UNIQUENESS OF BLOWING UP SOLUTIONS ON THE BOUNDARY IN QUASILINEAR ELLIPTIC EQUATIONS WITH NON HOMOGENEOUS TERM IN L^∞

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Abstract

We consider the quasilinear elliptic equation

$$-\Delta u + H(x, u, Du) = f, \text{ in } \Omega \quad (E_H)$$

being $\Omega \subset \mathbb{R}^N$, $N \geq 1$, an open set with regular bounded boundary and $f \in L^\infty(\Omega)$. In this work we obtain upper estimates of the growth, near to the boundary, for any classic solution of (E_H) and the exact behavior for any non negative solution of (E_H) verifying

$$\lim_{\text{dist}(x, \partial\Omega) \rightarrow 0} u(x) = +\infty.$$

Finally, we prove the uniqueness of blow-up solutions of (E_H) .

Key words and phrases: *Nonlinear elliptic equations, interior bounds, blow-up solutions, uniqueness.*

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1. Introduction

In this work, we show upper estimates and uniqueness of unbounded solutions, in a sense that we will precise later, of the PDE in non divergent form

$$-\Delta u + H(x, u, Du) = f, \text{ in } \Omega \quad (E_H)$$

being Ω an open set with bounded boundary of $\mathbb{R}^N$, $N \geq 1$, $f \in L^\infty(\Omega)$, $f \geq 0$, and $H : \Omega \times \overline{\mathbb{R}_+} \times \mathbb{R}^N \rightarrow \mathbb{R}_+$ is an onto function verifying the following structural hypotheses (1):

- i) $x \rightarrow H(x, r, q)$ belongs to $L^\infty(\Omega)$ and $(r, q) \rightarrow H(x, r, q)$ is continuous.
- ii) $|H(x, r, q) - H(x, r, q')| \leq \mu \rho(|q - q'|)$ almost everywhere on Ω , $\forall r, \forall q, q'$, for some $\mu \geq 0$, and $\rho : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is an onto increasing continuous function with $\rho(0^+) = 0$.
- iii) $H(x, r, q) - H(x, s, q) \geq \lambda[\beta(r) - \beta(s)]$, $r \geq s$, almost everywhere on Ω , $\forall r, \forall q, q'$, $\lambda > 0$ and $\beta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is an onto increasing continuous function with $\beta(0^+) = 0$.

Remark 1

Some particular choices of H lead to the equations:

$$-\Delta u + \sum_{i=1}^N a_i(x) \frac{\partial u}{\partial x_i} + \beta(u) = f \quad (E_s)$$

$$-\Delta u + h(x, \nabla u) + \beta(u) = f \quad (E_q)$$

where $k > 0$, $\beta, h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ are continuous functions verifying i), ii), iii), and a_i, f are locally bounded measurable functions on Ω .

Definition

Let $1 < s \leq \infty$. We say that $u \in W_{loc}^{2,s}(\Omega)$ is a **strong solution** of (E_H) if

$$-\Delta u(x) + H(x, u(x), Du(x)) = f \quad \text{a.e. } x \in \Omega.$$

Our main results are,

- i) **Upper estimates of the growth, near the boundary, for any strong solution of (E_H)** , particularly, these hold for every function u which verifies (E_H) on Ω and $u|_{\partial\Omega}$ bounded.

ii) **Existence of a unique behavior on the boundary**, for any u , solution of the problem

$$-\Delta u + H(x, u, Du) = f, \text{ in } \Omega$$

$$\lim_{\text{dist}(x, \partial\Omega) \rightarrow 0} u(x) = +\infty. \quad (P_H)$$

Therefore, a posteriori, the behavior on the boundary of the functions which verifies (P_H) , in strong sense, cannot be as general as

$$\lim_{\text{dist}(x, \partial\Omega) \rightarrow 0} u(x) = +\infty,$$

but only one behavior, which is determined by the interior data of (E_H) .

iii) **Uniqueness of strong solutions of (P_H) .**

The following comparison result over strong solutions of (E_H) , will be very helpful.

