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Universidad Católica del Norte
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ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF THE FUNCTIONAL DIFFERENTIAL EQUATION $x'(t) = a(t)x(r(t)) + bx(t)$ *

Manuel Pinto

ABSTRACT

We study the global existence, the stability and the asymptotic behavior of solutions of the functional differential equations $x'(t) = a(t)\, x\,(r(t)) + bx(t)$, $b \in \mathbb{R}$ where r is a continuous contraction at infinity.

1. INTRODUCTION

We study the asymptotic behavior of the solutions of the functional differential equation :

$$x'(t) = a(t)\, x\,(r(t)) + bx(t) \qquad b \in \mathbb{R}\ , \qquad (1.1)$$

where $a : [0,\infty) \to [0,\infty)$ and $r : [0,\infty) \to [0,\infty)$ are continuous functions.

Particular cases of this equation have been studied by a number of authors. BELLMAN-COOKE [1], DRIVER [4] and HALE [8] are excellents references for knowing its history. The work of DE BRUIJN [2,3] treats several particular cases. KRASOVSKI [12] studied the case $r(t) = t - \tau(t)$, $0 \le \tau(t) \le \tau_0$. KATO-Mc LEOD [10, 11, 15] and MAHLER [14] considered

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the case $r(t) = \lambda t$, $\lambda > 0$ $(\lambda \neq 1)$. HEARD [9] treated the case $r(t) = t^{\alpha}$, $\alpha > 1$, etc.

For our, r will verify the general retarded condition (c) :
(c) $r(t) \leq t$ on $[0,\infty)$ and there exists $T \geq 0$ for which r is an increasing and unbouded function such that $r(t) < t$ on the interval $[T, \infty)$.

Thus $r(t) = \lambda t + o(t)$, $0 < \lambda < 1$; $r(t) = t^{\lambda}$, $0 < \lambda < 1$; $r(t) = t - \log(t+1)$; $r(t) = \log(\lambda t^n + o(t^n)$, $\lambda > 0$, $n > 0$ $r(t) = \exp(\sqrt{\log t} - c)^2$, $c > 0$; $r(t) = \sqrt[n]{\lambda t^m + o(t^m)}$, $\lambda > 0$, $n > m \geq 1$; $r(t) = \log(\lambda e^{t/2} + t^3 e^{t/4})$, $\lambda \geq 1$; $r(t) = t - \sqrt[n]{\lambda t^n + o(t^{n-1})}$, $\lambda > 0$; etc., are some examples of delay r considered here (see section 2).

A general study of this type of equations is not known at present to the author (see DRIVER [5]).

The main results obtained are :

(i) For $b = 0$. If $a = a(t) \in L_1([0,\infty))$, then any solution x of (1.1) is defined, bounded and stable on $[T,\infty)$ and it verifies $x(t) = L + o(1)$, $t \to \infty$ for some constant L.

(ii) For $b > 0$. If $a = a(t) \in L_1([0,\infty))$ and r satisfied condition (c), then any solution x of (1.1) is defined on $[T,\infty)$ and it verifies $x(t) = e^{bt}(L + o(1))$, $t \to \infty$.

This result is also true if a is a bounded function and r is a contraction at infinity (see Definition 2).

(iii) For $b < 0$. We give some conditions under which any solution of (1.1) is $O(t^{\alpha})$ for some α. In particular, we prove that any solution x of (1.1) is bounded on $[T,\infty)$ if r is a contraction at infinity and

$$\|a\|_{[T,\infty)} =: \sup\ \{|a(s)|/s \in [T,\infty)\} \leq -b$$

.

If further $a \in L_1([0,\infty))$ then $x(t) \to 0$ as $t \to \infty$. Moreover, for r a contraction at infinity, any solution x of (1.1) tends to zero as $t \to \infty$ if $\|a\|_{[T,\infty)} < -b$ more precisely $x = O(t^{-\beta})$, $\beta > 0$. This is hence the case if $a(t) \to 0$ as $t \to \infty$.

So, in particular, the linear equation with constant coefficients :

$$x'(t) = ax(r(t)) + bx(t) \ , \quad b < 0 \ , \qquad |a| < |b| \ ,$$

where r is a contraction at infinity, is asymptotically stable and $x = O(t^{-\beta})$, $\beta > 0$. For $r(t) = t-\tau(t)$, $0 \leqslant \tau(t) \leqslant \tau_0$, the stability is a classical result obtained by KRASOVSKI [12].

