A NEW PETROV-GALERKIN SCHEME FOR THE TWO-POINT BOUNDARY VALUE PROBLEM*

RODOLFO A. ARAYA and GABRIEL N. GATICA
Universidad de Concepción, Concepción, Chile

Abstract

This paper introduces a new asymptotically convergent Petrov-Galerkin method for solving the two-point boundary value problem. The procedure is based on an appropriate combination of a Dirichlet-Neumann mapping and a saddle point variational formulation of the original problem, which yields a bilinear form satisfying the usual Gårding inequality. The corresponding discrete scheme is defined in the usual manner but replacing the Dirichlet-Neumann mapping by its associated finite element approximation. The solvability of the Galerkin equations is proved, and asymptotic error estimates are provided. Further, a particular case of this new scheme can be viewed as a preconditioning technique for the discrete formulation of the saddle point problem. Finally, the matrix formulation is discussed and several numerical experiments are included.

* Partially supported by FONDECYT through the project 91-386.

1. Introduction.

Let us consider the following two-point boundary value problem

$$\begin{aligned} -(pu')' + qu &= f \quad \text{en } \Omega := (0, 1) \\ u(0) &= g_0, \quad u(1) = g_1, \end{aligned} \quad (1.1)$$

where $f \in L^2(\Omega)$, $q \in C(\bar{\Omega})$, $p \in C^1(\bar{\Omega})$, and $g := (g_0, g_1) \in \mathbf{R}^2$. It is further assumed that there exists $p_0 > 0$ such that $p(x) \geq p_0 \forall x \in \Omega$, and that (1.1) has a unique solution. Throughout the paper, we will utilize a saddle point variational formulation of (1.1) in which the boundary flows are also included as unknowns. Indeed, setting $\lambda = (\lambda_0, \lambda_1) := (p(0)u'(0), -p(1)u'(1)) \in \mathbf{R}^2$, and performing the usual integration by parts, we arrive at the following weak formulation of our problem: *Find $(u, \lambda) \in H^1(\Omega) \times \mathbf{R}^2$ such that*

$$\begin{aligned} a(u, v) + b(v, \lambda) &= \int_0^1 f v \, dx \quad \forall v \in H^1(\Omega) \\ b(u, \mu) &= \langle g, \mu \rangle \quad \forall \mu \in \mathbf{R}^2, \end{aligned} \quad (1.2)$$

where $a : H^1(\Omega) \times H^1(\Omega) \rightarrow \mathbf{R}$ and $b : H^1(\Omega) \times \mathbf{R}^2 \rightarrow \mathbf{R}$ are the bounded bilinear forms defined by

$$a(u, v) := \int_0^1 [pu'v' + quv] \, dx$$

and

$$b(v, \lambda) := \langle \gamma_0 v, \lambda \rangle \quad (1.3)$$

Here, $\langle \cdot, \cdot \rangle$ is the inner product in $\mathbf{R}^2$, and $\gamma_0 : H^1(\Omega) \rightarrow \mathbf{R}^2$ is the trace operator of order zero. Recall that γ_0 is linear, bounded, and satisfies the identities, $\gamma_0(H^1(\Omega)) = \mathbf{R}^2$, and $\text{Ker}(\gamma_0) = H_0^1(\Omega)$.

For our purposes, it is more convenient to rewrite (1.2) as follows: *Find $(u, \lambda) \in \mathcal{H} := H^1(\Omega) \times \mathbf{R}^2$ such that*

$$B((u, \lambda), (v, \mu)) = F(v, \mu) \quad \forall (v, \mu) \in \mathcal{H}, \quad (1.4)$$

where $B : \mathcal{H} \times \mathcal{H} \rightarrow \mathbf{R}$ is the bounded bilinear form defined by

$$B((u, \lambda), (v, \mu)) := a(u, v) + b(v, \lambda) + b(u, \mu), \quad (1.5)$$

and $F \in \mathcal{H}'$ is given by

$$F(v, \mu) := \int_0^1 f v \, dx + \langle g, \mu \rangle .$$

The aim of this paper is to present a new asymptotically convergent Petrov-Galerkin scheme for solving (1.4). To this end, we first introduce in Section 2 a linear mapping $L : \mathbf{R}^2 \rightarrow H^1(\Omega)$ that allows us to transform (1.4) into an equivalent formulation whose bilinear form satisfies the Gårding inequality. The new Petrov-Galerkin scheme is then defined by replacing L by its finite element approximation. In Section 3, we give some preliminary results and provide the mapping properties of L . The solvability and approximation results of the discrete scheme are proved in Section 4. Finally, in Section 5 we discuss the matrix formulation of the method and present several numerical experiments.

2. The new Petrov-Galerkin scheme. Main results.

Let us introduce the Dirichlet-Neumann mapping

$$\begin{aligned} L : \mathbf{R}^2 &\rightarrow H^1(\Omega) \\ \mu &:= (\mu_0, \mu_1) \rightarrow L\mu , \end{aligned} \tag{2.1}$$

where $z := L\mu \in H^1(\Omega)$ is the unique solution of the boundary value problem

$$\begin{aligned} -(pz')' + z &= 0 \quad \text{in } \Omega \\ z'(0) &= \frac{\mu_0}{2p(0)} \quad , \quad z'(1) = -\frac{\mu_1}{2p(1)} . \end{aligned} \tag{2.2}$$

It is easily seen that L is linear and that the weak formulation of (2.2) reduces to:
Find $z \in H^1(\Omega)$ such that

$$\tilde{a}(v, z) = -\frac{1}{2} b(v, \mu) \quad \forall v \in H^1(\Omega) , \tag{2.3}$$

where b is given by (1.3) and $\tilde{a} : H^1(\Omega) \times H^1(\Omega) \rightarrow \mathbf{R}$ is the bounded bilinear form defined by

$$\tilde{a}(v, z) := \int_0^1 [p v' z' + v z] \, dx .$$

Since $\tilde{a}$ is clearly $H^1(\Omega)$ -elliptic, the well known Lax-Milgram Lemma provides the existence and uniqueness of z in (2.3). Moreover, because of the

corresponding continuous-dependence result, we deduce that there exists $C_0 > 0$ such that

$$\|L\mu\|_{H^1(\Omega)} \leq C_0 \|\mu\|_{\mathbf{R}^2} \quad \forall \mu \in \mathbf{R}^2.$$

With this, we can now define the bounded linear operator $T : \mathcal{H} \rightarrow \mathcal{H}$ as

$$T(v, \mu) := (v - L\mu, -\frac{1}{2}\mu) \quad \forall (v, \mu) \in \mathcal{H}. \quad (2.4)$$

We will show below in Section 3 that T is 1-1 and onto. Hence, instead of (1.4) we consider from now on the equivalent formulation: *Find $(w, \sigma) \in \mathcal{H}$ such that*

$$A((w, \sigma), (v, \mu)) = F(v, \mu) \quad \forall (v, \mu) \in \mathcal{H}, \quad (P)$$

where $A : \mathcal{H} \times \mathcal{H} \rightarrow \mathbf{R}$ is the bounded bilinear form defined by

$$A((w, \sigma), (v, \mu)) := B(T(w, \sigma), (v, \mu)). \quad (2.5)$$

In this way the solution $(u, \lambda) \in \mathcal{H}$ of the original problem (1.4) is given by $(u, \lambda) = T(w, \sigma)$. Let us observe from (1.5), (2.4) and (2.5) that

$$A((w, \sigma), (v, \mu)) = a(w - L\sigma, v) - \frac{1}{2}b(v, \sigma) + b(w - L\sigma, \mu), \quad (2.6)$$

which clearly shows that A becomes non-symmetric. However, we will prove in Section 4 that A satisfies the Gårding inequality, and hence, under the uniqueness assumption, the usual results on the Fredholm alternative will be employed (see e.g. Mikhlin (1965), Stummel (1969), Hildebrandt & Wienholtz (1964)).

