow R$ such that for each $t \in J$ and $i = 1, \dots, r$

$$L_X(w^i)(X_t(x)) = \sum_{j=1}^r a_{ij}(t) w_{X_t(x)}^j \quad \square$$

Motivated by the co-distributions defined in relation to the problem of the observability of control systems, each distribution will be assumed to be differentiable at x for each $x \in M$.

Together with the concepts above, the following lemma is obtained.

Lemma 2.4. If $X \in X(M)$ preserves a co-distribution θ infinitesimally at x along the parametrization $\delta = (w^1, \dots, w^r)$ then, for each $t \in J$

$$X_t^* \theta(X_t(x)) = \theta(x).$$

Proof. . It is sufficient to show that for each $t \in J$ and for each $i = 1, \dots, r$

$$\alpha_i(t) = X_t^*(x)w_{X_t(x)}^i \in \theta(x).$$

Consider an application $\lambda \in T_x^*M$ such that $\theta(x) \subset \text{Ker}(\lambda)$. Let us denote, for $i = 1, \dots, r$

$$\gamma_i(t) = \lambda(\alpha_i(t)).$$

After a derivation, we obtain:

$$\begin{aligned} \frac{d}{dt}\alpha_i(t) &= X_t^*L_X(w^i)(X_t(x)) \\ &= X_t^*\left(\sum_{j=1}^r a_{ij}(t)w_{X_t(x)}^j\right) \\ &= \sum_{j=1}^r a_{ij}(t)\alpha_j. \end{aligned}$$

Then

$$\frac{d}{dt}\gamma_i(t) = \sum_{j=1}^r a_{ij}(t)\lambda(\alpha_j(t))$$

In particular if γ denotes $(\gamma_1, \dots, \gamma_r)$ as column vector, we obtain

$$\begin{aligned} \dot{\gamma} &= (a_{ij}) \cdot \gamma \\ 1 \leq i, j &\leq r \end{aligned}$$

then γ satisfies a linear differential equation with Lebesgue-integrable coefficients, and thus the existence and uniqueness of the solutions are warranted. Now,

$$\alpha_i(0) = w_x^i \in \theta(x)$$

and by the construction of λ , we obtain $\gamma(0) = 0$ and consequently $\gamma \equiv 0$. Otherwise, for each $i = 1, \dots, r$

$$\lambda(\alpha_i(t)) = 0$$

therefore for each $i = 1, \dots, r$ $\alpha_i(t) \in \theta(x)$. In fact, the reasoning above is valid for each λ satisfying $\theta(x) \subset \text{Ker}\lambda$ $\square$

Remarks 2.5.

Lemma 2.4. establishes that if $X \in X(M)$ preserves θ infinitesimally at x along δ , then X preserves θ over the set

$$\{X_t(x) \mid t \in J\}.$$

This situation is still insufficient since the parametrization δ , explains only a "piece of trajectory through x " and not a neighborhood of point x .

Definition 2.6. A parametrization $\delta = (w^1, \dots, w^r)$ of θ is called exhaustive over a set $A \subset M$, if for each $y \in A$

$$\theta(y) = \text{Span} \{w_y^1, \dots, w_y^r\} \quad \square$$

Now, we are really in a position to make the best of definition 2.3.

Lemma 2.7. Let us suppose that $\delta = (w^1, \dots, w^r)$ is an exhaustive parametrization of θ over an open V of M and that $X \in X(M)$ preserves θ infinitesimally along δ at all points of V . Then, X preserves θ in V .

