SADDLE CONNECTIONS IN PARABOLIC DIFFERENTIAL EQUATIONS

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Abstract

We assume the existence of a saddle connection between two hyperbolic equilibrium points. Necessary and sufficient conditions are given for the existence of a connection for the perturbed equations. These connections are obtained from the zeros of a finite number of bifurcation functions.

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1. Introduction

In the study of autonomous equations which present closed heteroclinic contours or homoclinic solutions, two approaches have become very interesting: the generation of periodic orbits when an autonomous perturbation is added to the equation [1], [3], [4], [12] and the existence of solutions which present chaotic behaviour, when considering the equation under suitable non autonomous periodical perturbations [2], [5].

In order to determine if the connections are or are not maintained for the perturbed equations, it is necessary to study the conditions for the perturbations.

Hale and Lin [7] have already considered autonomous perturbations to retarded functional differential equations. In [1], [2] autonomous and periodic perturbations were considered to parabolic equations, when the unstable manifolds of the equilibrium points have the same finite dimension.

Here we study the existence of saddle connections under perturbations of evolution equations, where the unstable manifold of the equilibrium points have not necessarily the same dimension.

For the case of the existence of a heteroclinic contour, it can be deduced from (3.5) that the condition of the dimension equality of the unstable manifolds is convenient for the existence of isolated saddle connections in that contour.

A linear operator A on a Banach space X is said to be sectorial if A is a closed, densely defined operator, such that, for some $\phi \in (0, \pi/2)$ and some $\mu \geq 1$ and real a , the sector

$$S_{a,\phi} = \{\lambda | \phi \leq |\arg(\lambda - a)| \leq \pi, \lambda \neq a\}$$

is in the resolvent set of A , i.e. $S_{a,\phi} \cap \sigma(A) = \emptyset$, and

$$\|(\lambda - A)^{-1}\| \leq \mu/|\lambda - a| \quad \text{for all } \lambda \in S_{a,\phi}.$$

Many elliptic boundary-value problems define sectorial operators as shown by Friedman [6] and Pazy [11].

For any sectorial operator A , there is a real number a , such that, if $A_1 = A + aI$, then $\operatorname{Re} \sigma(A_1) > 0$. We can define fractional powers A_1^α of A_1 . Consider the set $X^\alpha \subseteq X$ defined by the domain of A_1^α , $X^\alpha = D(A_1^\alpha)$, with the graph norm $\|x\|_\alpha = \|A_1^\alpha x\|$, $x \in X^\alpha$. Different choices of a give equivalent norms on X^α [9, Theorem 1.4.6]. These spaces X^α are Banach spaces in the norm $\|\cdot\|_\alpha$ for $\alpha \geq 0$. X^α is a dense subspace of X^β with continuous inclusion for $\alpha \geq \beta$ [9, Theorem 1.4.8].

We assume that the equation

$$\dot{x} + Ax = g(x) \tag{1.1}$$

where A is a sectorial operator on a Banach space X , $g \in C^k(X^\alpha, X)$, $k \geq 2$, has two hyperbolic equilibrium points x_1 and x_2 , such that its linear approximation

has a finite number of eigenvalues with negative real part and the rest of the spectrum has positive real part, i.e. $\dim W^u(x_i) = n_i$; $i = 1, 2$. We also assume the existence of a solution of (1.1), which is a saddle connection $\Gamma = \{p(t) | t \in \mathbb{R}\}$ from x_1 to x_2 , i.e.

$$\lim_{t \rightarrow -\infty} p(t) = x_1, \quad \lim_{t \rightarrow +\infty} p(t) = x_2$$

which approaches the equilibrium point tangent to eigenvectors corresponding to real and simple eigenvalues. Our aim is to determine the conditions under which, the perturbed equation of (1.1):

$$\dot{x} + Ax = f(x) \quad (1.2)$$

where $f \in C^k(X^\alpha, X)$, $k \geq 2$, in a neighbourhood of g has a heteroclinic connection Γ_f close to Γ from an equilibrium point x_{1f} to an equilibrium point x_{2f} , close to x_1 and x_2 respectively, and the T -periodical perturbed equation

$$\dot{x} + Ax = q(t, x) \quad (1.3)$$

where $q \in C^k(\mathbb{R} \times X^\alpha, X)$, $k \geq 2$, in a neighbourhood of g has a heteroclinic connection Γ_q close to Γ , from a periodical solution $x_{1q}(t)$ to a periodic solution $x_{2q}(t)$ in neighbourhoods of x_1 and x_2 respectively, which are given in (3.5) and (4.4) respectively. This last relation is valid in general for bounded perturbations.