In order to derive the discrete scheme, we now let

$$\{H_h\}_{h>0} := \{S_h^{k,r}\}_{h>0} \subseteq H^r(\Omega)$$

be a given family of finite element subspaces in the sense of Babuska & Aziz (1972), providing the *approximation property*

(AP)_h for all $m \leq r < k$, $m \leq s \leq k$, and for all $v \in H^s(\Omega)$

$$\inf_{v \in H_h} \|v - v_h\|_{H^m(\Omega)} \leq Ch^{s-m} \|v\|_{H^s(\Omega)}.$$

Here, k and r are positive integers satisfying $k \geq 2$ and $r \geq 1$. In particular, for linear finite elements we obtain $k = 2$ and $r = 1$ (see Ciarlet (1978, chapter II)).

Then the usual Galerkin scheme for (P) is defined as: *Find a pair $(w_h, \sigma_h) \in \mathcal{H}_h := H_h \times \mathbf{R}^2$ such that*

$$A((w_h, \sigma_h), (v_h, \mu_h)) = F(v_h, \mu_h) \quad \forall (v_h, \mu_h) \in \mathcal{H}_h. \quad (P_h)$$

However, we notice that from the numerical point of view (P_h) is not convenient since in order to evaluate the bilinear form A one requires to solve the continuous problem (2.3). To overcome this difficulty, we choose $\tilde{h} > 0$ and define the mapping $L_{\tilde{h}} : \mathbf{R}^2 \rightarrow H_{\tilde{h}}$, where for each $\mu \in \mathbf{R}^2$, $L_{\tilde{h}}\mu \in H_{\tilde{h}}$ is the unique solution of the discrete problem

$$\tilde{a}(v_{\tilde{h}}, L_{\tilde{h}}\mu) = -\frac{1}{2}b(v_{\tilde{h}}, \mu) \quad \forall v_{\tilde{h}} \in H_{\tilde{h}}. \quad (2.7)$$

We now replace L by $L_{\tilde{h}}$ in (2.4) and introduce the operator $T_{\tilde{h}} : \mathcal{H} \rightarrow \mathcal{H}$

$$T_{\tilde{h}}(v, \mu) := (v - L_{\tilde{h}}\mu, -\frac{1}{2}\mu) \quad \forall (v, \mu) \in \mathcal{H}, \quad (2.8)$$

and the discrete bilinear form $A_{\tilde{h}} : \mathcal{H} \times \mathcal{H} \rightarrow \mathbf{R}$

$$A_{\tilde{h}}((z, \delta), (v, \mu)) := B(T_{\tilde{h}}(z, \delta), (v, \mu)) \quad \forall (z, \delta), (v, \mu) \in \mathcal{H}. \quad (2.9)$$

Thus, we consider the following *Petrov-Galerkin scheme* for (P) : *Find a pair $(\bar{w}_h, \bar{\sigma}_h) \in \mathcal{H}_h := H_h \times \mathbf{R}^2$ such that*

$$A_{\tilde{h}}((\bar{w}_h, \bar{\sigma}_h), (v_h, \mu_h)) = F(v_h, \mu_h) \quad \forall (v_h, \mu_h) \in \mathcal{H}_h. \quad (P_{h, \tilde{h}})$$

Note here that we do not necessarily assume that $\tilde{h}$ and h coincide. However, the particular case $\tilde{h} = h$ will be commented later on. Now, since $(u, \lambda) = T(w, \sigma)$ is the solution of (P) , the pair $(u_{h, \tilde{h}}, \lambda_h) \in (H_h + H_{\tilde{h}}) \times \mathbf{R}^2$ defined by

$$(u_{h, \tilde{h}}, \lambda_h) := T_{\tilde{h}}(\bar{w}_h, \bar{\sigma}_h) = (\bar{w}_h - L_{\tilde{h}}\bar{\sigma}_h, -\frac{1}{2}\bar{\sigma}_h) \quad (2.10)$$

constitutes the Petrov-Galerkin approximation of (u, λ) .

The main result to be proved in this paper regarding the solvability and approximation properties of $(P_{h, \tilde{h}})$ is stated as follows.

Theorem 2.1. *Suppose that (1.4) (and hence (P)) is uniquely solvable. Let $\{H_h\}_{h>0} := \{S_h^{k,r}\}_{h>0}$ be a numerable family of finite element subspaces satisfying $(\mathbf{AP})_h$. Then there exists a positive constant h_0 such that for all $h, \tilde{h} \leq h_0$,*

the Petrov-Galerkin scheme $(P_{\tilde{h}, \tilde{h}})$ has a unique solution $(\tilde{w}_h, \tilde{\sigma}_h) \in \mathcal{H}_h$. Moreover, if $(u, \lambda) \in \mathcal{H}$ is the unique solution of (1.4) and if $(u_{h, \tilde{h}}, \lambda_h)$ is the Petrov - Galerkin approximation defined by (2.10), we have Cea's estimate,

$$\|(u, \lambda) - (u_{h, \tilde{h}}, \lambda_h)\|_{\mathcal{H}} \leq C \{ \text{dist}(u, H_h) + (h + \tilde{h}) \|\lambda\|_{\mathbf{R}^2} \}. \quad (2.11)$$

Furthermore, if $u, L\lambda \in H^s(\Omega)$, $s \geq 2$, then

$$\|(u, \lambda) - (u_{h, \tilde{h}}, \lambda_h)\|_{\mathcal{H}} \leq C \{ h^{s-1} \|u\|_{H^s(\Omega)} + (h^{s-1} + \tilde{h}^{s-1}) \|L\lambda\|_{H^s(\Omega)} \}. \quad (2.12)$$

The proof of Theorem 2.1 is given in Section 4 below. For this purpose, some preliminary results are collected in Section 3.