Theorem 1 ([Di-Le1])

The structural hypotheses (1) implies,

If $u_1, u_2 \in W_{loc}^{2,\infty}(\Omega)$ verifies in strong sense

$$-\Delta u_1 + H(x, u_1, Du_1) \leq -\Delta u_2 + H(x, u_2, Du_2), \text{ in } \Omega,$$

and

$$\limsup (u_1 - u_2)(x) \leq 0, \text{ as } \text{dist}(x, \partial\Omega) \rightarrow 0,$$

then

$$u_1(x) \leq u_2(x), \quad x \in \Omega.$$

Remark 2

The previous theorem holds, in particular, when u_1 and/or u_2 are defined only in the interior, even if they are unbounded on $\partial\Omega$.

Corollary 2 ([Di-Le1])

The structural hypotheses (1) implies that:

if $u_1, u_2 \in W_{loc}^{2,\infty}(\Omega)$, verifies, in strong sense,

$$-\Delta u_1 + H(x, u_1, Du_1) = -\Delta u_2 + H(x, u_2, Du_2), \text{ in } \Omega$$

and

$$\limsup (u_1 - u_2)(x) = 0, \text{ as } \text{dist}(x, \partial\Omega) \rightarrow 0,$$

then

$$u_1 \equiv u_2 \text{ in } \Omega.$$

Remark 3

According to Corollary 2, the uniqueness of behavior on the boundary of bounded domains of unbounded strong solutions u , in the class $W_{loc}^{2,\infty}(\Omega)$, of (E_H) , implies the uniqueness of unbounded solutions of (E_H) . See [Di-Le2].

Remark 4

About the existence of strong solutions of (P_H) . See [Le-Or].

The argumental work line of [Le-Or] is based on the results of the type Leray-Schauder fixed point theorems (see [Gi-Tr], cap. 11), which need interior estimates of u in $W_{loc}^{1,\infty}(\Omega)$. In particular, the interior estimates of u in $L_{loc}^\infty(\Omega)$ are found through comparison arguments as the one indicated in Theorem 1 in [Di-Le1]. See also [Di-Le2]. In these papers the need to build adequate unbounded families of functions to compare them with the solutions of the problem (P_H) are shown.

Taking into account the structure of (E_H) , a sufficient condition to obtain interior estimates of u in $W_{loc}^{1,\infty}(\Omega)$, see [Di-Le1], is the existence of an onto continuous function $\gamma : IR_+ \rightarrow IR_+$, with $\gamma(0^+) = 0$, verifying

$$\int_r^\infty \frac{ds}{(\Gamma(s))^{\frac{1}{2}}} < \infty, \quad (\mathcal{H}_\gamma)$$

where,

$$\Gamma(r) = \int_0^r \gamma(s) ds.$$

Thus, through all this paper, we will suppose that (H_γ) holds. Obviously the test functions will depend on γ . In particular, if $\gamma(r) = r^m$, $(\mathcal{H}_\gamma)$ holds when $m > 1$.

Results

To simplify this work, we only give the details for the following illustrative case,

Let $u \in W_{loc}^{2,\infty}(\Omega)$ be a strong solution of the problem (P) ,

$$-\Delta u(x) + |\nabla u(x)|^k + \lambda u^m(x) = f(x), \quad x \in \Omega$$

$$\lim_{\text{dist}(x, \partial\Omega) \rightarrow 0} u(x) = +\infty, \quad (P)$$

where Ω is an open set with bounded boundary of C^2 -class, $\lambda > 0$, $m > 1$, $0 < k < 2$ and $f \in L^\infty(\Omega)$, $f \geq 0$.

Lemma 3 ([Gi-Tr])

Let Ω be a bounded domain of $\mathbb{R}^N$, $N \geq 1$, with non void $\partial\Omega$. Then if we write,

$$d(x) = \text{dist}(x, \partial\Omega), x \in \mathbb{R}^N.$$

We have that,

$d(\cdot)$ is Lipschitz-continuous on $\mathbb{R}^N$. Moreover, if $\partial\Omega \in C^k$, $k \geq 2$, then there exists $\delta_0 = \delta_0(\Omega)$ depending only on Ω , such that,

$$d(\cdot) \in C^k(B_{\delta_0}^+(\partial\Omega))$$

and

$$|\nabla d(\cdot)| \equiv 1, \text{ en } B_{\delta_0}^+(\partial\Omega),$$

where $B_{\delta_0}^+(\partial\Omega) = \{x \in \Omega, 0 < d(x) < \delta_0\}$.