In this way, our results extends the results about asymptotic behavoir of KATO and Mc LEOD [10], KATO [1], Mc LEOD [15], who considered $r(t) = \lambda t$, $0 < \lambda < 1$. Moreover, our basic results are also valid if (1.1) is a system, that is, where $x(t)$ is an n-dimensional column vector and a,b are two $n \times n$ matrices. Thus they generalize also some results of LIM [13] and FOX, MAYERS, OCKENSON and TAYLOR [6]. However, we wish to treat these systems in other papers. Furthermore, our results can be extended to functional equations with different r ([16,17]).

Throught this paper, we adopte the following basic definitions.

For $t_0 \geqslant 0$ such that $r(t_0) < t_0$ and a continuous function $\varphi : [r(t_0), t_0] \to \mathbb{R}$, we denote by $x(t; t_0, \varphi)$ a solution of (1.1) for which $x(s; t_0, \varphi) = \varphi(s)$ for $s \in [r(t_0), t_0]$. We will say that the solution x of equation (1.1) is defined on $[r(t_0), \infty)$ if $x = x(t;t_0;\varphi)$ is defined on $[r(t_0),\infty)$ for any φ. The initial function φ will not be explicitely mentioned. Thus the stability on $[r(t_0),\infty)$ obtained will be expresed as

$$|x(t)| \leqslant K \sup_{s\in[r(t_0),t_0]} |x(s)| \qquad \text{for} \quad t \geqslant t_0$$

2. THE PRELIMINARY FACTS

In the following, we use a special sequence:

Definition 1. (Retarding sequence). Let $r : [0,\infty) \to \mathbb{R}$ be a function. The sequence $\{\xi_1\}_{i=1}^{\infty} \subset [0,\infty)$ is called a retarding sequence for r if $r(\xi_i) = \xi_{i=1}$ for any i and $\{\xi_i\}$ is an unbounded increasing sequence.

Example 1.

For any t, the sequence $\{\xi_i\}_{i=1}^{\infty}$ given by $\xi_i = t + i\tau$, $i \in \mathbb{N}$, is a retarding sequence for the function $r(t) = t - \tau$, $\tau > 0$ a constant. If $\xi_1 = 2$, $\xi_2 = 2 + 2^2$, ..., $\xi_i = 2 + \ldots\ldots 2^i$, $i \in \mathbb{N}$, then $\{\xi_i\}_{i=1}^{\infty}$ is a retarding sequence for $r(t) = (t-2)/2$. If $r(t) = \lambda t$, $\lambda > 0$, then $\xi_i = \lambda^{-i}$, $i \in \mathbb{N}$, verifies always $r(\xi_i) = \xi_{i=1}$, but it is a retarding sequence only if $\lambda < 1$. If $\xi_i = -1 + \sqrt{a+i}$, $i \in \mathbb{N}$, $a \geqslant 0$

then $\{\xi_i\}_{i=1}^{\infty}$ is a retarding sequence for $r(t) = \sqrt{t^2 + 2t - 1}$ but $r(t) = \sqrt{t^2 + 1}$ has not a retarding sequence.

The class of functions considered by condition (c) includes the "contractions at infinity".

Definition 2.

Let r be a function satisfying condition (c). We will say that r is a contraction at infinity if

$$\gamma = \limsup_{t\to\infty} r(t)/t < 1 \tag{2.1}$$

If $\gamma = 0$, we will say that r is a 0-contraction at infinity.

Remark 1.

If r is a contraction at infinity, then there is $\gamma \in (0,1)$ such that $r(t) < \gamma t$ for any $t \geq T$ (some $T > 0$); however r has not in general a retarding sequence of the form λ^{-i} with $0 < \lambda < 1$. This is shown by $r(t) = (t-1)/2$ and $\xi_i = 2(2^i - 1)$, $i \in \mathbb{N}$ or $r(t) = \sqrt{t}$ (an 0-contraction) and $\xi_i = 2^{2^i}$. Actually, that is exclusive of $r(t) = \lambda t$, $0 < \lambda < 1$.

Lemma 1.

Let r be a function satisfying condition (c) on $[T,\infty)$ and denote by r_{-1} its inverse function. Then there exists a retarding sequence $\{\xi_i\}_{i=1}$ given by the iterations of

$$\xi_i = r_{-1}^{i}(\xi) , \quad i \in \mathbb{N} \tag{2.2}$$

where ξ_1 is any number in $[T,\infty)$ and $\xi_0 = r(\xi_1)$.

