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FUBINI'S THEOREM VIA MIKUSINSKI'S CHARACTERIZATION OF THE LEBESGUE INTEGRAL

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Abstract

We generalize a characterization of the Lebesgue integral in $\mathbb{R}^n$ due to J Mikusinski to abstract measure spaces and use the characterization to give a proof of Fubini's Theorem

1. Introduction.

In addition to the general convergence theorems which lead to the completeness of the L^p -spaces for the Lebesgue integral, one of the great successes of the Lebesgue theory is the validity of the general Fubini theorems relating the equality of the product and iterated integrals. If the product measure is defined by means of the outer measure induced by the product of the two given measures on the measurable rectangles, one of the common approaches to the Fubini Theorem for the Lebesgue integral is to first establish the result for characteristic functions in the product space and then to establish the result for general integrable functions by approximating them by simple functions ([1] §22, [6] §12.4). The proof of Fubini's Theorem for characteristic functions is somewhat tedious and usually requires the use of several approximation theorems for sets which are measurable with respect to an outer measure ([1] 22.4, [6] 12.4.18). In this note we would like to point out that a characterization of the Lebesgue integral due to Jan Mikusinski can be used to bypass the proof of Fubini's Theorem for characteristic functions, and the proof of the Fubini Theorem for integrable functions is no more difficult than the usual proofs based on approximation by simple functions since Mikusinski's characterization is via a series of simple functions. Mikusinski's characterization of the Lebesgue integral is of independent interest since it is quite elementary and a development of the Lebesgue integral can actually be based on the characterization ([2], [4], [5]).

Mikusinski has only given a characterization of the Lebesgue integral for functions defined in $\mathbb{R}^n$ so in the next section we give an analogous characterization for functions which are Lebesgue integrable with respect to an abstract measure.

2. Mikusinski's Characterization of the Lebesgue Integral.

Let S be a non-empty set, $\mathcal{S}$ a semi-ring of subsets of S and μ a countably additive measure on $\mathcal{S}$ ([1] §9, 10). Then μ induces an outer measure μ^* on S ,

$$\mu^*(A) = \inf \left\{ \sum_{k=1}^{\infty} \mu(I_k) : I_k \in \mathcal{S}, A \subseteq \bigcup I_k \right\},$$

and the restriction of μ^* , still denoted by μ , to the class of μ^* -measurable sets is a complete countably additive measure ([1] §11).

In $\mathbb{R}^n$, let the set $\mathcal{S}$ consisting of all half-closed intervals of the form $I = [a_1, b_1) \times \cdots \times [a_n, b_n)$ form a semi-ring and the set function $\lambda(I) = (b_1 - a_1) \cdots (b_n - a_n)$ generates Lebesgue measure on $\mathbb{R}^n$.

An $\mathcal{S}$ -simple function is a function of the form $\varphi = \sum_{k=1}^n a_k \chi_{I_k}$ where $a_k \in \mathbb{R}$, $I_k \in \mathcal{S}$ and χ_I denotes the characteristic function of a set I .

For Mikusinski's description of the Lebesgue integral we require two preliminary lemmas.

Lemma 1. Let E be μ -null and $\epsilon > 0$. There exists $\{J_k\} \subseteq \mathcal{S}$ such that $\sum_{k=1}^{\infty} \mu(J_k) < \epsilon$ and $\sum_{k=1}^{\infty} \chi_{J_k}(s) = \infty$ for $s \in E$ [so s belong to infinitely many J_k for $s \in E$].

Proof. For each i , there is a sequence $\{I_{ij} : j \in \mathbb{N}\} \subseteq \mathcal{S}$ covering E such that $\sum_{j=1}^{\infty} \mu(I_{ij}) < \epsilon/2^i$. Now arrange the double sequence $\{I_{ij} : i, j \in \mathbb{N}\}$ into a sequence $\{J_k\}$. If $s \in E$ s belongs to infinitely many $\{J_k\}$ so $\{\chi_{J_k}(s)\}$ is 1 infinitely often and

$$\sum_{k=1}^{\infty} \mu(J_k) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \mu(I_{ij}) < \epsilon$$

We say that a series $\sum a_k$ is absolutely convergent to x if the series is absolutely convergent and $\sum_{k=1}^{\infty} a_k = x$.