Lemma 4

If $\lambda > 0$, $m > 1$, then the following polynomials

$$\begin{aligned} P_1(r) &= r \left[\lambda r^{m-1} - \left(\frac{2}{m-1} \right) \left(\frac{2}{m-1} + 1 \right) \right] & ; k > 0 \\ P_2(r) &= r \left[\lambda r^{m-1} - \left(\frac{2}{m-1} \right) \left(\frac{2}{m-1} + 1 \right) + \left(\frac{2}{m-1} \right)^k r^{k-1} \right] & ; k > 1 \\ P_3(r) &= r \left[\left(\frac{2-k}{k-1} \right)^k r^{k-1} - \left(\frac{2-k}{k-1} \right) \left(\frac{2-k}{k-1} + 1 \right) \right] & ; 1 < k < 2 \end{aligned}$$

have a unique positive root.

Proof:

Let $P(r) \in \{P_1(r), P_2(r), P_3(r)\}$. Then $P(r)$ verifies

1. $P(0) = 0$
2. $P'(0) < 0$
3. $P''(r) > 0, \forall r > 0$, (i.e. $P(r)$ is strictly convex).

From here, it follows that each polynomial $P(r)$ has a unique positive root. Furthermore, in the cases of P_1 and P_3 their roots can be explicitly calculated, obtaining

$$c_1 = \left[\frac{1}{\lambda} \left(\frac{2}{m-1} \right) \left(\frac{2}{m-1} + 1 \right) \right]^{\frac{1}{m-1}}$$

this the unique positive root of P_1 , and

$$c_3 = \frac{(k-1)^{-\frac{(2-k)}{(k-1)}}}{(2-k)}$$

is the unique positive root of P_3 .

Remark 5

It is easy to prove the inequalities

$$\forall r \geq 0 : P_1(r) \leq P_2(r) \text{ and } P_3(r) \leq P_2(r),$$

therefore,

$$c_2 \leq c_1 \text{ and } c_2 \leq c_3.$$

Theorem 5 (upper estimate of any solution of (E))

Let Ω be an open set with bounded boundary of C^2 -class and $f \in L^\infty(\Omega)$, $f \geq 0$. Assume that $\lambda > 0$, $m > 1$, $0 < k < 2$ and c_1, c_2, c_3 the roots obtained in Lemma 4.

If $u \in W_{loc}^{2,\infty}(\Omega)$ is a strong solution of the problem (P), then

$$\lim_{\text{dist}(x, \partial\Omega) \rightarrow 0} \sup \frac{u(x)}{w(x)} \leq C,$$

where $w = w(\text{dist}(x, \partial\Omega), k, m, \lambda, f)$ is a known function which is obtained explicitly and $C = \max\{c_1, c_2, c_3\}$.

In particular, we have the following estimates,

- 1.- $0 < k < \frac{2m}{m+1} \Rightarrow \lim_{\text{dist}(x, \partial\Omega) \rightarrow 0} \sup u(x) d(x)^{\frac{2}{m-1}} \leq c_1$
- 2.- $k = \frac{2m}{m+1} \Rightarrow \lim_{\text{dist}(x, \partial\Omega) \rightarrow 0} \sup u(x) d(x)^{\frac{2}{m-1}} \leq c_2$
- 3.- $\frac{2m}{m+1} < k < 2 \Rightarrow \lim_{\text{dist}(x, \partial\Omega) \rightarrow 0} \sup u(x) d(x)^{\frac{2-k}{k-1}} \leq c_3.$

Proof:

Let $\delta_0 > 0$ be the constant given in Lemma 3, $\delta \in]0, \delta_0[$ and $\epsilon > 0$.

Now, we define the function

$$W(x) = W_{\epsilon, \delta}(x) = (c_i + \epsilon) [d(x) - \delta]^{-\alpha} + c_\epsilon, \quad x \in \Omega_{-\delta},$$

where $\alpha \in \left\{ \left(\frac{2}{m-1} \right), \left(\frac{2-k}{k-1} \right) \right\}$, c_i is the positive root of P_i , $c_\epsilon > 0$ is a constant to be determined, and

$$\Omega_{-\delta} = \{x \in \Omega : \text{dist}(x, \partial\Omega) > \delta\}, \quad \Gamma_\delta = \Omega_\delta \setminus \Omega_{-\delta}.$$