Proof. Let be $\xi_1 \in [T,\infty)$ and $\xi_0 = r(\xi_1)$. Then $\xi_1 > r(\xi_1)$. Define $\xi_i = r_{-1}^{i}(\xi_0)$, $i \in \mathbb{N}$. Thus $\xi_1 > \xi_0$ and by induction $\xi_{i+1} = r_{-1}(\xi_i) > r_{-1}(\xi_{i-1}) = \xi_i$ because r_{-1} is an increasing function on $[r(T),\infty)$. Moreover, $\xi_i \to \infty$ as $i \to \infty$ since if $\ell = \lim_{i\to\infty} \xi_i < \infty$, then is a fixed point of r. Thus, as $r(\xi_i) = \xi_{i-1}$, $\{\xi_i\}_{i=1}^{\infty}$ is a retarding sequence of r.

Remark 2.

i) If $r(t) \geq t$ on $[T,\infty)$ for some $T > 0$ then the function r has no retarding sequence. More generally, r has no retarding sequence if there exists a time-sequence $\{t_i\} \to \infty$ as $i \to \infty$ such that $r(t_i) \geq t_i$ (although $r(t) < t$ on (t_i, t_{i+1})) .

ii) The computation of sequences $\{\xi_i\}_{i=1}^{\infty}$ given by (2.2) is difficult as it can be verified with $r(t) = \log(t+1)$ or even $r(t) = \sqrt{2t+1}$. However, a retarding sequence is only a tool to obtain the results which depende on ξ_1 and on a general property.

Definition 3.

We will call the retarding sequence of r on $[T,\infty)$ the sequence given by (2.2) with $\xi_0 = r(T)$.

Examples of r.

We show now several classes of delay r which satisfy condition (c). Since we are interested on the behavior of the solutions $x = x(t)$ of (1.1) for t big enough, we present delay - functions r which are defined or positive only for t sufficiently large.

a) The contraction at infinity $r(t) = \lambda t + o(t)$, $0 < \lambda < 1$ ($o(f(t))$ means as usual a function such that $o(f(t)) / f(t) \to 0$ as $t \to \infty$).

b) The 0-contractions at infinity $r(t) = \sqrt[n]{t^m + c}$, $m < n$, $c \in \mathbb{R}$. If $m = n$ and $c < 0$, r is not a contraction at infinity but is satisfies condition (c).

c) More generally (to case $m = n$ in b), let f be an increassing continuous, positive and unbounded function on $[0,\infty)$. For $c > 0$, the function, defined for $t \geq f(c)$, $r(t) = f(f^{-1}(t) - c)$ satisfied condition (c) and for any $t \geq 0$, the sequence $\{f(t+ic)\}_{i=1}^{\infty}$ is a retarding sequence for r. Thus, for $c > 0$, the following functions r satisfy condition (c) :

i) $r(t) = t - c$; ii) $r(t) = \log(e^t - c)$;

iii) $r(t) = \exp(\log t - c) = \lambda t$; $0 < \lambda < 1$;

iv) $r(t) = (\sqrt[n]{t - c})^n$, $n \in \mathbb{N}$; v) $r(t) = \sqrt[n]{t^n - c}$, $n \in \mathbb{N}$;

vi) $r(t) = \exp\,(\sqrt{\log t - c})^2$; vii) $r(t) = [\log\,(\exp(t^2) - c)]^{1/2}$

d) Rational expresions which are contractions at infinity :

$$r(t) = \frac{\lambda_1 t^n + o(t^n)}{\lambda_2 t^{n-1} + o(t^{n-1})}\ , \qquad 0 < \lambda_1 < \lambda$$

e) The irrational expresions as

$$r(t) = \sqrt[n]{\lambda_1 t^m + o(t^m)} \quad , \quad n > m \geq 1 \quad , \quad \lambda_1 > 0 \quad ;$$

$$r(t) = \sqrt[n]{\lambda_1 t^m + o(t^m)} - \sqrt[n]{\lambda_2 t^m + o(t^m)} \quad , \quad n > m \geq 1 \quad ,$$

$\lambda_1 \quad \lambda_2 > 0$, which are 0-contraction at infinity and

$$r(t) = \sqrt[n]{\lambda_1 t^n + o(t^n)} - \sqrt[n]{\lambda_2 t^n + o(t^n)} \quad , \quad n \in \mathbb{N} \ , \ \lambda_2 > 0 \ ,$$

$0 < \lambda_1 - \lambda_2 < 1$, which are contractions at infinity.

f) The logarithmic expresions which are 0-contractions at infinity as

$$r(t) = \log\,(\lambda t^n + o(t^n)) \ , \ \lambda > 0 \quad , \quad n \in \mathbb{N} \ .$$