2. Preliminaries

Suppose A_0 is a sectorial operator in X , $0 \leq \alpha < 1$, $t \mapsto A(t) - A_0 : \mathbb{R} \rightarrow L(X^\alpha, X)$ is locally Hölder continuous with exponent greater than α and let $T(t, s)$ be the solution operator for $\dot{x} + A(t)x = 0$.

Definition 2.1 The equation $\dot{x} + A(t)x = 0$ is said to have an exponential dichotomy on $J \subseteq \mathbb{R}$ with exponent $\beta > 0$ and bound M if there exist projections $P(t), Q(t) = I - P(t)$, such that

- (a) $T(t, s)P(s) = P(t)T(t, s), \quad t \geq s \text{ in } J.$
- (b) The restriction of $T(t, s)$ to the range of $Q(s)$, i.e. $T(t, s)|_{R(Q(s))}, t \geq s$ is an isomorphism onto $R(Q(t))$ and we define the inverse map from $R(Q(t))$ to $R(Q(s))$ by $T(s, t)$.
- (c) $\|T(t, s)P(s)\| \leq Me^{-\beta(t-s)}$ for $t \geq s$ in J .
- (d) $\|T(t, s)Q(s)\| \leq Me^{\beta(t-s)}$ for $s \geq t$ in J .

Remarks

1. We note that if $A(t) = A$ is a constant operator and, for some $\delta > 0$, $\sigma(A)$ is disjoint from the strip $\{|\lambda| - \delta \leq \operatorname{Re} \lambda \leq \delta\}$ then the equation has

an exponential dichotomy on $\mathbb{R}$. This case holds for the linear variational equations

$$\dot{x} + (A - g_x(x_i))x = 0 \quad i = 1, 2. \quad (2.1)$$

because x_i are hyperbolic points.

2. If we consider the equation

$$\dot{x} + (A - g_x(p(t)))x = 0 \quad (2.2)$$

where $\Gamma = \{\rho(t)\}$ is the saddle connection from x_1 , to x_2 , then by Roughness Theorem ([9, Theorem 7.6.11]) there exists a $T_0 \in \mathbb{R}$, such that this equation has an exponential dichotomy on $(-\infty, -T_0]$ and on $[T_0, \infty)$. But for a general non-autonomous linear semigroup for which $R(Q(t))$ is finite dimensional, Lin [10] proved that the exponential dichotomy can be extended from $J = (-\infty, T_0]$ to $J = (-\infty, t_0]$, $T_0 \leq t_0$, under the assumption that $T(t_0, T_0)\phi \neq 0$ for any $\phi \neq 0$ $\phi \in R(Q(T))$. Also, the dichotomy can be extended from $J = [T_0, \infty)$ to $J = [t_0, \infty)$, $t_0 \leq T_0$, provided that $T^*(t_0, T_0)\psi \neq 0$ for any $\psi \neq 0$, $\psi \in R(Q^*(T_0))$. these assumptions are satisfied in many application.

In what follows we suppose that the above assumptions are satisfied.

A result, for the existence of bounded solutions is given by

Theorem 2.2 Let us suppose that the equation

$$\dot{x} + A(t)x = 0$$

has an exponential dichotomy on $(-\infty, \infty)$ with constant M and exponent β , where $A(t) = A - B(t)$, A is a sectorial operator and $t \mapsto B(t)$ is Hölder continuous.

If $f(t, x)$ is a continuous function such that $f(t, 0) = 0$ and $\|f(t, x_1) - f(t, x_2)\| \leq \gamma \|x_1 - x_2\|_\alpha$ for $\|x_1\|_\alpha, \|x_2\|_\alpha \leq \rho$, t is real and $\gamma\eta M < 1$, $b(t)$ is a continuous function such that

$$\|b(t)\| = \sup |b(t)| \leq \frac{\rho(1 - \gamma\eta M)}{\eta M},$$

with

$$\eta = \max \left\{ \frac{2}{\beta}, \frac{1}{\beta} + \frac{\Gamma(1 - \alpha)}{\beta^{-(1+\alpha)}} \right\},$$

where $\Gamma(t)$ is the gamma function.