Proof. . : We use a compactness argument. Let $T > 0$ and $y \in \bigcap_i \text{dom}(X_i)$. Let us define the set S by :

$$S = \{t \in [0, T] \mid X_\sigma^* (\theta(X_\sigma(y))) = \theta(y), \forall \sigma \in [0, T]\}.$$

Lemma 2.4 states that S is not empty. Let $s = \text{Sup}(S)$. Since X preserves θ infinitesimally in $X_s(y) \in V$ along δ , there exists $\varepsilon > 0$ satisfying :

$$-\varepsilon < t < \varepsilon \implies X_t^* \theta(X_{t+s}(y)) = \theta(X_s(y)).$$

Then, $s = T$ and X preserves θ $\square$

Let F a transitive family of vector fields on M and θ a co-distribution. The Lemmas in the above paragraph can be summarized as follows: if Lie's derivatives of exhaustive parametrizations of θ in the direction of the elements of F can be written as linear combinations with the Lebesgue-integrable coefficients of the parametrization, then F preserves θ . We must study, therefore, co-distributions having this type of parametrizations.

3. INTERGRABILITY

We start with a new concept:

Definition 3.0. Let $F \subset X(M)$ and θ a co-distribution in M . θ is said to be F -involutive if

$$L_X(w) \in \theta, \forall X \in F, \forall w \in \theta \quad \square$$

Remarks 3.1.

1. If F preserves D , then θ is F -involutive. In particular, ∇ is D_Σ -involutive. In fact,

$$\nabla = \text{Span}\{d(h_i \circ \phi) \mid \phi \in G_\Sigma, i = 1, \dots, s\}$$

and, if $\beta \in G_\Sigma$ and $d(h_i \circ \phi) \in \nabla$, we have:

$$\beta^*(d(h_i \circ \phi)) = d(h_i \circ (\phi \circ \beta)).$$

Since $\phi \circ \beta \in G_\Sigma$ it follows that $\beta^*(d(h_i \circ \phi)) \in \nabla$ for each $i = 1, \dots, s$. Consequently, $F = D_\Sigma$ preserves ∇ . In fact, the elements that generate ∇ form exhaustive parametrizations of this co-distribution over opens sets of M .

2. From the definition of Λ itself, we conclude that this co-distribution is D_Σ -involutive.
3. ∇^+ is D_Σ -involutive, however, as seen before, ∇^+ may not be preserved by D_Σ .

Theorem 3.2. Let F a transitive family over M and θ a co-distribution in M . Then, F preserves $\theta \Leftrightarrow \theta$ is regular and F -involutive.

Proof: Suppose that θ is regular and F -involutive and let $x \in M$ and $\delta = (w^1, \dots, w^r)$ a parametrization of θ centered in x (in a neighborhood of x). θ is regular and since the 1-forms w^j , $j = 1, \dots, r$ are continuous, there must be a neighborhood V_o of x such that

$$\theta(y) = \text{Span}\{w_y^1, \dots, w_y^r\}, \forall y \in V_o$$

(The argument used here is of the following type: if a continuous function is not null in a specific point, then it is not null in a certain neighborhood of the point considered).

Consequently, δ is an exhaustive parametrization of θ over V_o . By hypothesis θ is F -involutive and then

$$\forall w \in \theta, \forall X \in F, L_X(w) \in \theta.$$

Thus, for $y \in V_o$, $i = 1, \dots, r$ we have:

$$L_X(w^i) \mid_y = \sum_{j=1}^r a_{ij} w_y^j.$$

As $\{w^1, \dots, w^r\}$ can be considered as a linearly independent set in V_o , we conclude that the functions $a_{ij} : V_o \longrightarrow \mathbb{R}$ are determined.

Now, if we prove that these functions are Lebesgue-integrable, then since the reasoning above is independent from the point $y \in V_o$ and $X \in F$, Lemma 2.7 applies and hence F would preserve θ in V_o . As the exhaustive parametrizations of θ cover M thoroughly, the result would be proved. It is possible to prove something stronger: each a_{ij} is a differentiable function.