Moreover, let $g : [0,\infty) \to [1,\infty)$ be a differentiable function such that $g' \geq 0$, $g(\infty) = \infty$. If $g'(t)/g(t) \to 0$ as $t \to \infty$, then

$$r(t) = \log\, g(t)$$

is a 0-contraction at infinity, while if for $0 < \varepsilon \leq \lambda < 1$ we have $e^{\varepsilon t} \leq g(t) \leq e^{\varepsilon t}$, then r is a contraction at infinity. Thus

$$r(t) = \log\,(e^{t/2} - te^{t/4})$$

is a contraction at infinity which is not a 0-contraction at infinity.

We can also study functions r of the form $r(t) = t - \tau(t)$. $\tau(t) \geq 0$. For that, we examine the possible operations between the admisible functions r.

g) Addition. Let be r_i , $i = 1,2$ two functions satisfying condition (c). Of course, the addition $r = r_1 + r_2$ is not, in general, a function satisfying condition (c), even if r_1 and r_2 are contractions at infinity . However, any finite sum of 0-contractions at infinity.

Further, the addition between a contraction at infinity and a finite sum of 0-contraction at infinity is against a contraction at infinity. Thus

$$\sum_{i=1}^{n} \lambda_i t^{\alpha_i} \ , \quad 0 < \alpha_i < 1 \ , \ \lambda_i > 0 \ , \quad n \in \mathbb{N} \ ; \quad \text{and}$$

$$\sqrt[n]{\lambda_1 t^{n-1} + O(t^{n-1})} + \log(\lambda_2 t^{\alpha} + O(t^{\alpha})) \ , \quad \lambda_1, \lambda_2 > 0 \ , \ \alpha > 0 \ ,$$

are 0-contractions at infinity and if r_0 denotes a 0-contraction at infinity, then the following functions are contractions at infinity

$$\log(e^{\lambda t} + O(e^{\lambda t})) + r_0 \quad , \quad 0 < \lambda < 1 \ ;$$

$$\frac{\lambda_1 t^n + O(t^n)}{\lambda_2 t^{n-1} + O(t^{n-1})} + r_0 \quad , \quad 0 < \lambda_1 < \lambda_2 \quad , \quad \text{etc.}$$

h) Multiplication. Even the multiplication of two 0-contractions at infinity is not always a function satisfying condition (c) .

i) Composition. The composition between two functions satisfying condition (c) is against a function satisfying condition (c). Moreover, $r = r_1 \circ r_2$ is a contraction or a 0-contraction at infinity if at least r_1 or r_2 it is so. Thus the classes of functions verifying condition (c) or being contraction or even a 0-contraction at infinity are very large. Then we know that the complicated expresions as

$$\left[\frac{[\log(t+1)]^2 + 5|\log(t+1) -3|}{2\log(t+1) + 3\sqrt{\log(t+1)}}\right]^{1/2}$$

are 0-contractions at infinity, or expresions as

$$\frac{(t^2-1)^{3/2} + (t^2-1)^{1/2}}{5\sqrt{t^2-1} + 2\sqrt[4]{t^2-1}}$$

are contractions at infinity, or expresions as

$$\exp\left(\frac{1}{2}\log(t^2+2t) - 3\right)$$

satisfy condition (c) (but they are not contractions at infinity).

j) The function $r(t) = t - \tau(t)$. This is a very important type of delay-functions, which has been largely studied for $\tau = \tau(t)$ bounded on $[0,\infty)$ (see [8,19]). Here, τ is not necessarily bounded. For instance, if τ is a differentiable function on $[0,\infty)$ then it is enough that τ verifies $\tau'(t) < 1$. Thus the examples of this type are very divers. Some simple but representatif examples are the following :

$$r(t) = t-\log(t+1) \ ; \quad r(t) = t - te^{-t} \ ; \quad r(t) = t - \sqrt{t} \ ;$$

$$r(t) = t - \frac{2t^2 - 5t + 1}{3t + 6} \ ; \quad r(t) = t-\log(e^{t/2} - t^n e^{t/3}) \ , \quad n \in \mathbb{N} \ ;$$

$$r(t) = t - \lambda\sqrt{t^2-t+1} \ , \quad 0 < \lambda < 1 \ ; \quad r(t) = t - 1/t+1 \ , \quad \text{etc.}$$

3. THE CASE $b \geq 0$

In this section we study equation (1.1) for $b = 0$ and $b > 0$, although the case $b < 0$ is sometimes also included. Consider the equations

$$x' = a(t)\,x\,(r(t)) \tag{3.1}$$

and

$$x' = bx(t) + a(t)\,x\,(r(t)) \tag{3.2}$$

where $r = r(t)$ satisfies condition (c) on $[T,\infty)$.