Lemma 2. If f is μ -integrable, there is a sequence of $\mathcal{S}$ -simple functions $\{\alpha_k\}$ such that the series $\sum_k \alpha_k$ is absolutely convergent to f μ -a.e. and $\sum_{k=1}^{\infty} \int_S |\alpha_k| d\mu < \infty$.

Proof. There exists a sequence of $\mathcal{S}$ -simple functions $\{\varphi_k\}$ such that $\|\varphi_k - f\|_1 \rightarrow 0$. ([1] 21.2) so there is a subsequence, which we continue to denote by $\{\varphi_k\}$, such that $\varphi_k \rightarrow f$ μ -a.e., and we may additionally assume that $\|\varphi_{k+1} - \varphi_k\|_1 < 1/2^k$. Set $\varphi_0 = 0$ and $\alpha_k = \varphi_k - \varphi_{k-1}$ for $k \geq 1$. Then $\sum_{k=1}^n \alpha_k = \varphi_n \rightarrow f$ μ -a.e. or $\sum_{k=1}^{\infty} \alpha_k = f$ μ -a.e. Since $\sum_{k=1}^{\infty} \int_S |\alpha_k| d\mu < \infty$, the Monotone Convergence Theorem implies that $\sum_{k=1}^{\infty} |\alpha_k|$ converges in $\mathbb{R}$ μ -a.e. so the series $\sum \alpha_k$ is absolutely convergent to f μ -a.e.

We now give Mikusinski's characterization of the Lebesgue integral.

Theorem 3. $f : S \rightarrow \mathbb{R}$ is μ -integrable if and only if there is a sequence of $\mathcal{S}$ -simple functions $\{\psi_k\}$ satisfying

$$(a) \quad \sum_{k=1}^{\infty} \int_S |\psi_k| d\mu < \infty,$$

$$(b) \quad f(s) = \sum_{k=1}^{\infty} \psi_k(s) \text{ for any } s \text{ for which } \sum_{k=1}^{\infty} |\psi_k(s)| < \infty$$

In this case, $\int_S f d\mu = \sum_{k=1}^{\infty} \int_S \psi_k d\mu$

Proof. If the conditions hold, the Monotone Convergence Theorem implies that $\sum |\psi_k|$ is μ -integrable so the series $\sum |\psi_k|$ converges in $\mathbb{R}$ μ -a.e. By hypothesis,

$$f(s) \sum_{k=1}^{\infty} \psi_k(s)$$

for such points so f is μ -integrable and the Dominated Convergence Theorem implies that $\int_S f d\mu = \sum_{k=1}^{\infty} \int_S \psi_k d\mu$

Conversely, assume that f is μ -integrable. Let $\{\alpha_k\}$ be in Lemma 2 and let E be a μ -null set where $\sum_{k=1}^{\infty} |\alpha_k(s)| < \infty$ and $f(s) = \sum_{k=1}^{\infty} \alpha_k(s)$ for $s \notin E$. Let $\{J_k\}$ be as in Lemma 1 (with respect to E) and set $\beta_k = \chi_{J_k}$. Define a sequence of $\mathcal{S}$ -simple functions, $\{\psi_k\}$, by

$$\alpha_1, \beta_1, -\beta_1, \alpha_2, \beta_2, -\beta_2, \quad [\psi_{3k-2} = \alpha_k, \psi_{3k-1} = \beta_k, \psi_{3k} = -\beta_k]$$

If $s \in E$ the series $\sum |\psi_k(s)|$ diverges since $\beta_k(s) = 1$ infinitely often. If $\sum_{k=1}^{\infty} |\psi_k(s)| < \infty$, then $\sum_{k=1}^{\infty} \beta_k(s) < \infty$ [the series actually only has a finite number of non-zero terms] so $s \notin E$ and $\sum_{k=1}^{\infty} \psi_k(s) = \sum_{k=1}^{\infty} \alpha_k(s) = f(s)$. Moreover,

$$\sum_{k=1}^{\infty} \int_S |\psi_k| d\mu \leq \sum_{k=1}^{\infty} \int_S |\alpha_k| d\mu + 2 \sum_{k=1}^{\infty} \int_S |\beta_k| d\mu < \infty$$

by Lemmas 1 and 2

Note, in particular, if conditions (a) and (b) are satisfied, the function f is μ -measurable and the series $\sum \psi_k$ converges to f μ -a.e.