Then the equation

$$\dot{x} + A(t)x = b(t) + f(t, x)$$

has a unique bounded solution $x(t)$ such that $\|x(t)\|_\alpha \leq \rho$.

Proof.

Let

$$H_\rho = \{x : \mathbb{R} \rightarrow X^\alpha | x \text{ is bounded, continuous and } \|x\|_\alpha \leq \rho\}$$

and for $x \in H_\rho$. We define

$$R(x)(t) = \int_{-\infty}^t T(t, s)P(s)(b(s) + f(s, x(s)))ds - \int_t^\infty T(t, s)Q(s)(b(s) + f(s, x(s)))ds$$

Then:

1. $R(x) \in H_\rho$, because [9, Lemma 7.6.2]

$$\begin{aligned} \|R(x)(t)\|_\alpha &\leq \int_{-\infty}^t M e^{-\beta(t-s)} \max\{1, (t-s)^{-\alpha}\} \|b(s) + f(s, x(s))\| ds \\ &\quad + \int_t^\infty M e^{-\beta(s-t)} \|b(s) + f(s, x(s))\| ds \leq \\ M \left(\frac{\rho(1-\gamma\eta M)}{\eta M} + \gamma\rho \right) & \left[\int_{-\infty}^t e^{-\beta(t-s)} \max\{1, (t-s)^{-\alpha}\} ds + \int_t^\infty e^{-\beta(s-t)} ds \right] \leq \\ \frac{\rho}{\eta} \left(\int_{-\infty}^t e^{-\beta(t-s)} \max\{1, (t-s)^{-\alpha}\} ds + \int_t^\infty e^{-\beta(s-t)} ds \right) &\leq \\ \frac{\rho}{\eta} \left(\int_0^\infty e^{-\beta u} \max\{1, u^{-\alpha}\} du + \int_0^\infty e^{-\beta u} du \right) &\leq \rho \end{aligned}$$

2. R is a contraction, because

$$\begin{aligned} \|R(x)(t) - R(y)(t)\| &\leq \int_{-\infty}^t \max\{1, (t-s)^{-\alpha}\} M e^{-\beta(t-s)} \|f(s, x(s)) - f(s, y(s))\| ds - \\ &\quad \int_t^\infty M e^{-\beta(s-t)} \|f(s, x(s)) - f(s, y(s))\| ds \leq \\ M\gamma\|x(t) - y(t)\|_\alpha &\left\{ \int_0^\infty e^{-\beta u} \max\{1, u^{-\alpha}\} + \int_0^\infty e^{-\beta u} du \right\} \leq M\eta\gamma\|x - y\|_\alpha \end{aligned}$$

As a consequence of this theorem, if the equation

$$\dot{z} + Az = g(t, z) \tag{2.3}$$

has a bounded solution $\xi(t)$ on $(-\infty, \infty)$, such that the variational equation

$$\dot{z} + Az = g_x(t, \xi(t))z$$

has an exponential dichotomy on $(-\infty, \infty)$, then the perturbed equation:

$$\dot{z} + Az = b(t, z)$$

where b is a bounded function in t , in a neighbourhood of g , has a unique bounded solution near $\xi(t)$.

However, if the equation (2.3) is autonomous and $\xi(t)$ is not an equilibrium point, then the Theorem 2.2 can not be applied, because the variational equation

has now the nontrivial bounded solution $\dot{\xi}(t)$ and hence cannot have an exponential dichotomy on $(-\infty, \infty)$.

Finally we require an important lemma ([2]).

Lemma 2.3

If A is a sectorial operator, $t \mapsto A(t) - A : \mathbb{R} \rightarrow L(X^\alpha, X)$ is locally Hölder continuous, such that the equation

$$\dot{x} + A(t)x = 0$$

has an exponential dichotomy on $\mathbb{R}^+$ and $\mathbb{R}^-$ with projections $P^+(t)$, $Q^+(t)$ and $P^-(t)$, $Q^-(t)$, respectively, with $R(Q^+(t))$ and $R(Q^-(t))$ of finite dimension. Then, for the linear operator

$$L : C^{k+1}(\mathbb{R}, X^\alpha) \rightarrow C^k(R, X), \quad k \geq 0,$$

defined by

$$(Lx)(t) = \dot{x}(t) + A(t)x(t)$$

we have:

(a) L is Fredholm:

(b) $N(L) = \{g \in C^{k+1}(\mathbb{R}, X^\alpha) : \dot{g}(t) + A(t)g(t) = 0, g(0) \in R(P^+(0)) \cap R(Q^-(0))\};$

(c) $R(L) = \{f \in C^k(\mathbb{R}, X) : \int_{-\infty}^{\infty} \langle \psi(t), f(t) \rangle dt = 0\},$

for all $\psi \in C_b^{k+1}(\mathbb{R}, X^{\alpha*})$ satisfying the adjoint equation

$$\dot{x} - A^*(t)x = 0 \tag{2.4}$$

and index $L = \dim R(Q^-(0)) - \dim R(Q^+(0))$.

Where $N(L)$ and $R(L)$ are the kernel and the range of the linear operator L .

3. Autonomous perturbation case

Since the saddle points x_i are hyperbolic then equation (2.1) has an exponential dichotomy. By Theorem 2.2 the perturbed equation has two hyperbolic, isolated saddle point, x_{1f}, x_{2f} close to x_1 and x_2 respectively, which without loss of generality we consider that they coincide, i.e. $x_{1f} \equiv x_1$ and $x_{2f} \equiv x_2$.

By the remark 2, after definition 2.1, we have that the variational equation (2.2)

$$\dot{x} + [A - g_x(p(t))]x = 0$$

has an exponential dichotomy only on $\mathbb{R}^-$ and $\mathbb{R}^+$.

We apply the results from the above section to obtain a bifurcation function for heteroclinic orbits of (1.2), for f near g . To this end, let $x(t) = p(t) + z(t)$ then

$$\dot{z} + A(t)z = F(t, z) \tag{3.1}$$

where

$$A(t) = A - \frac{\partial g}{\partial x}(p(t)) \quad \text{and} \quad F(t, z) = f(p + z) - g(p) - \frac{\partial g}{\partial x}(p(t))z$$

Obtaining a heteroclinic orbit of the equation (1.4) is equivalent to obtain a small solution $z(t)$ of (3.1).

From the Roughness Theorem and the extension given by Lin [10], (see Remark 2), we have that the equation $\dot{z} + A(t)z = 0$ has an exponential dichotomy on $\mathbb{R}^+$ and $\mathbb{R}^-$.

We consider (3.1) as a perturbation of the equation

$$(Lz)(t) = 0 \quad L = \frac{\partial}{\partial t} + A(\cdot).$$

and the operator L as a continuous linear operator from $C^{k+1}(\mathbb{R}, X^\alpha)$ to $C^k(\mathbb{R}, X)$, $k \geq 0$.

We construct the desired solution $z(t)$ of (3.1) by applying the Liapunov - Schmidt method to obtain the bifurcation equations for bounded solutions of (3.1) (see Hale [8]).

Let E_2 a projection from $C^k(\mathbb{R}, X)$ to $R(L)$, and E_1 a projection from $C^{k+1}(\mathbb{R}, X^\alpha)$ to $N(L)$.

Then, the equation (3.1) is equivalent to the systems

$$\dot{z} + A(t)z = E_2 F(t, z) \tag{3.2}$$

$$0 = (I - E_2)F(t, z) \tag{3.3}$$

Since L is Fredholm, then it is known that the equation (3.2) can be replaced by

$$(I - E_1)z = K E_2 F(t, z) \tag{3.4}$$

where $K : R(E_2) \rightarrow R(I - E_1)$, is the right hand side inverse of L .

We want to obtain bifurcation functions such that its zeros are in 1 - 1 correspondence with the heteroclinic connections of equation (1.2).

Let $\dim N(L) = d$, then we can consider $\{\hat{p}, y_1, y_2, \dots, y_{d-1}\}$ as a basis for $N(L)$, so $z = \hat{z} + \sum_{i=1}^{d-1} k_i y_i$ where

$$\hat{z} \in R(I - E_1) = C^{k+1}(\mathbb{R}, X^\alpha) \setminus N(L)$$

The equation (3.4) is equivalent to

$$\hat{z} = K E_2 F(t, \hat{z} + \sum k_i y_i).$$

Using the Implicit Function Theorem we obtain the existence of a solution $z^*(s)$, with $s = (k_1, \dots, k_{d-1})$.