As it is usual for linear equations, the existence of the solutions of (3.1) and (3.2) can be treated with sucesive approximations. However, in every one of our results we prove that any solutions is bounded (and stable) in any compact interval on $[T,\infty)$, hence it is defined on $[r(T),\infty)$. The study of the definition of x on any compact interval $[0, T_1]$ can be done by the classical techniques.

Theorem 1.

Assume that $a \in L_1([0,\infty))$. Then the solutions x of (3.1) are defined on $[r(T),\infty)$, converge as $t \to \infty$ and the zero solution is stable respect of the compact initial interval $[r(T), T]$, namely :

$$|x(t)| \leq K \sup \{|x(s)| / s \in [r(T), T]\} \quad , \qquad t \geq T \quad ,$$

where $K \geq 1$ is a constant.

Proof. Let $\{\xi_i\}_{i=0}^{\infty}$ be the retarding sequence of r on $[T,\infty)$, that is $\xi_0 = r(T)$. Define $I_i = [\xi_{i-1}, \xi_i]$, $i \geq 1$ and

$$\|x\|_{I_i} = \sup \{|x(s)|/s \in I_i\} \quad , \qquad i \in \mathbb{N}$$

Since r is an increasing function we have $r(I_i) = [r(\xi_{i-1}), r(\xi_i)] = [\xi_{i-2}, \xi_{i-1}] = I_{i-1}$, because $\{\xi_i\}$ is a retarding sequence of r. Moreover, for $t \in I_i$ we get $|x(r(t))| \leq \|x\|_{I_{i-1}}$ Take $s \in [\xi_{i-1}, \xi_i] = I_i$ and integrate (3.1) on $[\xi_{i-1}, s]$. We obtain

$$x(s) = x(\xi_{i-1}) + \int_{\xi_{i-1}}^{s} a(t)x(r(t))dt \tag{3.3}$$

Then

$$|x(s)| \leq |x(\xi_{i-1})| + \|x\|_{I_{i-1}} \cdot \int_{I_i} |a(t)|dt$$

$$\leq (1+\alpha_i)\|x\|_{I_{i-1}} \quad , \quad \alpha_i = \int_{I_i} |a(t)|dt \quad .$$

Hence

$$\|x\|_{I_i} \leq (1+\alpha_i)\|x\|_{I_{i-1}} \quad , \qquad i \geq 1 \quad .$$

Thus for any $i \in \mathbb{N}$,

$$\|x\|_{I_i} \leq \prod_{k=1}^{i} (1+\alpha_i) \|x\|_{I_1} \quad .$$

Since $\sum^{\infty} \alpha_k$ converges, then $\|x\|_{I_i} \leq K\|x\|_{I_1}$ for any $i \in \mathbb{N}$ and $K \geq 1$ a constant. Hence

$$|x(s)| \leq K \sup \{|x(t)|/t \in [r(T),T]\} \quad , \qquad s \geq T$$

because $I_1 = [\xi_0, \xi_1] = [r(T),T]$. We have also the stability of zero solution with respect of the initial interval $I_1 = [r(T),T]$. Moreover, since x is bounded and $a \in L_1$, (3.3) implies that $\lim_{s\to\infty} x(s)$ exists.

Example 2.

The equation

$$x'(t) = e^{-t}\, x(2t)$$

and its solutions $x = e^t$ show that the conclusion of Theorem 1 is not true if r does not satisfy condition (c).

Example 3.

The equation

$$x'(t) = 3t^{-1}\, x(t/3)$$

and its solution $x = t$ show that the integrability condition of $a = a(t)$ is essential in Theorem 1. Even for $a(t) =$ constant, the existence of the $\lim_{t\to\infty} x(t)$ is not assured as it is shown by the equation

$$x'(t) = -x(t - \pi/2)$$

and its solution $x = \sin t$.

The case $b > 0$ is actually a consequence of the case $b = 0$ treated in Theorem 1. In fact, if x is a solution of equation (3.2) then $v(t) = x(t)e^{-bt}$ satisfies

$$v'(t) = a(t)e^{-b(t-r(t))}v(r(t))$$

Thus from Theorem 1 we have :

Theorem 2.