3. Preliminary results.

Let us first see that the operator $T : \mathcal{H} \rightarrow \mathcal{H}$ (cf. (2.4)) is bijective. In fact, if $T(v, \mu) = (0, 0)$ then $v - L\mu = 0$ and $\mu = 0$, whence $L\mu = 0$ and $v = 0$. This shows that T is 1-1. Also, given $(z, \delta) \in \mathcal{H}$ it is easily seen that $T(z - 2L\delta, -2\delta) = (z, \delta)$ and therefore T is onto. Then in virtue of the bounded inverse Theorem (see e.g. Schechter (1971, chapter III)) we deduce that T^{-1} is also bounded, and hence there exist positive constants C_1, C_2 such that

$$C_1 \|(u, \lambda)\|_{\mathcal{H}} \leq \|T(u, \lambda)\|_{\mathcal{H}} \leq C_2 \|(u, \lambda)\|_{\mathcal{H}} \quad \forall (u, \lambda) \in \mathcal{H}. \quad (3.1)$$

We now derive some important properties of the operators $L_{\tilde{h}}, T_{\tilde{h}}$ and L . Taking $v_{\tilde{h}} = L_{\tilde{h}} \mu$ in (2.7) and using the strong coerciveness of $\tilde{a}$ and the boundedness of b , we obtain

$$\|L_{\tilde{h}} \mu\|_{H^1(\Omega)} \leq C \|\mu\|_{\mathbf{R}^2} \quad \forall \mu \in \mathbf{R}^2,$$

with a positive constant C independent of $\tilde{h}$. It follows that the family of operators $\{T_{\tilde{h}}\}_{\tilde{h} > 0}$ is uniformly bounded. Thus, since B is bounded there exists $\tilde{C} > 0$, independent of $\tilde{h}$, such that

$$|A_{\tilde{h}}((z, \delta), (v, \mu))| \leq \tilde{C} \|(z, \delta)\|_{\mathcal{H}} \|(v, \mu)\|_{\mathcal{H}} \quad \forall (z, \delta), (v, \mu) \in \mathcal{H}. \quad (3.2)$$

In addition, since $L_{\tilde{h}} \mu$ (cf. (2.7)) is clearly the finite element approximation of $L\mu$ in the space $H_{\tilde{h}}$, we have the usual Cea estimate,

$$\|L\mu - L_{\tilde{h}} \mu\|_{H^1(\Omega)} \leq C \text{dist}(L\mu, H_{\tilde{h}}) \quad \forall \mu \in \mathbf{R}^2. \quad (3.3)$$

Hence, from (2.4), (2.8) and (3.3) we get

$$\|T(v, \mu) - T_{\tilde{h}}(v, \mu)\|_{\mathcal{H}} \leq C \operatorname{dist}(L\mu, H_{\tilde{h}}) \quad \forall (v, \mu) \in \mathcal{H}, \quad (3.4)$$

which yields,

$$|A((z, \delta), (v, \mu)) - A_{\tilde{h}}((z, \delta), (v, \mu))| \leq \bar{C} \operatorname{dist}(L\delta, H_{\tilde{h}}) \|(v, \mu)\|_{\mathcal{H}}, \quad (3.5)$$

for all $(z, \delta), (v, \mu) \in \mathcal{H}$.

In what follows we prove some useful lemmas. The letter C will denote a generic constant with different values in different places.

Lemma 3.1. *There exists a positive constant C such that*

$$\|\mu\|_{\mathbf{R}^2} \leq C \|L\mu\|_{H^1(\Omega)} \quad \forall \mu \in \mathbf{R}^2.$$

Proof: We first show that b satisfies the continuous Babuska-Brezzi condition. Let $\tilde{H}^1(\Omega)$ be the orthogonal of $H_0^1(\Omega)$. Then, since $\operatorname{Ker}(\gamma_0) = H_0^1(\Omega)$ we deduce that $\gamma_0 : \tilde{H}^1(\Omega) \rightarrow \mathbf{R}^2$ is an isomorphism. Thus, there exists a positive constant β such that

$$\|v\|_{H^1(\Omega)} \leq \beta \|\gamma_0 v\|_{\mathbf{R}^2} \quad \forall v \in \tilde{H}^1(\Omega).$$

It follows that

$$\begin{aligned} \|\mu\|_{\mathbf{R}^2} &= \sup_{\substack{\lambda \in \mathbf{R}^2 \\ \lambda \neq 0}} \frac{|\langle \lambda, \mu \rangle|}{\|\lambda\|_{\mathbf{R}^2}} = \sup_{\substack{v \in \tilde{H}^1(\Omega) \\ v \neq 0}} \frac{|\langle \gamma_0 v, \mu \rangle|}{\|\gamma_0 v\|_{\mathbf{R}^2}} \\ &= \sup_{\substack{v \in \tilde{H}^1(\Omega) \\ v \neq 0}} \frac{|b(v, \mu)|}{\|\gamma_0 v\|_{\mathbf{R}^2}} \leq \beta \sup_{\substack{v \in \tilde{H}^1(\Omega) \\ v \neq 0}} \frac{|b(v, \mu)|}{\|v\|_{H^1(\Omega)}}, \end{aligned}$$

and hence

$$\|\mu\|_{\mathbf{R}^2} \leq \beta \sup_{\substack{v \in H^1(\Omega) \\ v \neq 0}} \frac{|b(v, \mu)|}{\|v\|_{H^1(\Omega)}} \quad \forall \mu \in \mathbf{R}^2.$$

Finally, since from (2.3) we have $b(v, \mu) = -2\tilde{a}(v, L\mu)$ we conclude that

$$\|\mu\|_{\mathbf{R}^2} \leq \beta \sup_{\substack{v \in H^1(\Omega) \\ v \neq 0}} \frac{2|\tilde{a}(v, L\mu)|}{\|v\|_{H^1(\Omega)}} \leq C \|L\mu\|_{H^1(\Omega)},$$

which completes the proof. •

We now consider the trace operator $\gamma_1 : H^2(\Omega) \rightarrow \mathbf{R}^4$, where

$$\gamma_1 v := (v(0), v(1), v'(0), v'(1)) \quad \forall v \in C^2(\bar{\Omega}).$$

Then we have the following result.

Lemma 3.2. *For all $\lambda = (\lambda_0, \lambda_1) \in \mathbf{R}^2$ there exists $v \in H^2(\Omega)$ such that $\gamma_1 v = (0, 0, \lambda_0, \lambda_1)$ and*

$$\|v\|_{H^2(\Omega)} \leq C \|\lambda\|_{\mathbf{R}^2},$$

where C is a positive constant independent of λ and v .

Proof: Let us recall from the trace Theorem for one-dimensional Sobolev spaces (see e.g. Marti (1986)) that γ_1 is a linear and bounded operator with

$$\gamma_1(H^2(\Omega)) = \mathbf{R}^4 \quad \text{and} \quad \text{Ker}(\gamma_1) = H_0^2(\Omega).$$

Thus, similarly as in Lemma 3.1, if $\tilde{H}^2(\Omega)$ denotes the orthogonal of $H_0^2(\Omega)$, we obtain that $\gamma_1 : \tilde{H}^2(\Omega) \rightarrow \mathbf{R}^4$ is an isomorphism. This implies that there exists $C > 0$ such that

$$\|v\|_{H^2(\Omega)} \leq C \|\gamma_1 v\|_{\mathbf{R}^4} \quad \forall v \in \tilde{H}^2(\Omega).$$

Therefore, given a vector $\lambda := (\lambda_0, \lambda_1) \in \mathbf{R}^2$, it suffices to take

$$v := \gamma_1^{-1}(0, 0, \lambda_0, \lambda_1) \in \tilde{H}^2(\Omega)$$

to complete the proof. •

The following lemma provides a well known regularity result.