For each $y \in V_o$ we consider the linear mapping

$$\delta(y) : \mathbb{R}^r \longrightarrow \theta(y) \subset T_y^* M$$

defined by

$$\delta(y)(t_1, \dots, t_r) = \sum_{j=1}^r t_j w_y^j.$$

Then, δ is differentiable in V_o and $\delta(y)$ has a range r . Let us also observe that

$$L_X(w_i) |_y = \delta(y)(a_{i1}(y), \dots, a_{ir}(y)) = \delta(y)(a_i(y))$$

is differentiable in V_o . Due to arguments on range, there are matrices $P(y) \in GL_r(\mathbb{R})$, $Q(y) \in M_{(m-y)xy}(\mathbb{R})$, (where m , is of course the dimension of M and hence the dimension of the co-tangent space $T_y^* M$), such that

$$\delta(y) = \begin{pmatrix} P(y) \\ Q(y) \end{pmatrix}$$

The differentiability of $P(\cdot)$ is warranted by the differentiability of $\delta(\cdot)$. As for each $y \in V_o$, $P(y)^{-1}$ exist, we conclude that $P^{-1}(\cdot)$ is also differentiable. Now

$$a_i(y) = (P(y)^{-1}O) \begin{pmatrix} P(y) \\ Q(y) \end{pmatrix} a_i(y) = (P(y)^{-1}O) L_X(w^i) |_y$$

So, the applications $a_i = (a_{i1}, \dots, a_{ir})$ are differentiable in a neighborhood of x . Since the reciprocal was already proved, the proof of the theorem is finished $\square$

Extensions of Theorem 3.2.

1. The differentiability of the matrix (a_{ij}) in the above theorem is a condition quite stronger than the restriction Lebesgue-integrable. When explaining this situation, the concept below comes up naturally.

Definition 3.3. θ is said to be F -weakly involutive, if for each $x \in M$, there is a neighborhood U of x and an exhaustive parametrization δ in U , such that, each $X \in F$ preserves θ infinitesimally in x along δ , with Lebesgue-integrable coefficients $\square$

Theorem 3.4. Let F a transitive family of vector fields over M and θ an F -weakly-involutive co-distribution. Then F preserves θ .

Proof : From the hypothesis we deduce that θ is a regular co-distribution and the rest of the proof is contained in the proof of the theorem 3.2 $\square$

2. In lemma 2.4 the curves considered were

$$\alpha_j(t) = X_t^*(w^j)(X_t(x))$$

The proof of this lemma results from the fact that $\beta = \lambda(\alpha)$ satisfies a linear differential equation with Lebesgue-integrable coefficients and with initial condition $\beta(0) = 0$. Let us consider again the definition 2.3 (3) but this time with restrictions on derivatives of highest order, that is, condition (ii) is replaced by

$$L_X^{(s)}(w^i)(X_t(x)) = \sum_{j=1}^r a_{ij}(t) w_{X_t(x)}^j$$

with a_{ij} Lebesgue-integrable and $s \in N$.
Since,

$$\alpha_i^{(s)}(t) = X_t^* L_X^{(s)}(w^i) X_t(s)$$

then,

$$\beta_i^{(s)}(t) = \sum_{j=1}^r a_{ij}(t) \cdot \beta_j(t), \quad 1 \leq i \leq r$$

however, this equation is equivalent to the system

$$\begin{aligned} \beta^{(1)}(t) &= \gamma_1(t) \\ \gamma^{(1)}(t) &= \gamma_2(t) \\ &\vdots \\ &\vdots \\ \gamma_{s-1}^{(1)}(t) &= (a_{ij})\beta(t). \end{aligned}$$

If we assume that $L_X^{(p)}(w^i)(x) \in \theta(x)$, $s \geq p \geq 1$, that is, if $\gamma_l = 0$ for $s-1 \geq l \geq 1$ we will have:

$\beta(s)$ satisfies a linear differential equation with Lebesgue-integrable coefficients and $\beta^{(s)}(0) = 0$. Due to the uniqueness of the solution we conclude that $\beta^{(s)} \equiv 0$. Inductively, we obtain

$$\beta^{(p)} \equiv 0 \text{ for } s \geq p \geq 1.$$

Then, $\beta \equiv 0$ and again the conclusion of the lemma 2.4 will be valid. $\square$

Definition 3.5. An exhaustive parametrization $\delta = (w^1, \dots, w^r)$ of θ in U is said to be F-weakly-involutive of order $n \in N$, if for each $X \in F$, there are integers $l_1, \dots, l_r \in \{1, \dots, n\}$ such that for each $p \in \{1, \dots, n\}$ and each $i = 1, \dots, r$

$$\mathbf{i}) L_X^{(p)}(w^i)(y) \in \theta(y), \quad \forall y \in U$$

ii) The equation

$$\delta(y)a_i(y) = L_X^{(li)}(w^i)(y)$$

admits solution with $a_i : U \longrightarrow \mathbb{R}$ Lebesgue-integrable $\square$

Theorem 3.6. Let F a transitive family of vector fields on M , θ a co-distribution and $n \in \mathbb{N}$. If the domains of the parametrizations F -weakly- involutive of order n of cover M , then F preserves θ .

Proof : It comes directly from the comment above and the definition 3.5 itself $\square$

Let M a manifold, X a vector field and w a 1-form, defined over M . Matsuda's condition [6] is transposed to our context as follows:

The series

$$\sum_{k=0}^{\infty} (-1)^k \frac{t^k}{k!} L_X^{(k)}(w)(x)$$

is of class C^1 in a neighborhood U_o of $(0, x) \in \mathbb{R} \times M$ and the derivatives may be computed by differentiating the series term by term.

With this idea, we get the following result.

Theorem 3.7. Let F a transitive family of vector fields on M and θ an F -involutive co-distribution. If the domains of the exhaustive parametrizations of θ that satisfy Matsuda's condition in relation to F cover M , then F preserves θ .

Proof : Let $\delta = (w^1, \dots, w^r)$ be an exhaustive parametrization of θ in a neighborhood U of x . Let us define for $j = 1, \dots, r$

$$\alpha_j(t, \tau) = \sum_{k=0}^{\infty} (-1)^k \frac{t^k}{k!} (w^j)(X_\tau(x))$$

$$\beta_j(t, \tau) = X_\tau^* \alpha_j(t, \tau).$$

Since θ is F -involutive, we can see that $\alpha_j(t, \tau) \in \theta(X_j(x))$ By taking partial derivative, we get

$$\frac{\partial}{\partial t} \beta_j = \sum_{k=1}^{\infty} (-1)^k \frac{t^{k-1}}{(k-1)!} X_\tau^* \tau L_X(k)(w^j)(X_\tau(x))$$

$$\frac{\partial}{\partial t} \beta_j = \sum_{k=0}^{\infty} (-1)^k \frac{t^k}{k!} X_\tau^* L_X^{(k+1)}(w^j)(X_\tau(x))$$

In particular, for each $j = 1, \dots, r$, $\frac{\partial}{\partial t} \beta_j + \frac{\partial}{\partial \tau} \beta_j = 0$.

Since $\beta_j(0, 0) = w_x^j$ then $\beta_j(t, t) = w_x^j, \forall t$. Thus $X_{-\tau}^* w_x^j = \alpha_j(t, t) \in \theta(X_\tau(x))$ and then Σ preserves θ $\square$

ANALYTICAL DISTRIBUTIONS

Let X and w be analytical sections of the bundles TM and T^*M respectively. To study the curve

$$\alpha(t) = X_t^* w_{X_t}(x)$$

locally, it is sufficient to know the successive Lie's derivatives of the 1-form w with respect to the vector field X , at one point.

Theorem 3.8. Suppose that F is a transitive family of vector fields and θ is an analytical F -involutive co-distribution over M , such that the exhaustive analytical parametrization of θ cover M . Then F preserves θ .