To simplify, we write $\beta(x) = d(x) - \delta$. Thus through direct calculation over the function $W \in C^2(\Omega_{-\delta})$, we obtain the equality,

$$\begin{aligned} & -\Delta W(x) + |\nabla W(x)|^k + \lambda W^m(x) - f(x) = \\ & -\alpha(\alpha+1)(c_i + \epsilon)\beta^{-(\alpha+2)}|\nabla d|^2 + \alpha(c_i + \epsilon)\beta^{-(\alpha+1)}\Delta d \\ & + \alpha^k(c_i + \epsilon)^k\beta^{-(\alpha+1)k}|\nabla d|^k + \lambda[(c_i + \epsilon)\beta^{-\alpha} + c_\epsilon]^m - f \end{aligned}$$

Since for $x \in \Omega_{-\delta} \cap \Gamma_{\delta_0}$, $|\nabla d(x)| = 1$ holds, then,

$$\begin{aligned} & -\Delta W(x) + |\nabla W(x)|^k + \lambda W^m(x) - f(x) \geq \\ & -\alpha(\alpha+1)(c_i + \epsilon)\beta^{-(\alpha+2)} - \alpha(c_i + \epsilon)\beta^{-(\alpha+1)}\|\Delta d\|_{L^\infty(\Omega)} \quad (i) \\ & + \alpha^k(c_i + \epsilon)^k\beta^{-(\alpha+1)k} + \lambda(c_i + \epsilon)^m\beta^{-\alpha m} - \|f\|_{L^\infty(\Omega)}, \quad x \in \Omega_{-\delta} \cap \Gamma_{\delta_0}. \end{aligned}$$

Let us analyze each case which appears in the theorem.

Case 1

Let

$$0 < k < \frac{2m}{m+1},$$

then choosing $\alpha = \frac{2}{m-1}$, we obtain

$$\alpha m = \alpha + 2 > (\alpha + 1)k$$

and therefore, from (i), we can write, in $\Omega_{-\delta} \cap \Gamma_{\delta_0}$

$$\begin{aligned} & -\Delta W(x) + |\nabla W(x)|^k + \lambda W^m(x) - f(x) \geq \\ & \beta^{-(\alpha+2)} \left[\lambda(c_1 + \epsilon)^m - \alpha(\alpha+1)(c_1 + \epsilon) - \alpha(c_1 + \epsilon)\beta\|\Delta d\|_{L^\infty(\Omega)} \quad (ii) \right. \\ & \left. + \alpha^k(c_1 + \epsilon)^k\beta^{(\alpha+2)-(\alpha+1)k} - \|f\|_{L^\infty(\Omega)}\beta^{(\alpha+2)} \right]. \end{aligned}$$

Since c_1 is the positive root of P_1 , for $\epsilon > 0$ small enough, we have the inequality

$$\lambda(c_1 + \epsilon)^m - \alpha(\alpha+1)(c_1 + \epsilon) > 0,$$

and then, $\forall \epsilon > 0$, there exists $\delta_1 = \delta_1(\lambda, \epsilon) \in]0, \delta_0[$, such that,

$$\begin{aligned} & \left[\lambda(c_1 + \epsilon)^m - \alpha(\alpha+1)(c_1 + \epsilon) - \alpha(c_1 + \epsilon)\delta_1\|\Delta d\|_{L^\infty(\Omega)} + \right. \\ & \left. \alpha^k(c_1 + \epsilon)^k\delta_1^{(\alpha+2)-(\alpha+1)k} - \|f\|_{L^\infty(\Omega)}\delta_1^{(\alpha+2)} \right] > 0. \end{aligned}$$

Thus if we choose $\delta \in]0, \delta_1/2[$, such that

$$0 < \delta < d(x) < \delta_1 < \delta_0$$

then we have that

$$0 < \beta(x) = d(x) - \delta < \delta_1,$$

and therefore, from (ii),

$$\begin{aligned} & -\Delta W(x) + |\nabla W(x)|^k + \lambda W^m(x) - f(x) \geq \\ & \beta^{-(\alpha+2)} \left[\lambda(c_1 + \epsilon)^m - \alpha(\alpha+1)(c_1 + \epsilon) - \alpha(c_1 + \epsilon)\beta \|\Delta d\|_{L^\infty(\Omega)} \right. \\ & \left. + \alpha^k (c_1 + \epsilon)^k \beta^{(\alpha+2) - (\alpha+1)k} - \|f\|_{L^\infty(\Omega)} \beta^{(\alpha+2)} \right] > 0, \text{ in } \Omega_{-\delta} \cap \Gamma_{\delta_1}. \end{aligned}$$