Lemma 3.3. *Let $p \in C^1(\bar{\Omega})$ and suppose that there exists $p_0 > 0$ such that $p(x) \geq p_0 \forall x \in \Omega$. Further, let $u \in H^1(\Omega)$ be the weak solution of the boundary value problem,*

$$\begin{aligned} -(p u')' + u &= \tilde{f} \quad \text{in } \Omega \\ u'(0) &= u'(1) = 0, \end{aligned}$$

with $\tilde{f} \in L^2(\Omega)$. Then $u \in H^2(\Omega)$ and there exists a positive constant C independent of u and $\tilde{f}$ such that

$$\|u\|_{H^2(\Omega)} \leq C \|\tilde{f}\|_{L^2(\Omega)}. \quad (3.6)$$

Proof: It is easily seen that u is characterized by the formulation:

$$\tilde{a}(u, v) = \tilde{F}(v) := \int_0^1 \tilde{f} v \quad \forall v \in H^1(\Omega).$$

Hence, the Lax-Milgram lemma provides the existence and uniqueness of u and the corresponding continuous-dependence result gives

$$\|u\|_{H^1(\Omega)} \leq C \|\tilde{f}\|_{L^2(\Omega)}. \quad (3.7)$$

On the other hand, since we have in the distributional sense

$$(p u')' = u - \tilde{f},$$

we deduce that $(p u')' \in L^2(\Omega)$, and therefore $p u' \in H^1(\Omega)$. Now, from the fact that $H^1(\Omega)$ is a Banach algebra (see e.g. Brézis (1984, chapter VIII)) and since clearly $\frac{1}{p} \in H^1(\Omega)$, we conclude that $u' = \frac{1}{p} (p u') \in H^1(\Omega)$. Thus, $u \in H^2(\Omega)$ and

$$\begin{aligned} \|u''\|_{L^2(\Omega)} &= \|(\frac{1}{p})' (p u') + \frac{1}{p} (p u')'\|_{L^2(\Omega)} \\ &= \|(\frac{1}{p})' (p u') + \frac{1}{p} (u - \tilde{f})\|_{L^2(\Omega)} \leq C \{ \|u\|_{H^1(\Omega)} + \|\tilde{f}\|_{L^2(\Omega)} \}, \end{aligned} \quad (3.8)$$

where C in (3.8) depends only on p . Finally, (3.7) and (3.8) yields (3.6). •

We are now in a position to prove the main regularity property of the operator L .

Lemma 3.4. *Let p be as in Lemma 3.3 and let L be the operator defined by (2.1) - (2.3). Then, $R(L) \subseteq H^2(\Omega)$ and there exists a positive constant $\tilde{C}$ such that*

$$\|L\mu\|_{H^2(\Omega)} \leq \tilde{C} \|\mu\|_{\mathbf{R}^2} \quad \forall \mu \in \mathbf{R}^2.$$

Proof: Let $\mu = (\mu_0, \mu_1) \in \mathbf{R}^2$ and define $\lambda = (\lambda_0, \lambda_1) := (\frac{\mu_0}{2p(0)}, -\frac{\mu_1}{2p(1)}) \in \mathbf{R}^2$. According to Lemma 3.2 we can choose $v \in H^2(\Omega)$ such that

$$\gamma_1 v = (0, 0, \lambda_0, \lambda_1) \quad \text{and} \quad \|v\|_{H^2(\Omega)} \leq C \|\lambda\|_{\mathbf{R}^2}. \quad (3.9)$$

It follows from (2.2) and (3.9) that $u := v - L\mu$ satisfies the boundary value problem

$$\begin{aligned} -(p u')' + u &= \tilde{f} \quad \text{in } \Omega \\ u'(0) &= u'(1) = 0, \end{aligned}$$

where $\tilde{f} := -(p v')' + v$. Furthermore, since $v \in H^2(\Omega)$ we have $\tilde{f} \in L^2(\Omega)$, with

$$\|\tilde{f}\|_{L^2(\Omega)} \leq C \|v\|_{H^2(\Omega)}. \quad (3.10)$$

Hence, in virtue of Lemma 3.3 we deduce that $u \in H^2(\Omega)$ and

$$\|u\|_{H^2(\Omega)} \leq C \|\tilde{f}\|_{L^2(\Omega)}. \quad (3.11)$$

This implies that $L\mu = v - u \in H^2(\Omega)$, and using (3.9), (3.10) and (3.11), we get

$$\begin{aligned} \|L\mu\|_{H^2(\Omega)} &\leq \|v\|_{H^2(\Omega)} + \|u\|_{H^2(\Omega)} \leq \|v\|_{H^2(\Omega)} + C\|\tilde{f}\|_{L^2(\Omega)} \\ &\leq C\|v\|_{H^2(\Omega)} \leq C\|\lambda\|_{\mathbf{R}^2} \leq \tilde{C}\|\mu\|_{\mathbf{R}^2}, \end{aligned}$$

which completes the proof. •

4. Solvability and approximation of $(P_{h,\tilde{h}})$.

In this section we give a detailed proof of Theorem 2.1. To begin with, we recall from Mikhlin (1965) the following important theorem.

Theorem 4.1. *Let $\mathcal{H}$ be a Hilbert space. Given a bounded bilinear form $A : \mathcal{H} \times \mathcal{H} \rightarrow \mathbf{R}$ and a linear functional $F \in \mathcal{H}'$, we consider the variational problem: Find $u \in \mathcal{H}$ such that*

$$A(u, v) = F(v) \quad \forall v \in \mathcal{H}. \quad (4.1)$$

Let $\{\mathcal{H}_h\}_{h>0}$ be a sequence of approximating finite dimensional subspaces of $\mathcal{H}$, and consider the Galerkin scheme: Find $u_h \in \mathcal{H}_h$ such that

$$A(u_h, v_h) = F(v_h) \quad \forall v_h \in \mathcal{H}_h. \quad (4.2)$$

Assume that the approximation property $\lim_{h \rightarrow 0} \text{dist}(v, \mathcal{H}_h) = 0$ holds for all $v \in \mathcal{H}$. Also, suppose that A satisfies Gårding's inequality and that (4.1) is uniquely solvable. Then there exist positive constants h_0 , $\tilde{C}$, independent of h , such that for all $h \leq h_0$,

$$\sup_{\substack{v_h \in \mathcal{H}_h \\ v_h \neq 0}} \frac{|A(z_h, v_h)|}{\|v_h\|_{\mathcal{H}}} \geq \tilde{C}\|z_h\|_{\mathcal{H}} \quad \forall z_h \in \mathcal{H}_h.$$

Moreover, the Cea estimate,

$$\|u - u_h\|_{\mathcal{H}} \leq C \text{dist}(u, \mathcal{H}_h) \quad (4.3)$$

holds for all $h \leq h_0$, where $u \in \mathcal{H}$ and $u_h \in \mathcal{H}_h$ denote the unique solutions of (4.1) and (4.2), respectively.