Suppose that r satisfies condition (c) and $a(t)\cdot\exp[-b(t-r(t))] \in L_1([0,\infty))$. Then for any solution x of equation (3.2), there exists a constant L such that

$$x(t) = e^{bt}(L + o(1)), \qquad t \to \infty \tag{3.4}$$

In particular, (3.4) is true for :

(i) Any $b \in \mathbb{R}$ if $t-r(t)$ is bounded and $a \in L_1([0,\infty))$.

(ii) Any $b \geq 0$ and any r satisfying (c) if $a \in L_1([0,\infty))$.

(iii) Any $b > 0$ if r is a contraction at infinity and $a = a(t)$ is bounded.

Corollary 1.

Assume that $t-r(t)$ is bounded, $b > 0$ and $a \in L_1([0,\infty))$. Then any solution x of

$$x'(t) = -bx(t) + a(t)x(r(t))$$

tends exponentially to zero as $t \to \infty$. Moreover, the solution are asymptotically stable.

In particular,

Corollary 2.

The solutions of the equation

$$x'(t) = bx(t) + a(t)x(t-\tau) , \qquad \tau > 0$$

where $b \in \mathbb{R}$, and $a \in L_1([0,\infty))$, verify (3.4) and hence for $b < 0$ the solutions are asymptotically stable. The same result is true for

$$x'(t) = bx(t) + a(t)x(r(t))$$

where

(i) $r(t) = \sqrt[n]{t^n - \tau}, \quad n \geq 1, \quad \tau > 0$

(ii) $r(t) = \sqrt[n]{\log [\exp(t^n) - \tau]}, \quad n \geq 1, \quad \tau > 0$

In fact, in any case $t - r(t)$ is bounded (for $n > 1$, it tends to zero as $t \to \infty$).

Corollary 3.

Assume that $r(t) = t - \tau(t)$ satisfied condition (c), $a = a(t)$ is bounded and $b > 0$. Then the solutions of the equation

$$x'(t) = bx(t) + a(t)x(t-\tau(t))$$

verify (3.4) for :

(i) $\tau(t) \geq \sqrt[n]{t}, \quad n \geq 2$;

(ii) $\tau(t) \geq \frac{\alpha}{b} \log t, \quad \alpha > 1$;

(iii) $\tau(t) \geq \sqrt[n]{t^n + O(t^n)}, \quad n \geq 1$.

In fact, in these cases $\exp[-b(t-r(t))] \in L_1([T,\infty))$.

Corollary 4.

Assume that $b > 0$ and r is a γ-contraction at infinity, where γ is given by (2.1). Suppose also that there exist constants $K > 0$ and λ, $0 < \lambda < 1-\gamma$ such that $|a(t)| \leq K \exp(\lambda bt)$ for $t \geq T$. Then formule (3.4) holds for equation (3.2). In particular, the solution of

$$x'(t) = bx(t) + p(t)x(r(t)) , \qquad (3.5)$$

where $p = p(t)$ is a polynomial function, satisfy (3.4) .

In fact, in this case $a(t) \cdot \exp[-b(t-r(t))] \in L_1([0,\infty))$, because $|a(t)| \exp[-b(t-r(t))] \leq K \exp[-bt(1-\lambda-t^{-1}r(t))]$ and $\lambda + t^{-1}r(t) \leq \lambda + \gamma < 1$. The later assertion follows since for any polinomial function p we have $|p(t)| \leq Ke^{\varepsilon t}$ for any $\varepsilon > 0$.

4. THE CASE b < 0

When $b < 0$, the solutions of equation (1.1)

$$x'(t) = bx(t) + a(t)x(r(t) \qquad (4.1)$$

does not behave asymptotically as exponentials. For instance, the equation

$$x'(t) = 2x\left[\frac{t-2}{2}\right] - x(t)$$

has the solution $x = 2(t+3)$. Furthermore, this is not an isolated case. For $b < 0$, the solutions of (4.1) are rather dominated by polynomial functions. In this section, we yield some conditions under which any solution of (4.1) is defined on $[T,\infty)$ and it is $O(t^{\alpha})$ for some α.

Before, we prove two simple facts :

Proposition 1.

Let $b < 0$ and x be a solution of equation (4.1) defined on $[0,\infty)$

(i) If either $a \in L_1([0,\infty))$ or $a(t) \to 0$ as $t \to \infty$, then any bounded solution x of (4.1) satisfies $x(t) \to 0$ as $t \to \infty$.