Let

$$c_\epsilon = \max\{u(x) : d(x) \geq \delta_1\}$$

then

$$\lim_{d(x) \rightarrow \delta} \sup [W - u](x) = +\infty,$$

and

$$W(x) \geq u(x), \quad x \in \partial\Omega_{-\delta_1}.$$

Finally, from Theorem 1 we obtain,

$$u(x) \leq (c_1 + \epsilon) [d(x) - \delta]^{\frac{-2}{m-1}} + \max\{u(x) : d(x) \geq \delta_1\}, \quad x \in \Omega_{-\delta},$$

therefore,

$$u(x)[d(x) - \delta]^{\frac{2}{m-1}} \leq (c_1 + \epsilon) + c_\epsilon [d(x) - \delta]^{\frac{2}{m-1}}, \quad x \in \Omega_{-\delta}.$$

Firstly, taking $\delta \rightarrow 0$ and then making $d(x) \rightarrow 0$, in the previous inequality we have that $\forall \epsilon > 0$:

$$\lim_{\text{dist}(x, \partial\Omega) \rightarrow 0} \sup u(x) d(x)^{\frac{2}{m-1}} \leq (c_1 + \epsilon),$$

therefore, we conclude that

$$\lim_{\text{dist}(x, \partial\Omega) \rightarrow 0} \sup u(x) d(x)^{\frac{2}{m-1}} \leq c_1.$$

For the other cases, we proceed in a similar way obtaining the following results.

Case 2

If $k = \frac{2m}{m+1}$, then for $\alpha = \frac{2}{m-1}$, we have

$$\alpha m = \alpha + 2 = (\alpha + 1)k$$

and furthermore

$$\lim_{\text{dist}(x, \partial\Omega) \rightarrow 0} \sup u(x) d(x)^{\frac{2}{m-1}} \leq c_2.$$

Note that when the structure of the equation is given for this choice of k , the polynomial $P_2(r)$ can be written as,

$$\begin{aligned} P_2(r) &= r \left[\lambda r^{m-1} - \left(\frac{2}{m-1} \right) \left(\frac{2}{m-1} + 1 \right) + \left(\frac{2}{m-1} \right)^{\frac{2m}{m+1}} r^{\frac{m-1}{m+1}} \right] \\ &= r \left[\lambda r^{\frac{2k-2}{2-k}} - \left(\frac{2-k}{k-1} \right) \left(\frac{2-k}{k-1} + 1 \right) + \left(\frac{2-k}{k-1} \right)^k r^{k-1} \right]. \end{aligned}$$

Case 3

If $\frac{2m}{m+1} < k < 2$, then for $\alpha = \frac{2-k}{k-1}$, we have that

$$\alpha m < \alpha + 2 = (\alpha + 1)k.$$

Furthermore

$$\lim_{\text{dist}(x, \partial\Omega) \rightarrow 0} \sup u(x) d(x)^{\frac{2-k}{k-1}} \leq c_3.$$

Remark 6

The upper estimates obtained in Theorem 5 allow us to show, in particular, results of non existence of strong solutions of

$$\begin{aligned} -\Delta u + H(x, u, Du) &= f \quad , \quad \text{in } \Omega \\ \lim_{\text{dist}(x, \partial\Omega) \rightarrow 0} \frac{u(x)}{\varphi(x)} &= 1 \end{aligned}$$

assuming that $\varphi \in C(\Omega)$ is unbounded, and increases faster than the upper estimate obtained in Theorem 5, on the boundary.

Lemma 6 [Gi-Tr],[Lieb].

Let $\Omega \subseteq \mathbb{R}^n$ be an open set with bounded boundary of C^2 -class. For $\delta > 0$, we define the following sets,

$$\Omega_\delta = \{x \in \mathbb{R}^n / \text{dist}(x, \overline{\Omega}) < \delta\} \quad \text{and} \quad \Gamma_\delta = \Omega_\delta \setminus \Omega_{-\delta}$$

where $\Omega_{-\delta}$ was defined in the proof of Theorem 5. Therefore, we can write $\Omega_0 = \Omega$.

We have that, there exists a function $d \in C^k(\mathbb{R}^n, \mathbb{R})$, such that

- i) $d(x) = \text{dist}(x, \partial\Omega)$, $x \in B_{\delta_0}^+(\partial\Omega)$
- ii) $d(x) = -\text{dist}(x, \partial\Omega)$, $x \in \Omega_{\delta_0} \setminus \Omega$
- iii) $d(x) \geq \delta_0$, $x \in \Omega_{-\delta_0}$
- iv) $d(x) \leq -\delta_0$, $x \in \mathbb{R}^n \setminus \Omega_{\delta_0}$.