For a proof of this theorem we may also refer to Wendland (1988).

We now show that the bilinear form A (cf. (2.5), (2.6)) satisfies the usual Gårding inequality. It is important to remark that another way of verifying this fact consists of proving that the bilinear form B defined by (1.5) satisfies

a generalized Gårding inequality (for details we refer to *Gatica & Hsiao* (1994), *Araya* (1992), *Araya & Gatica* (1993)).

Theorem 4.2. *Let $\mathcal{H} := H^1(\Omega) \times \mathbf{R}^2$ and $\mathcal{V} := L^2(\Omega) \times \mathbf{R}^2$. Then there exist a positive constant α and a bounded bilinear form $K : \mathcal{H} \times \mathcal{V} \rightarrow \mathbf{R}$ such that*

$$A((w, \sigma), (w, \sigma)) \geq \alpha \|(w, \sigma)\|_{\mathcal{H}}^2 - K((w, \sigma), (w, \sigma)) \quad \forall (w, \sigma) \in \mathcal{H}. \quad (4.4)$$

Proof: Given $(w, \sigma) \in \mathcal{H}$ we get from (2.6)

$$\begin{aligned} A((w, \sigma), (w, \sigma)) &= a(w - L\sigma, w) - \frac{1}{2} b(w, \sigma) + b(w - L\sigma, \sigma) \\ &= a(w - L\sigma, w) + \frac{1}{2} b(w, \sigma) - b(L\sigma, \sigma). \end{aligned}$$

Moreover, since from (2.3) we clearly have $b(w, \sigma) = -2\tilde{a}(w, L\sigma)$ and also $b(L\sigma, \sigma) = -2\tilde{a}(L\sigma, L\sigma)$, the above expression reduces to

$$A((w, \sigma), (w, \sigma)) = a(w - L\sigma, w) - \tilde{a}(w, L\sigma) + 2\tilde{a}(L\sigma, L\sigma). \quad (4.5)$$

Then, if we define the bounded bilinear form $a_0 : L^2(\Omega) \times L^2(\Omega) \rightarrow \mathbf{R}$,

$$a_0(v, z) := \int_0^1 (1 - q) v z \, dx \quad \forall v, z \in L^2(\Omega),$$

the equation (4.5) can be rewritten as

$$\begin{aligned} A((w, \sigma), (w, \sigma)) &= \tilde{a}(w - L\sigma, w) - \tilde{a}(w, L\sigma) + \tilde{a}(L\sigma, L\sigma) \\ &\quad + \tilde{a}(L\sigma, L\sigma) - a_0(w - L\sigma, w), \end{aligned}$$

which is,

$$A((w, \sigma), (w, \sigma)) = \tilde{a}(w - L\sigma, w - L\sigma) + \tilde{a}(L\sigma, L\sigma) - a_0(w - L\sigma, w).$$

Hence, using the $H^1(\Omega)$ -ellipticity of $\tilde{a}$, Lemma 3.1, the definition of T (cf. (2.4)) and (3.1), we obtain that

$$\begin{aligned} A((w, \sigma), (w, \sigma)) &\geq C \{ \|w - L\sigma\|_{H^1(\Omega)}^2 + \|L\sigma\|_{H^1(\Omega)}^2 \} - a_0(w - L\sigma, w) \\ &\geq C \{ \|w - L\sigma\|_{H^1(\Omega)}^2 + \|\sigma\|_{\mathbf{R}^2}^2 \} - a_0(w - L\sigma, w) \\ &\geq C \|T(w, \sigma)\|_{\mathcal{H}}^2 - a_0(w - L\sigma, w) \end{aligned}$$

$$\geq \alpha \|(w, \sigma)\|_{\mathcal{H}}^2 - a_0(w - L\sigma, w).$$

Therefore, in order to complete the proof it suffices to define the bounded bilinear form $K : \mathcal{H} \times \mathcal{V} \rightarrow \mathbf{R}$,

$$K((w, \sigma), (v, \mu)) := a_0(w - L\sigma, v) \quad \forall (w, \sigma), (v, \mu) \in \mathcal{H},$$

which clearly verifies (4.4). •

Now, since (P) is assumed to be uniquely solvable, a direct application of Theorems 4.2 and 4.1 implies that there exist $h_0 > 0$ and $\tilde{C} > 0$, independent of h , such that for all $h \leq h_0$,

$$\sup_{\substack{(v_h, \mu_h) \in \mathcal{H}_h \\ (v_h, \mu_h) \neq 0}} \frac{|A((z_h, \delta_h), (v_h, \mu_h))|}{\|(v_h, \mu_h)\|_{\mathcal{H}}} \geq \tilde{C} \|(z_h, \delta_h)\|_{\mathcal{H}} \quad (4.6)$$

for all $(z_h, \delta_h) \in \mathcal{H}_h$. It follows that (P_h) has a unique solution $(w_h, \sigma_h) \in \mathcal{H}_h$ for all $h \leq h_0$, and if $(w, \sigma) \in \mathcal{H}$ denotes the unique solution of (P) , we have the Cea estimate from (4.3),

$$\|(w, \sigma) - (w_h, \sigma_h)\|_{\mathcal{H}} \leq C \operatorname{dist}(w, H_h) \quad \forall h \leq h_0. \quad (4.7)$$

Here, we have used the fact that

$$\operatorname{dist}((w, \sigma), \mathcal{H}_h) := \operatorname{dist}(w, H_h) + \operatorname{dist}(\sigma, \mathbf{R}^2) = \operatorname{dist}(w, H_h).$$

On the other hand, the approximation property $(\mathbf{AP})_h$ together with Lemma 3.4 provide,

$$\operatorname{dist}(L\delta_h, H_{\tilde{h}}) \leq C\tilde{h} \|L\delta_h\|_{H^2(\Omega)} \leq C\tilde{h} \|\delta_h\|_{\mathbf{R}^2} \leq C\tilde{h} \|(z_h, \delta_h)\|_{\mathcal{H}},$$

and hence, using (4.6) and (3.5) we conclude that for all $(z_h, \delta_h) \in \mathcal{H}_h$,

$$\sup_{\substack{(v_h, \mu_h) \in \mathcal{H}_h \\ (v_h, \mu_h) \neq 0}} \frac{|A_{\tilde{h}}((z_h, \delta_h), (v_h, \mu_h))|}{\|(v_h, \mu_h)\|_{\mathcal{H}}} \geq (\tilde{C} - C\tilde{h}) \|(z_h, \delta_h)\|_{\mathcal{H}}.$$

Thus, there exist positive constants h_0, β , independent of h and $\tilde{h}$, such that for all $h, \tilde{h} \leq h_0$,

$$\sup_{\substack{(v_h, \mu_h) \in \mathcal{H}_h \\ (v_h, \mu_h) \neq 0}} \frac{|A_{\tilde{h}}((z_h, \delta_h), (v_h, \mu_h))|}{\|(v_h, \mu_h)\|_{\mathcal{H}}} \geq \beta \|(z_h, \delta_h)\|_{\mathcal{H}} \quad (4.8)$$

for all $(z_h, \delta_h) \in \mathcal{H}_h$, and therefore the discrete problem $(P_{h,\tilde{h}})$ has a unique solution $(\bar{w}_h, \bar{\sigma}_h) \in \mathcal{H}_h$.