(ii) If $a = a(t)$ is a non-negative function such that $a \in L_1([0,\infty))$ and either $x \geq 0$ or $x \leq 0$, then $x(t) \to 0$ $t \to \infty$. In particular, in this case, any non-oscillatory solution tends to zero as $t \to \infty$.

Proof. (i) If x is a solution of (4.1), then for $t \geq t_0 \geq 0$:

$$x(t) = x(t_0)e^{b(t-t_0)} + \int_{t_0}^{t} e^{b(t-s)} a(s)x(r(s))ds$$

and since $b < 0$ and x is bounded, the last integral tends to zero as $t \to \infty$. In fact, if $a \in L_1([0,\infty))$ this follows from

$$\int_{t_0}^{t} e^{b(t-s)} |a(s)| ds \leq e^{bt} \int_{t_0}^{T} e^{-bs} |a(s)|ds + \int_{T}^{\infty} |a(s)|ds$$

and if $a(t) \to 0$ as $t \to \infty$, this follows from :

$$\int_{t_0}^{t} e^{b(t-s)}|a(s)|ds \leq e^{bt} \int_{t_0}^{T} e^{-bs}|a(s)|ds + \|a\|_{[T,\infty)}$$

(ii) If now $x \geq 0$ and $a \geq 0$, we integrate directly equation (4.1) to obtain

$$x(t) = x(t_0) + \int_{t_0}^{t} a(s)x(r(s))ds + \int_{t_0}^{t} bx(s)ds$$

$$\leq x(t_0) + \int_{t_0}^{t} a(s)x(r(s))ds \quad ,$$

because $b < 0$. Since $x(t) \geq 0$, $a(t) \geq 0$ and $a \in L_1([0,\infty))$, we can apply the proof of Theorem 1 to deduce that x is bounded. Moreover, by (i) above, $x(t) \to 0$ as $t \to \infty$.

Theorem 3.

Let be $b < 0$ and r a function satisfying condition (c) on an interval $[T,\infty)$ such that $\sum^{\infty} \xi_i^{-1}$ converges, where $\{\xi_i\}_{i=0}^{\infty}$ is the retarding sequence of r on $[T,\infty)$. Assume that there exists $\alpha \in \mathbb{R}$ such that $|a(t)|(r(t)t^{-1})^{\alpha} \leq -b$ on $[T,\infty)$. Then any solution of (4.1) is $O(t^{\alpha})$.

Proof. Let be $T_1 = T_1(\alpha) \geq T$ such that $\alpha s - be^s \geq 1/2$ for $s > T_1$. Therefore $\exp(\alpha s - be^s)$ is an increasing function on $[T_1,\infty)$. Using integration by parts, for $[t_0, s] \subset [T_1,\infty)$ we have

$$\int_{t_0}^{s} -be^t \cdot e^{\alpha t-be^t} dt = [e^{\alpha t-be^t}]_{t_0}^{s} + O(e^{\alpha s-be^s-t_0}) \qquad (4.2)$$

where the O-term is uniform in s, α, and t_0.

Let x be a solution of (4.1). We make the transformation $z(s) = t^{-\alpha}x(t)$, $t = e^s$ in (4.1). We obtain

$$\frac{d}{ds} [e^{\alpha s-be^s} z(s)] = a(t)(r(t)t^{-1})^{\alpha} e^{s+\alpha s-be^s} z(\ell n r(t))$$

We remark that $\ln r(t)$ is positif for $t \geq T_2$, sufficiently large. We suppose that $T_2 = T_1$. We may also suppose that $\xi_0 \geq T_1$ because if $\xi_{i_0} \geq T_1$, then the sequence $\{\tilde{\xi}_i\}_{i=0}^{\infty}$ given by $\tilde{\xi}_i = \xi_{i+i_0}$, $i \geq 0$, satisfies $\tilde{\xi}_0 \geq T_1$.

Define now $r_1(s) = \ln r(e^s)$ and $\eta_i = \ln \xi_i$. Then $\{\eta_i\}_{i=0}^{\infty}$ is a retarding sequence of r_1 because $r_1(\eta_i) = \ln r(\xi_i) = \ln \xi_{i-1} = \eta_{i-1}$.