One of the features of Mikusinski's characterization of the Lebesgue integral given in Theorem 3 is that null sets are never mentioned. This is used to good advantage in the proof of Fubini's Theorem given in Theorem 4

Mikusinski's original characterization of the Lebesgue integral was for functions defined in $\mathbb{R}^n$ in this case the characterization has an even simpler form. The $\mathcal{S}$ -simple functions $\{\psi_k\}$ in Theorem 3 are replaced by characteristic functions, χ_I , of half-closed intervals $I = [a_1, b_1) \times \dots \times [a_n, b_n)$ (called **brick functions** by Mikusinski, see [5]). In this case the value of the integral only requires that the measure of such a half-closed interval be defined so if the definition of the Lebesgue integral is based on this characterization, it is not necessary to develop properties of the Lebesgue measure, indeed, they follow from properties of the integral (for details, see [2] or [5]). It is also very easy to define the integral for functions with values in a Banach space and the resulting integral is just the Bochner integral

([5]). It should be pointed out that this characterization of the Lebesgue integral was also given earlier by MacNeille ([3]).

3. Fubini's Theorem.

In this section we use Mikusinski's characterization of the Lebesgue integral to give a proof of Fubini's Theorem. Let $(S, \mathcal{M}, \mu)$ and $(T, \mathcal{N}, \nu)$ be complete measure spaces and let $\mathcal{S}$ be the semi-ring consisting of all rectangles $A \times B$ for $A \in \mathcal{M}$, $B \in \mathcal{N}$. Define the product measure $\mu \times \nu$ on $\mathcal{S}$ by $\mu \times \nu(A \times B) = \mu(A)\nu(B)$ [here $0 \cdot \infty = 0$ as usual]. Then $\mu \times \nu$ is countably additive on $\mathcal{S}$ ([1] 22.1) and has a complete countably additive extension, denoted by $\mu \times \nu$, to the σ -algebra, $\sum$, of $(\mu \times \nu)^*$ -measurable sets.

If $f : S \times T \longrightarrow \mathbb{R}$, for $s \in S$ the function $f(s, \cdot) : T \longrightarrow \mathbb{R}$ is defined by $f(s, \cdot)(t) = f(s, t)$, $t \in T$. If f is a $\mu \times \nu$ -integrable, $\mathcal{S}$ -simple function, then by definition of the product measure

$$\int_{S \times T} f d\mu \times \nu = \int_S \int_T f(s, t) d\nu(t) d\mu(s),$$

i.e., Fubini's Theorem is immediate for such functions.

In our statement of Fubini's Theorem we follow the usual convention of agreeing that if a function g is only defined μ -a.e. in S then it is extended to S , by setting $g = 0$ on the μ -null set where it is not defined.

Theorem 4 (Fubini). Let $f : S \times T \longrightarrow \mathbb{R}$ be $\mu \times \nu$ -integrable. Then

- (i) $f(s, \cdot)$ is ν -integrable for μ almost all $s \in S$,
- (ii) $s \longrightarrow \int_T f(s, \cdot) d\nu$ is μ -integrable, and
- (iii) $\int_{S \times T} f d\mu \times \nu = \int_S \int_T f(s, \cdot) d\nu d\mu(s) = \int_S \int_T f(s, t) d\nu(t) d\mu(s)$.