Furthermore, applying (4.8) to $(z_h, \delta_h) := (w_h, \sigma_h) - (\bar{w}_h, \bar{\sigma}_h) \in \mathcal{H}_h$ and using that $A_{\tilde{h}}((\bar{w}_h, \bar{\sigma}_h), (v_h, \mu_h)) = F(v_h, \mu_h) = A((w, \sigma), (v_h, \mu_h))$, we find after some computations,

$$\begin{aligned} \beta ||(w_h, \sigma_h) - (\bar{w}_h, \bar{\sigma}_h)||_{\mathcal{H}} &\leq \sup_{\substack{(v_h, \mu_h) \in \mathcal{H}_h \\ (v_h, \mu_h) \neq 0}} \frac{|A_{\tilde{h}}((w_h, \sigma_h) - (w, \sigma), (v_h, \mu_h))|}{||(v_h, \mu_h)||_{\mathcal{H}}} \\ &+ \sup_{\substack{(v_h, \mu_h) \in \mathcal{H}_h \\ (v_h, \mu_h) \neq 0}} \frac{|A_{\tilde{h}}((w, \sigma), (v_h, \mu_h)) - A((w, \sigma), (v_h, \mu_h))|}{||(v_h, \mu_h)||_{\mathcal{H}}}. \end{aligned}$$

Then, using (3.2) and (3.5) we get

$$\beta ||(w_h, \sigma_h) - (\bar{w}_h, \bar{\sigma}_h)||_{\mathcal{H}} \leq C \{ ||(w, \sigma) - (w_h, \sigma_h)||_{\mathcal{H}} + \text{dist}(L\sigma, H_{\tilde{h}}) \},$$

which together with (4.7) and the triangle inequality give,

$$||(w, \sigma) - (\bar{w}_h, \bar{\sigma}_h)||_{\mathcal{H}} \leq C \{ \text{dist}(w, H_h) + \text{dist}(L\sigma, H_{\tilde{h}}) \}. \quad (4.9)$$

Now, since $(u, \lambda) = T(w, \sigma)$ and $(u_{h,\tilde{h}}, \lambda_h) = T_{\tilde{h}}(\bar{w}_h, \bar{\sigma}_h)$, we obtain

$$\begin{aligned} ||(u, \lambda) - (u_{h,\tilde{h}}, \lambda_h)||_{\mathcal{H}} &= ||T(w, \sigma) - T_{\tilde{h}}(\bar{w}_h, \bar{\sigma}_h)||_{\mathcal{H}} \\ &\leq ||T(w, \sigma) - T_{\tilde{h}}(w, \sigma)||_{\mathcal{H}} + ||T_{\tilde{h}}(w, \sigma) - T_{\tilde{h}}(\bar{w}_h, \bar{\sigma}_h)||_{\mathcal{H}} \\ &\leq C \{ \text{dist}(L\sigma, H_{\tilde{h}}) + ||(w, \sigma) - (\bar{w}_h, \bar{\sigma}_h)||_{\mathcal{H}} \}, \end{aligned} \quad (4.10)$$

where the estimate (3.4) and the uniform boundedness of $\{T_{\tilde{h}}\}$ have been used in the last inequality. But, since $w = u - 2L\lambda$ and $\sigma = -2\lambda$, we deduce that

$$\begin{aligned} &\text{dist}(w, H_h) + \text{dist}(L\sigma, H_{\tilde{h}}) \\ &\leq C \{ \text{dist}(u, H_h) + \text{dist}(L\lambda, H_h) + \text{dist}(L\lambda, H_{\tilde{h}}) \} \\ &\leq C \{ \text{dist}(u, H_h) + (h + \tilde{h}) ||\lambda||_{\mathbf{R}^2} \}. \end{aligned} \quad (4.11)$$

Consequently, (4.9), (4.10) and (4.11) provide the Cea estimate (2.11). Similarly, the asymptotic rate of convergence (2.12) follows easily from (4.11) and $(\mathbf{AP})_{\mathbf{h}}$. This completes the proof of Theorem 2.1.

It is important to remark here that, in virtue of Theorem 2.1, the mesh sizes h and $\tilde{h}$ need not to be related, and hence they can be chosen freely, as long as they belong to the interval $(0, h_0)$. Now, for the particular case $\tilde{h} = h$, we have from (2.7) and (2.8),

$$T_h : \mathcal{H} \rightarrow \mathcal{H}$$

$$T_h(v, \mu) := (v - L_h \mu, -\frac{1}{2} \mu) \quad \forall (v, \mu) \in \mathcal{H},$$

where for each $\mu \in \mathbf{R}^2$, $L_h \mu \in H_h$ is the unique solution of

$$\bar{a}(v_h, L_h \mu) = -\frac{1}{2} b(v_h, \mu) \quad \forall v_h \in H_h.$$

Then, it is not difficult to prove, similarly as we did for T , that in this case $T_h : \mathcal{H}_h \rightarrow \mathcal{H}_h$ becomes an isomorphism, and therefore the pair

$$(u_h, \lambda_h) := T_h(\bar{w}_h, \bar{\sigma}_h) \in \mathcal{H}_h$$

constitutes the Galerkin solution of the formulation (1.2) (or (1.4)). In other words, (u_h, λ_h) is the unique element in $\mathcal{H}_h$ satisfying

$$B((u_h, \lambda_h), (v_h, \mu_h)) = F(v_h, \mu_h) \quad \forall (v_h, \mu_h) \in \mathcal{H}_h. \quad (4.12)$$

The above remarks show that when h and $\tilde{h}$ coincide the mapping T_h provides, through the Petrov-Galerkin scheme $(P_{h, \tilde{h}})$, a sort of preconditioning technique to compute the solution of (4.12). For further details on preconditioning methods for saddle point problems we may refer to *Bramble & Pasciak* (1988).