Note that i) and ii) implies that

$$|\nabla d(x)| = 1, \text{ when } x \in \Gamma_{\delta_0}.$$

Remark 7

Recall that, from Lemma 3 it is known that if $\Omega \subseteq \mathbb{R}^n$ is an open set with bounded boundary of C^k -class, then there exists a constant $\delta_0 > 0$ such that $\text{dist}(x, \partial\Omega)$ is of C^k -class on $\Omega - \Omega_{-\delta_0}$.

Theorem 7 (Growth rate of the unbounded solutions of (E))

Let $\Omega \subseteq \mathbb{R}^n$ be an open set with bounded boundary of C^2 -class, $\lambda > 0$, $m > 1$, $0 < k < 2$ and $f \in L^\infty(\Omega)$, $f \geq 0$.

If $u \in W_{loc}^{2,\infty}(\Omega)$ is a strong solution of (P), then the growth rate of u is,

$$\lim_{\text{dist}(x, \partial\Omega) \rightarrow 0} u(x) d(x)^{\frac{2}{m-1}} = C,$$

where $C = C(\lambda, m, f, k) \in \mathbb{R}^+$ is a known constant and $\alpha_i \in \left\{ \frac{2}{m-1}, \frac{2-k}{k-1} \right\}$.

In particular, we prove that if c_i is the positive root of P_i , given in Lemma 4, then

- 1.- $0 < k < \frac{2m}{m+1} \Rightarrow \lim_{\text{dist}(x, \partial\Omega) \rightarrow 0} u(x) d(x)^{\frac{2}{m-1}} = c_1$
- 2.- $k = \frac{2m}{m+1} \Rightarrow \lim_{\text{dist}(x, \partial\Omega) \rightarrow 0} u(x) d(x)^{\frac{2}{m-1}} = c_2$
- 3.- $\frac{2m}{m+1} < k < 2 \Rightarrow \lim_{\text{dist}(x, \partial\Omega) \rightarrow 0} u(x) d(x)^{\frac{2-k}{k-1}} = c_3.$

Proof:

Let $\delta_0 > 0$ be the constant given in Lemma 3. For $\delta \in]0, \delta_0[$ and $\epsilon \in]0, c_i[$, fixed but arbitrary, we define, in Ω_δ , the function

$$w(x) = w_{\epsilon, \delta}(x) = (c_i - \epsilon)[d(x) + \delta]^{-\delta} - c_\epsilon, \forall x \in \Omega_\delta,$$

where $\alpha \in \left\{ \frac{2}{m-1}, \frac{2-k}{k-1} \right\}$, c_i is the positive root of the polynomial P_i and $c_\epsilon > 0$ is a constant to determine.

We write $\beta = [d + \delta]$ and via direct calculations over the function $w \in C^2(\Omega_\delta)$, we obtain

$$-\Delta w(x) + |\nabla w(x)|^k + \lambda w^m(x) - f(x) = -\alpha(\alpha + 1)(c_i - \epsilon)\beta^{-(\alpha+2)}|\nabla d|^2 +$$

$$\alpha(c_i - \epsilon)\beta^{-(\alpha+1)}\Delta d + \alpha^k(c_i - \epsilon)^k\beta^{-(\alpha+1)k}|\nabla d|^k + \lambda[(c_i - \epsilon)\beta^{-\alpha} - c_\epsilon]^m - f, \text{ in } \Omega.$$

Since in the region $\Omega \cap \Gamma_{\delta_0}$, we have that $|\nabla d(x)| = 1$, then

$$\begin{aligned} & -\Delta w(x) + |\nabla w(x)|^k + \lambda w^m(x) - f(x) \leq \\ & -\alpha(\alpha + 1)(c_i - \epsilon)\beta^{-(\alpha+2)} + \alpha(c_i - \epsilon)\beta^{-(\alpha+1)}\|\Delta d\|_{L^\infty(\Omega)} \\ & + \alpha^k(c_i - \epsilon)^k\beta^{-(\alpha+1)k} + \lambda[(c_i - \epsilon)\beta^{-\alpha} - c_\epsilon]^m - \|f\|_{L^\infty(\Omega)}, \text{ in } \Omega \cap \Gamma_{\delta_0}. \end{aligned} \quad (i)$$

Now, let us analyze each case given in the theorem.