Proof. Let $x \in M$ and U a neighborhood of x and the domain of an analytical exhaustive parametrization $\delta \doteq (w^1, \dots, w^r)$. For each $X \in F$ and $i = 1, \dots, r$

$$\alpha_i(t) = X_t^* w_{X_t}^i(x)$$

defines an analytical curve over T^*M . Since θ is F -involutive, we get

$$\alpha_i^{(k)}(0) = L_X^{(k)}(w^i)(x) \in \theta(x).$$

From the analyticity, we conclude the existence of $\varepsilon > 0 : |t| < \varepsilon \Leftrightarrow \alpha_i(t) \in \theta(x)$, δ is exhaustive over U and thus, by using the same type of argument utilized in lemma 2.7, we see that X and then F preserves θ in U $\square$

Theorem 3.8. is not valid in the C^∞ case. Indeed, consider the system Σ defined by $M = \mathbb{R}^2$, $D = \{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}$ and as in example 2.0 (2).

Then

$$\Lambda = \text{Span}\{f^{(k)} dx\}$$

is D -involutive and $\delta = f dx$ is a global C^∞ parametrization of Λ . However, D cannot preserve this co-distribution because of the fact that Λ is not integrable in any point of the y axis. In fact, the dimension of Λ varies along any integral curve of the vector field X .

This example also shows that in the C^∞ case, Λ may not be integrable, even though the vector fields of D are analytical and complete and the system Σ is symmetric and controllable. Indeed, the conclusion of the non-integrability of Λ remains invariable if we consider

$$D = \{\pm \frac{\partial}{\partial x}, \pm \frac{\partial}{\partial y}\}$$

instead of D as above.

Stefan's condition [8] is seen in the following concept.

Definition 3.9. Let $A \subset M$, a A co-distribution θ is said to be locally analytic along A if : for each $x \in A$ and $v \in \theta(x)$ there is an analytic form w defined in a neighborhood of A , $w \in \theta$ and such that $w_x = v$ $\square$

We observe that in the previous example there are exhaustive analytic parametrizations of Λ for $x > 0$ and for $x < 0$. However, it is not possible "to connect" these parametrizations via analytic continuation. In fact, Λ is not locally analytic along any piece of trajectory of the vector field $X = \frac{\partial}{\partial x}$ such that the set

$$A = \{X_t \mid t \in J\}$$

contains points with positive, and negative first coordinate. This is because, for $v = dx$ and $z = (x, y) \in A$ with $x > 0$ there is not $w \in \Lambda$ such that $w_z = v$. In fact, each

$$w = a(x, y)dx + b(x, y)dy \in \Lambda$$

is such that $a(x, y)$ is null over an open subset of $\mathbb{R}^2$ so that $a \equiv 0$ in the plane, because of analyticity.

Theorem 3.10. Let F a transitive family of vector fields on M and θ an analytic F -involutive co-distribution. Then, F preserves $\theta \iff$ for each $X \in F$, θ satisfies : if $x \in \text{dom}(X)$ there exist $\varepsilon > 0$ such that θ is locally analytic along the set

$$A(X, x, \varepsilon) = \{X_t(x) \mid |t| < \varepsilon\}.$$

Proof: Let $X \in F$ and $x \in \text{dom}(X)$. There is an analytic parametrization $\delta = (w^1, \dots, w^r)$ of θ centered in x and defined in U . Let $\varepsilon > 0$ so that

$$-\varepsilon < t < \varepsilon \Rightarrow X_t(x) \in U.$$

Let $y = X_{t_0}(x) \in A(X, x, \varepsilon)$ and consider $v \in \theta(y)$.

Since X preserves θ , there are $u_1, \dots, u_r$ such that

$$X_{t_0}^*(v) = \sum_{j=1}^r u_j W_x^j.$$

Then

$$w = X_{-t_0}^* \left(\sum_{j=1}^r u_j W^j \right)$$

is an analytic form defined in a neighborhood of $A(X, x, \varepsilon)$ and $w_x = v$.