5. Matrix formulation and numerical experiments.

Let us first provide the matrix formulation of $(P_{h, \tilde{h}})$. Let $\{e_1, \dots, e_N\}$ and $\{\tilde{e}_1, \dots, \tilde{e}_M\}$ be basis of the finite element subspaces H_h and $H_{\tilde{h}}$, respectively. We certainly assume that both H_h and $H_{\tilde{h}}$ belong to the family $S_h^{k,r}$, with $k \geq 2$ and $r \geq 1$. Then, the corresponding base of $\mathcal{H}_h := H_h \times \mathbf{R}^2$ is given by the set

$$\{(e_j, (0, 0)) / j = \overline{1, N}\} \cup \{(0, (1, 0)), (0, (0, 1))\}. \quad (5.1)$$

Now, let $\vec{\alpha} := (\alpha_1, \dots, \alpha_M)^T$ and $\vec{\beta} := (\beta_1, \dots, \beta_M)^T$ be vectors of $\mathbf{R}^M$ such that

$$L_{\tilde{h}}(1, 0) = \sum_{j=1}^M \alpha_j \tilde{e}_j \quad \text{and} \quad L_{\tilde{h}}(0, 1) = \sum_{j=1}^M \beta_j \tilde{e}_j,$$

and let $\tilde{\mathbf{A}} := (\tilde{\mathbf{a}}_{ij})_{M \times M}$ be the stiffness matrix associated to the bilinear form $\tilde{a}$, i.e.,

$$\tilde{\mathbf{a}}_{ij} = \tilde{a}(\tilde{e}_i, \tilde{e}_j) = \int_0^1 [p\tilde{e}'_i\tilde{e}'_j + \tilde{e}_i\tilde{e}_j] dx.$$

It follows from (1.3) and (2.7) that $\vec{\alpha}$ and $\vec{\beta}$ are, respectively, the solutions of the linear systems,

$$\tilde{\mathbf{A}} \vec{\alpha} = - \begin{bmatrix} 1/2 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \text{and} \quad \tilde{\mathbf{A}} \vec{\beta} = - \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1/2 \end{bmatrix}.$$

Here we have assumed, without loss of generality, that $\tilde{e}_1(0) = \tilde{e}_M(1) = 1$, $\tilde{e}_1(1) = \tilde{e}_M(0) = 0$, and $\tilde{e}_j(0) = \tilde{e}_j(1) = 0 \ \forall j = 2, M-1$. The same assumption is valid for the base $\{e_1, \dots, e_N\}$ of H_h .

On the other hand, in virtue of (2.8) and (2.9) we obtain for all $(z, \delta), (v, \mu) \in \mathcal{H}$,

$$A_{\tilde{h}}((z, \delta), (v, \mu)) = a(z - L_{\tilde{h}} \delta, v) - \frac{1}{2} b(v, \delta) + b(z - L_{\tilde{h}} \delta, \mu).$$

Hence, using the base of $\mathcal{H}_h$ given in (5.1), and denoting

$$\bar{w}_h := \sum_{j=1}^N \mathbf{w}_j e_j \in H_h \quad \text{and} \quad \bar{\sigma}_h := \begin{bmatrix} \sigma_{0h} \\ \sigma_{1h} \end{bmatrix} \in \mathbf{R}^2,$$

the discrete scheme $(P_{h, \tilde{h}})$ can be formulated in matrix form as follows: Find $\vec{\mathbf{w}} := (\mathbf{w}_1, \dots, \mathbf{w}_N)^T \in \mathbf{R}^N$ and $\bar{\sigma}_h \in \mathbf{R}^2$ such that

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \vec{\mathbf{w}} \\ \bar{\sigma}_h \end{bmatrix} = \begin{bmatrix} \mathbf{F} \\ \mathbf{G} \end{bmatrix}, \quad (5.2)$$

where $\mathbf{A} := (\mathbf{a}_{ij})_{N \times N}$ is the stiffness matrix of the bilinear form a , i.e.,

$$\mathbf{a}_{ij} = a(e_i, e_j) = \int_0^1 [pe'_i e'_j + qe_i e_j] dx,$$

$\mathbf{B} := (\mathbf{b}_{ij})_{N \times 2}$, with

$$\begin{aligned} \mathbf{b}_{i1} &= -a(L_{\tilde{h}}(1, 0), e_i) - \frac{1}{2} b(e_i, (1, 0)) = -\sum_{j=1}^M \alpha_j a(\tilde{e}_j, e_i) - \frac{1}{2} e_i(0) \\ \mathbf{b}_{i2} &= -a(L_{\tilde{h}}(0, 1), e_i) - \frac{1}{2} b(e_i, (0, 1)) = -\sum_{j=1}^M \beta_j a(\tilde{e}_j, e_i) - \frac{1}{2} e_i(1), \end{aligned} \quad (5.3)$$

$\mathbf{C} := (\mathbf{c}_{ij})_{2 \times N}$, with

$$\begin{aligned} \mathbf{c}_{1j} &= b(e_j, (1, 0)) = e_j(0) \\ \mathbf{c}_{2j} &= b(e_j, (0, 1)) = e_j(1), \end{aligned}$$

or,

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 1 \end{bmatrix},$$

and

$$\mathbf{D} = \begin{bmatrix} -L_{\tilde{h}}(1, 0)(0) & -L_{\tilde{h}}(0, 1)(0) \\ -L_{\tilde{h}}(1, 0)(1) & -L_{\tilde{h}}(0, 1)(1) \end{bmatrix} = \begin{bmatrix} -\alpha_1 & -\beta_1 \\ -\alpha_M & -\beta_M \end{bmatrix}$$

In addition, $\mathbf{G} = (g_0, g_1)^T \in \mathbf{R}^2$, and the vector $\mathbf{F} := (\mathbf{f}_j)_{N \times 1}$ is defined by

$$\mathbf{f}_j = \int_0^1 f e_j dx.$$

It is important to remark that Theorem 2.1 guarantees the unique solvability of the linear systems (5.2) for all $N, M \geq N_0 = O(h_0^{-1})$. Now, once $\vec{\mathbf{w}}$ and $\vec{\sigma}_h$ are obtained from (5.2), the Petrov-Galerkin approximation of the solution (u, λ) of (P) is computed according to (2.10),

$$u_{h, \tilde{h}} = \bar{w}_h - L_{\tilde{h}} \bar{\sigma}_h = \sum_{j=1}^N \mathbf{w}_j e_j - \sum_{j=1}^M (\sigma_{0h} \alpha_j + \sigma_{1h} \beta_j) \tilde{e}_j, \quad (5.4)$$

and

$$\lambda_h = (\lambda_{0h}, \lambda_{1h})^T = -\frac{1}{2} \bar{\sigma}_h = -\frac{1}{2} (\sigma_{0h}, \sigma_{1h})^T$$

In this way, the corresponding approximations of the derivatives $u'(0)$ and $u'(1)$ are given respectively, by

$$\frac{\lambda_{0h}}{p(0)} \quad \text{and} \quad -\frac{\lambda_{1h}}{p(1)}.$$

We notice that for the particular case $\tilde{h} = h$, the matrix $\mathbf{B}$ from (5.3) becomes

$$\mathbf{B} = -\mathbf{A} [\vec{\alpha} \vec{\beta}] - \mathbf{C}^T ,$$

and the computation of $u_{h,\tilde{h}}$ from (5.4) reduces to

$$u_{h,\tilde{h}} = \sum_{j=1}^N (\mathbf{w}_j - \sigma_{0h} \alpha_j - \sigma_{1h} \beta_j) e_j .$$