Case 1

Assume that $0 < k < \frac{2m}{m+1}$, therefore choosing,

$$c_\epsilon = (c_1 - \epsilon)(2\delta)^{-\delta} \text{ and } \alpha = \frac{2}{m-1}$$

we obtain

$$\alpha m = \alpha + 2 > (\alpha + 1)k$$

for every $x \in \Omega \cap \Gamma_\delta$, we have

$$0 < (c_1 - \epsilon)\beta(x)^{-\alpha} - c_\epsilon.$$

Thus,

$$[(c_1 - \epsilon)\beta^{-\alpha} - c_\epsilon]^m \leq [(c_1 - \epsilon)\beta^{-\alpha}]^m \quad x \in \Omega \cap \Gamma_\delta.$$

By using the previous inequality in $\Omega \cap \Gamma_\delta$, we have that,

$$\begin{aligned} & -\Delta w(x) + |\nabla w(x)|^k + \lambda w^m(x) - f(x) \leq \\ & \beta^{-(\alpha+2)} \left[\lambda(c_1 - \epsilon)^m - \alpha(\alpha + 1)(c_1 - \epsilon) + \alpha(c_1 - \epsilon)\beta\|\Delta d\|_{L^\infty(\Omega)} \right. \\ & \left. + \alpha^k(c_1 - \epsilon)^k\beta^{(\alpha+2)-(\alpha+1)k} + \|f\|_{L^\infty(\Omega)}\beta^{(\alpha+2)} \right], \quad x \in \Omega \cap \Gamma_\delta. \end{aligned} \quad (ii)$$

On the other hand, since c_1 is the unique positive root of the polynomial P_1 , and P_1 tends to ∞ as $r \rightarrow \infty$, we have that for $\epsilon > 0$, small enough, but arbitrary,

$$\lambda(c_1 - \epsilon)^m - \alpha(\alpha + 1)(c_1 - \epsilon) < 0$$

and then, there exists $\delta_1 = \delta_1(\lambda, \epsilon) \in]0, \delta_0[$, such that,

$$\begin{aligned} & [\lambda(c_1 - \epsilon)^m - \alpha(\alpha + 1)(c_1 - \epsilon) + \alpha(c_1 - \epsilon)\delta_1\|\Delta d\|_{L^\infty(\Omega)} + \\ & + \alpha^k(c_1 - \epsilon)^k\delta_1^{(\alpha+2)-(\alpha+1)k} + \|f\|_{L^\infty(\Omega)}\delta_1^{(\alpha+2)}] < 0. \end{aligned} \quad (iii)$$

Therefore, if $\delta \in]0, \delta_{1/2}[$, then

$$0 < d(x) < \delta$$

and

$$0 < \beta(x) = d(x) + \delta < 2\delta < \delta_1,$$

consequently from (ii) and (iii), if $x \in \Omega \cap \Gamma_\delta$ we have that,

$$\begin{aligned} & -\Delta w(x) + |\nabla w(x)|^k + \lambda w^m(x) - f(x) \leq \\ & \beta^{-(\alpha+2)} \left[\lambda(c_1 - \epsilon)^m - \alpha(\alpha + 1)(c_1 - \epsilon) + \alpha(c_1 - \epsilon)\beta \|\Delta d\|_{L^\infty(\Omega)} \right. \\ & \left. + \alpha^k (c_1 - \epsilon)^k \beta^{(\alpha+2) - (\alpha+1)k} + \|f\|_{L^\infty(\Omega)} \beta^{(\alpha+2)} \right] \leq 0, \quad x \in \Omega \cap \Gamma_\delta. \end{aligned}$$

Since w verifies,

$$\lim_{\text{dist}(x, \partial\Omega) \rightarrow 0} [w - u](x) = -\infty$$

and

$$w(x) \leq u(x), \quad \text{on } \partial\Omega_{-\delta},$$

theorem 1 allows us to conclude that, for every $x \in \Omega \cap \Gamma_\delta$

$$(c_1 - \epsilon)[d(x) + \delta]^{\frac{-2}{m-1}} - (c_1 - \epsilon)[2\delta]^{\frac{-2}{m-1}} \leq u(x).$$