Proof. By Theorem 3, there is a sequence $\{\psi_k\}$ of $\mathcal{S} \times T$ -simple functions such that $\sum_{k=1}^{\infty} \int_{S \times T} |\psi_k| d\mu \times \nu < \infty$ and $f(s, t) = \sum_{k=1}^{\infty} \psi_k(s, t)$ for all (s, t) for which the series $\sum_{k=1}^{\infty} |\psi_k(s, t)|$ converges and

$$(1) \quad \int_{S \times T} f d\mu \times \nu = \sum_{k=1}^{\infty} \int_{S \times T} \psi_k d\mu \times \nu.$$

By the Monotone Convergence Theorem,

$$(2) \quad \begin{aligned} \sum_{k=1}^{\infty} \int_{S \times T} |\psi_k| d\mu \times \nu &= \sum_{k=1}^{\infty} \int_S \int_T |\psi_k(s, t)| d\nu(t) d\mu(s) \\ &= \int_S \sum_{k=1}^{\infty} \int_T |\psi_k(s, t)| d\nu(t) d\mu(s) < \infty \end{aligned}$$

so there is a μ -null set E such that $\sum_{k=1}^{\infty} \int_T |\psi_k(s, t)| d\nu(t) < \infty$ for $s \notin E$. For $s \notin E$, the Monotone Convergence Theorem implies that

$$\sum_{k=1}^{\infty} \int_T |\psi_k(s, t)| d\nu(t) = \int_T \sum_{k=1}^{\infty} |\psi_k(s, t)| d\nu(t)$$

so $\sum_{k=1}^{\infty} \psi_k(s, t)$ is absolutely convergent in $\mathbb{R}$ for ν -almost all $t \in T$. [The ν -null set may depend on s .] For such s, t , $f(s, t) = \sum_{k=1}^{\infty} \psi_k(s, t)$; in particular, for $s \notin E$, $f(s, \cdot) = \sum_{k=1}^{\infty} \psi_k(s, \cdot)$ ν -a.e., and since

$$\left| \sum_{k=1}^n \psi_k(s, \cdot) \right| \leq \sum_{k=1}^{\infty} |\psi_k(s, \cdot)|,$$

the Dominated Convergence Theorem implies that $f(s, \cdot)$ is ν -integrable and

$$\int_T f(s, \cdot) d\nu = \sum_{k=1}^{\infty} \int_T \psi_k(s, \cdot) d\nu.$$

Further, for $s \notin E$, $|\sum_{k=1}^n \int_T \psi_k(s, \cdot) d\nu| \leq \sum_{k=1}^{\infty} \int_T |\psi_k(s, \cdot)| d\nu$ and the function on the right hand side of this inequality is μ -integrable by (2) so the Dominated Convergence Theorem implies that $s \mapsto \int_T f(s, \cdot) d\nu$ is μ -integrable with

$$\int_S \int_T f(s, \cdot) d\nu d\mu(s) = \sum_{k=1}^{\infty} \int_S \int_T \psi_k(s, t) d\nu(t) d\mu(s) = \int_{S \times T} f d\mu \times \nu$$

by (1).

The proof of Fubini's Theorem for $\mathbb{R}^{n+m}$ given by Mikusinski is even simpler than that given in Theorem 4 since brick functions are used and the partial integral in $\mathbb{R}^n$ of a brick function in $\mathbb{R}^{n+m}$ is a brick function in $\mathbb{R}^m$ (see [5]).

We have the following immediate corollary of Theorem 4.

Corollary 5. If $E \in \Sigma$ and $\mu \times \nu(E) < \infty$, then for μ -almost all $s \in S$ the sections $E_s = \{t \in T : (s, t) \in E\}$ belong to $\mathcal{N}$ and $\mu \times \nu(E) = \int_S \nu(E_s) d\mu(s)$.

In many of the standard real analysis texts, Corollary 5 is a preliminary step in the proof of Fubini's Theorem used in showing that the result is valid for $\mu \times \nu$ -integrable, Σ -simple functions ([1] 22.4, [6] 12.4.15-18), and the proof is completed by approximating a $\mu \times \nu$ -integrable function by a sequence of such simple functions ([1] 22.6, [6] 12.4.19). The proof above using Mikusinski's characterization is only slightly more technical than the usual proofs of Fubini's Theorem and does not require the somewhat lengthy proof of Corollary 5 in advance.

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