Let $\psi_1, \psi_2, \dots, \psi_{d^*}$ be a basis for the bounded solutions for the adjoint equation (2.4). From equations (3.3) and (3.4), we obtain:

$$G_j(s, F) \equiv \int_{-\infty}^{\infty} \langle \psi_j, F(t, z^* + \sum_{i=1}^{d-1} k_i y_i) \rangle dt = 0 \quad j = 1, 2, \dots, d^* \quad (3.5)$$

The functions G_j , $j = 1 \dots d^*$ are the bifurcation functions and the perturbed equations (1.2) has a heteroclinic connection in a neighbourhood of Γ if and only if $G_j(s, F) = 0$, $\forall j$ for some $\|s\| < s_0$. The desired heteroclinic solution is given by

$$x(t) = p(t) + z^*(s)(t) + \sum_{i=1}^{d-1} k_i y_i(t)$$

4. Bounded perturbation case

Let us suppose that $g \in C^k(X^\alpha, X)$, $k \geq 2$, and the equation $\dot{x} + Ax = g(x)$ has a nonconstant bounded solution $p(t)$ such that the variational equation (2.2)

$$\dot{x} + Ax = g_x(p(t))x \quad (4.1)$$

has an exponential dichotomy on $\mathbb{R}^+$ and $\mathbb{R}^-$ with projections $P^+(t)$ and $P^-(t)$ respectively.

So, according to Lemma 2.3 the operator L given by

$$L(x)(t) = \dot{x}(t) + \{A - g_x(p(t))\}x(t)$$

is a Fredholm operator.

Let us suppose now that $q(t, x) \in C^k(R \times X^\alpha; X)$, $k \geq 2$, in a neighbourhood of g and is bounded in t . We want to study the perturbed equation

$$\dot{x} + Ax = q(t, x). \quad (4.2)$$

We change variables in a neighbourhood of $p(t)$ by $x(t) = p(t + \alpha) + z(t + \alpha)$, $\alpha \in \mathbb{R}$.

If $x(t)$ is a solution of (4.2) and t is replaced by $t - \alpha$, then $z(t)$ satisfies

$$\dot{z} + A(t)z = F(t, z, \alpha) \quad (4.3)$$

where $A(t) = A - g_x(p(t))$ and

$$F(t, z, \alpha) = q(t - \alpha, p + z) - g_x(p)z - g(p)$$

i.e., $L(z)(t) = F(t, z, \alpha)$. Now we can apply the Liapunov-Schmidt method to obtain the bifurcation equations for bounded solutions of (4.3), then (4.3) is equivalent to

$$\dot{z} + A(t)z = E_2 F(t, z, \alpha), \quad 0 = (I - E_2)F(t, z, \alpha).$$

Since the operator L is Fredholm, there exists a bounded right inverse operator K of L , so the first equation can be replaced this time by

$$(I - E_1)z = KE_2F(t, z, \alpha)$$

where $K : R(E_2) \rightarrow R(I - E_1)$.

If we consider again $\{\dot{p}, y_1, \dots, y_{d-1}\}$ as a basis for $N(L)$, and $\{\psi_1, \psi_2, \dots, \psi_{d^*}\}$ as a basis for the bounded solutions of the adjoint equation, the bifurcation system is given by

$$G_j(k, \alpha, F) = \int_{-\infty}^{\infty} \langle \psi_j, F(z^* + \sum_{i=1}^{d-1} k_i y_i, \alpha) \rangle dt = 0 \quad j = 1, 2, \dots, d^* \quad (4.4)$$

To prove the existence of periodic orbit $x_1(t)$ and $x_2(t)$ close to x_1 and x_2 respectively we have to consider that the saddle points x_i ($i = 1, 2$) are hyperbolic, so the variational equations (2.1) have an exponential dichotomy on $\mathbb{R}$. From the roughness theorem we obtain that the equation (4.2) has an exponential on both half sides $\mathbb{R}^-$ and $\mathbb{R}^+$.

But, on the other hand, if we apply theorem 2.2, we obtain that the perturbed equation (4.2) has a unique bounded solution $x_i(t)$ ($i = 1, 2$) in a neighbourhood of x_i . Furthermore, if we apply, as in theorem 2.2, the operator R on the set H_ρ of all bounded continuous functions $x(t)$, such that $\|x(t)\| < \rho$, but with the extra condition that $x(t)$ is T -periodic, then, if q is a T -periodic function we have that the fixed point $x_i(t)$ ($i = 1, 2$) is also T -periodic.

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