On the other hand, since $2\delta < \delta_1$, then

$$(c_1 - \epsilon)[d(x) + \delta]^{\frac{-2}{m-1}} - (c_1 - \epsilon)[\delta_1]^{\frac{-2}{m-1}} \leq u(x), \quad x \in \Omega \cap \Gamma_\delta$$

that is,

$$(c_1 - \epsilon) - (c_1 - \epsilon)[\delta_1]^{\frac{-2}{m-1}} [d(x) + \delta]^{\frac{2}{m-1}} \leq u(x)[d(x) + \delta]^{\frac{2}{m-1}}, \quad x \in \Omega \cap \Gamma_\delta. \quad (iv)$$

Now, taking limit $\delta \rightarrow 0$ in (iv), and then making $d(x) \rightarrow 0$, for $\epsilon > 0$, small enough, we have that

$$\lim_{\text{dist}(x, \partial\Omega) \rightarrow 0} \inf u(x) d(x)^{\frac{2}{m-1}} \geq (c_1 - \epsilon),$$

hence,

$$\lim_{\text{dist}(x, \partial\Omega) \rightarrow 0} \inf u(x) d(x)^{\frac{2}{m-1}} \geq c_1. \quad (v)$$

Finally, from (v) and Theorem 5, it is immediately that

$$\lim_{\text{dist}(x, \partial\Omega) \rightarrow 0} u(x) d(x)^{\frac{2}{m-1}} = c_1.$$

In the same way, we obtain the following results,

Case 2

Let $k = \frac{2m}{m+1}$ then for $\alpha = \frac{2}{m-1}$.

We have that

$$\alpha m = \alpha + 2 = (\alpha + 1)k,$$

and

$$\liminf_{\text{dist}(x, \partial\Omega) \rightarrow 0} u(x) d(x)^{\frac{2}{m-1}} \geq c_2.$$

Case 3

Let $\frac{2m}{m+1} < k < 1$ then for $\alpha = \frac{2-k}{k-1}$.

We have that,

$$\alpha m < \alpha + 2 = (\alpha + 1)k$$

and

$$\liminf_{\text{dist}(x, \partial\Omega) \rightarrow 0} u(x) d(x)^{\frac{2-k}{k-1}} \geq c_3.$$

These results and Theorem 5 conclude the proof of the theorem.

Remark 8

In particular, the previous theorem shows a non-existence result of strong solutions for the problem

$$\begin{aligned} -\Delta u + H(x, u, Du) &= f, \quad \text{in } \Omega \\ \lim_{\text{dist}(x, \partial\Omega) \rightarrow 0} \frac{u(x)}{\varphi(x)} &= 1, \end{aligned}$$

assuming that φ is a unbounded continuous function on Ω and for some $q > 0$:

$$\lim_{\text{dist}(x, \partial\Omega) \rightarrow 0} \varphi(x) \text{dist}(x, \partial\Omega)^{-q} \notin \{c_1, c_2, c_3\}.$$

Theorem 8 (Uniqueness of the unbounded solutions of the equation (E))

If $\Omega \subseteq \mathbb{R}^n$ is an open set with bounded boundary of C^2 -class, $\lambda > 0$, $m > 1$, $0 < k < 2$ and $f \in L^\infty(\Omega)$, $f \geq 0$, then the problem (P) has, at most, a unique strong solution in the class $W_{loc}^{2,\infty}(\Omega)$.

Proof:

Let $u_1, u_2 \in W_{loc}^{2,\infty}(\Omega)$ be two strong solutions of the problem (P). Then from Theorem 4, we have that,

$$\lim_{\text{dist}(x, \partial\Omega) \rightarrow 0} \frac{u_1(x)}{u_2(x)} = \lim_{\text{dist}(x, \partial\Omega) \rightarrow 0} \frac{u_2(x)}{u_1(x)} = 1,$$

therefore

$$\lim_{\text{dist}(x, \partial\Omega) \rightarrow 0} (u_1 - u_2)(x) = 0,$$

since it is also true

$$\begin{aligned} & -\Delta u_1(x) + |\nabla u_1(x)|^k + \lambda u_1^m(x) - f(x) = \\ & -\Delta u_2(x) + |\nabla u_2(x)|^k + \lambda u_2^m(x) - f(x), \quad x \in \Omega, \end{aligned}$$

the comparison results allow us to obtain,

$$u_1(x) = u_2(x), \quad x \in \Omega,$$

and this completes the proof.

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