Conversely let $X \in F$ and $x \in \text{dom}(X)$. There exist $\varepsilon > 0$ such that θ is locally analytic along $A(X, x, \varepsilon)$. We must prove that

$$|t| < \varepsilon \Rightarrow X_t^* \theta(X_t(x)) \subset \theta(x).$$

For each $v \in \theta(X_t(x))$, there exists $w \in \theta$ with $v = W_{X_t}(x)$.

Let us define the analytic curve

$$\alpha(s) = X_s^* W_{X_s}(x)$$

By deriving α and evaluating at $s = 0$, we obtain: $\alpha(s) \in \theta(x)$ for each $s \in \text{dom}()$.

Thus, $X_t^* W_{X_t}(x) \in \theta(x)$ and consequently due to the arbitrariness of v we conclude that

$$X_t^* \theta(X_t(x)) \subset \theta(x), \text{ for } |t| < \varepsilon$$

The proof follows then in the same way as lemma 2.7. $\square$

This result generalizes theorem 3.8. In fact, in this case what is necessary is the exhaustivity of θ along the trajectories of the vector fields of F .

4. Λ , ∇^+ AND ∇

Some facts about Λ , ∇^+ and ∇ are deduced. Other results may be found in, [2].
Proposition 4.1. Let $\Sigma = (M, D, \mathbb{R}^5, h)$ a control system. If the dimension of Λ is constant, then

$$\Lambda = \nabla^+ = \nabla$$

Proof : is regular and D -involutive by definition, so the theorem 3.2. is applied and then D preserves Λ . Let $i = 1, \dots, s$ and $X \in D$. For each possible $t \in R$, the D - invariance of Λ ensures that

$$d(h_i \circ X_t) = X_t^*(dh_i(x)) \in \Lambda(x), \quad \forall x \in M.$$

The same reasoning is valid for any possible finite combination of elements in D . That is, if

$$\rho = X_{t_1} \dots X_{t_k} \in G_\Sigma$$

then, for each $X \in M$

$$d(h_i \circ \rho) = \rho^* dh_i(x) \in \Lambda(x).$$

Thus, $\nabla(x) \subset \Lambda(x)$, $\forall x \in M$ and the result comes from the fact that $\Lambda \subset \nabla^+ \subset \nabla$, due to the construction itself of those objects $\square$

In [1] was proved the following proposition

Let $\Sigma = (M, D, \mathbb{R}^5, h)$ a transitive control system then,

$$\dim(\nabla^+) \text{ constant} \Leftrightarrow \nabla^+ = \nabla \quad \square$$

In particular, we obtain:

Corollary 4.2. Let Σ a transitive control system. Then

- a) $\nabla^+ = \nabla \Leftrightarrow \nabla^+$ is integrable.
- b) Σ controllable $\Rightarrow \nabla^+ = \nabla$.
- c) Σ analytic $\Rightarrow \Lambda = \nabla^+ = \nabla$.

Proof :

a) Since Σ is transitive, we deduce that $\nabla = \nabla^+$ is integrable. Conversely, the integrability of ∇^+ implies that this co-distribution is regular and consequently $\nabla^+ = \nabla$ by the above result.

b) If Σ is controllable, then given $x, y \in M$, $\exists \rho \in S_\Sigma$ such that $y = \rho(x)$. Since ∇^+ is S_Σ -invariant, the dimension of ∇^+ is independent from the states in M , thus $\nabla^+ = \nabla$.

c) Under these conditions Λ is integrable and therefore regular. The result comes directly from proposition 4.1. $\square$

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Víctor Ayala Bravo
Departamento de Matemática
Facultad de Ciencias
Universidad Católica del Norte
Casilla 1280, Antofagasta-Chile.