On the other hand, a two-point boundary value problem solver, using the Petrov-Galerkin scheme (5.2) with linear finite elements, has been implemented and tested on a variety of examples. Hence, given the uniform partitions of Ω , $0 = x_1 < x_2 < \dots < x_N = 1$, and $0 = \tilde{x}_1 < \tilde{x}_2 < \dots < \tilde{x}_M = 1$, with $h = \frac{1}{N-1}$ and $\tilde{h} = \frac{1}{M-1}$, we consider

$$H_h = \{ u : \Omega \rightarrow \mathbf{R} / u|_{[x_j, x_{j+1}]} \text{ is affine } \forall j = \overline{1, N-1} \} ,$$

and similarly,

$$H_{\tilde{h}} = \{ u : \Omega \rightarrow \mathbf{R} / u|_{[\tilde{x}_j, \tilde{x}_{j+1}]} \text{ is affine } \forall j = \overline{1, M-1} \} .$$

Also, we adopt the notation,

$$\|u - u_{h,\tilde{h}}\|_{\infty} := \max\{|u(x_j) - u_{h,\tilde{h}}(x_j)| / j = \overline{1, N}\} .$$

The behaviour of the algorithm is demonstrated by the three examples whose coefficients are given in Table 5.1. The numerical results are shown from Table 5.2 through Table 5.7 below. Since, as commented in Section 4, h and $\tilde{h}$ can be choosen freely, we have taken for simplicity $\tilde{h} = h$ and also $\tilde{h} = 2h$. All calculations cited below were carried out in double precision on a 80286 IBM Compatible PC. As predicted by the error estimates (2.11) and (2.12) both cases $\tilde{h} = 2h$ and $\tilde{h} = h$ provide the same order of convergence. In addition, a nice feature of the method is the fast convergence of $\frac{\lambda_{0h}}{p(0)}$ and $-\frac{\lambda_{1h}}{p(1)}$ to $u'(0)$ and $u'(1)$, respectively.

Table 5.1

	Example 1	Example 2	Example 3
$p(x)$	1	e^{x-x^2}	$(x^3+2)^2$
$q(x)$	x	$-x$	$(x^3+2)e^x$
Exact Solution $u(x)$	$\sin(\pi x)$	e^{x^2-x}	$\frac{x+1}{x^3+2}$
(g_0, g_1)	$(0, 0)$	$(1, 1)$	$(\frac{1}{2}, \frac{2}{3})$
$(u'(0), u'(1))$	$(\pi, -\pi)$	$(-1, 1)$	$(\frac{1}{2}, -\frac{1}{3})$

Table 5.2: Example 1, $\tilde{h} = 2h$

N	$\ u - u_{h,\tilde{h}}\ _{H^1(\Omega)}$	$\ u - u_{h,\tilde{h}}\ _\infty$	$\lambda_{0h} / p(0)$	$-\lambda_{1h} / p(1)$
5	0.4999E+00	0.2403E-02	3.147802	-3.151168
9	0.2513E+00	0.6142E-03	3.143165	-3.143922
17	0.1258E+00	0.1564E-03	3.141987	-3.142171
33	0.6294E-01	0.3910E-04	3.141693	-3.141739
65	0.3147E-01	0.9770E-05	3.141617	-3.141626
129	0.1574E-01	0.2882E-05	3.141653	-3.141609

Table 5.3: Example 1, $\tilde{h} = h$

N	$\ u - u_{h,\tilde{h}}\ _{H^1(\Omega)}$	$\ u - u_{h,\tilde{h}}\ _{\infty}$	$\lambda_{0h} / p(0)$	$-\lambda_{1h} / p(1)$
5	0.4999E+00	0.2403E-02	3.147802	-3.151169
9	0.2514E+00	0.6145E-03	3.143165	-3.143923
17	0.1259E+00	0.1566E-03	3.141988	-3.142172
33	0.6295E-01	0.3928E-04	3.141693	-3.141738
65	0.3148E-01	0.9893E-05	3.141619	-3.141630
129	0.1574E-01	0.2543E-05	3.141600	-3.141603

Table 5.4: Example 2, $\tilde{h} = 2h$

N	$\ u - u_{h,\tilde{h}}\ _{H^1(\Omega)}$	$\ u - u_{h,\tilde{h}}\ _{\infty}$	$\lambda_{0h} / p(0)$	$-\lambda_{1h} / p(1)$
5	0.1464E+00	0.3475E-02	-0.997895	0.995793
9	0.7359E-01	0.8894E-03	-0.999473	0.998917
17	0.3685E-01	0.2239E-03	-0.999868	0.999727
33	0.1843E-01	0.5811E-04	-0.999958	0.999923
65	0.9215E-02	0.1683E-04	-0.999982	0.999959
99	0.6018E-02	0.4910E-05	-1.000007	1.000010

Table 5.5: Example 2, $\tilde{h} = h$

N	$\ u - u_{h,\tilde{h}}\ _{H^1(\Omega)}$	$\ u - u_{h,\tilde{h}}\ _{\infty}$	$\lambda_{0h} / p(0)$	$-\lambda_{1h} / p(1)$
5	0.1464E+00	0.3475E-02	-0.997895	0.995792
9	0.7359E-01	0.8897E-03	-0.999473	0.998917
17	0.3685E-01	0.2238E-03	-0.999869	0.999727
33	0.1843E-01	0.5609E-04	-0.999968	0.999932
65	0.9215E-02	0.1405E-04	-0.999992	0.999983
99	0.6018E-02	0.6005E-05	-0.999996	0.999993

Table 5.6: Example 3, $\tilde{h} = 2h$

N	$\ u - u_{h,\tilde{h}}\ _{H^1(\Omega)}$	$\ u - u_{h,\tilde{h}}\ _{\infty}$	$\lambda_{0h} / p(0)$	$-\lambda_{1h} / p(1)$
5	0.6665E-01	0.2402E-02	0.506168	-0.332139
9	0.3328E-01	0.5882E-03	0.501526	-0.333038
17	0.1664E-01	0.1475E-03	0.500385	-0.333262
33	0.8317E-02	0.3691E-04	0.500106	-0.333313
65	0.4158E-02	0.8260E-05	0.499995	-0.333341
99	0.2716E-02	0.4559E-05	0.500008	-0.333312

Table 5.7: Example 3, $\tilde{h} = h$

N	$\ u - u_{h,\tilde{h}}\ _{H^1(\Omega)}$	$\ u - u_{h,\tilde{h}}\ _{\infty}$	$\lambda_{0h} / p(0)$	$-\lambda_{1h} / p(1)$
5	0.6665E-01	0.2402E-02	0.506168	-0.332139
9	0.3328E-01	0.5883E-03	0.501526	-0.333038
17	0.1664E-01	0.1463E-03	0.500381	-0.333260
33	0.8317E-02	0.3668E-04	0.500095	-0.333315
65	0.4158E-02	0.9197E-05	0.500024	-0.333329
99	0.2716E-02	0.3951E-05	0.500010	-0.333332

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Received: April 20, 1993.

Rodolfo A. Araya and Gabriel N. Gatica
Departamento de Ingeniería Matemática
Facultad de Ciencias Físicas y Matemáticas
Universidad de Concepción
Casilla 4009, Concepción, Chile.