Take $s \in I_i = \{\eta_{i-1}, \eta_i\}$ and integrate on $[\eta_{n-1}, s]$ to obtain

$$|[e^{\alpha s - be^s} z(s)]_{\eta_{i-1}}^{s}| = |\int_{\eta_{i-1}}^{s} a(e^t)(r(e^t)/e^t)\, e^{t+\alpha t - be^t} z(r_1(t)dt|$$

$$\leq \|z\|_{I_{i-1}} \int_{\eta_{i-1}}^{s} -be^t\, e^{\alpha t - be^t} dt$$

Thus from (4.2) we have that for any i

$$\|z\|_{I_i} \leq \|z\|_{I_{i-1}} (1 + 0(e^{-\eta_{i-1}})) .$$

Then $z(s)$ is a bounded function because $\Sigma\, e^{-\eta_{i-1}} = \Sigma\, \xi_{i-1}^{-1}$ converges. So $x = 0(t^{\alpha})$ and the proof is complete.

Example 4.

If $a = o(t^{\varepsilon})$, $\varepsilon \in \mathbb{R}$, then any solution x of

$$x'(t) = a(t)x(\sqrt{t}) + bx(t) ,$$

where $b < 0$, is $0(t^{2\varepsilon})$ as $t \to \infty$.

In fact, $\alpha = 2\varepsilon$ verifies the condition of Theorem 3 because

$$a(t)\left[\frac{r(t)}{t}\right]^{2\varepsilon} = a(t)\left[\frac{\sqrt{t}}{t}\right]^{2\varepsilon} = a(t)\cdot t^{-\varepsilon} \to 0$$

as $t \to \infty$. Since $\{\xi_i\}_{i=0}^{\infty}$, given by $\xi_i = 2^{2^i}$, is a retarding sequence of $r(t) = \sqrt{t}$ satisfying that $\sum^{\infty} \xi_i^{-1}$ converges then Theorem 3 implies that $x = 0(t^{2\varepsilon})$.

Thus for any $b < 0$ and $\varepsilon > 0$, any solution x of

$$x'(t) = t^{-\varepsilon}\, x(\sqrt{t}) + bx(t)$$

is $0(t^{-\varepsilon})$ and $x \in L_1([1,\infty))$ for $\varepsilon > 1/2$.

Now, if $\alpha \in (0, 2\varepsilon)$ we get

$$a(t)\cdot\left[\frac{r(t)}{t}\right]^{-\alpha} = t^{-\varepsilon}\cdot t^{\alpha/2} = t^{-(\varepsilon-\alpha/2)} \to 0$$

as $t \to \infty$.

Then from Theorem 3 $x = O(t^{-\alpha})$. In particular, we can take $\alpha = \varepsilon$. Moreover, if $\varepsilon > 1/2$, we choose $\alpha \in (1, 2\varepsilon)$ and $x = O(t^{-\alpha})$ implies $x \in L_1([1,\infty))$.

Theorem 4.

Let be $b < 0$ and r a γ-contraction at infinity, where γ is given by (2.1). Assume that a is a bounded function on $[T,\infty)$. Denote

$$\alpha = \log(-b/a_0)/\log \gamma \tag{4.3}$$

where $a_0 = \|a\|_{[T,\infty)}$. Then any solution x of (4.1) is $O(t^{\alpha})$ as $t \to \infty$.

Proof. First, $\sum^{\infty} \xi_i^{-1}$ converges for any retarding sequences $\{\xi_i\}$ of r because $\gamma < 1$ and $r(t) \leq \gamma t$ implies $\xi_i^{-1} < \gamma^{-1}\, \xi_0^{i-1}$. Moreover, if $a_0 = \|a\|_{[T,\infty)}$ then $|a(t)|(r(t)t^{-1})^{\alpha} \leq a_0\gamma^{\alpha}$. Thus the condition of Theorem 3 holds if $a_0\gamma^{\alpha} = -b$, i.e. if α is given by (4.3).

Remark 3.

$\sum^{\infty} \xi_i^{-1}$ converges for delay-functions r which are not contractions at infinity. This is the case of $r(t) = (\sqrt[n]{t} - \tau)^n$, $\tau > 0$, $n > 1$, because its retarding sequence is $\{(i\tau)^n\}_{i=1}^{\infty}$. For others functions we can apply a comparison method. If $r_1 \leq r_2$, their corresponding retarding sequences $\{\xi_i^1\}$, $\{\xi_i^2\}$ verify $(\xi_i^1)^{-1} \leq (\xi_i^2)^{-1}$. Thus if $\sum^{\infty} (\xi_i^2)^{-1}$ converges then $\sum^{\infty} (\xi_i^1)^{-1}$ converges. For instance, $r(t) = t - \sqrt[n]{\lambda t^n}$, $\lambda > 0$, $n > 1$, satisfies the hypotheses of Theorem 3 because $r(t) \leq (\sqrt[n]{t} - \tau(\lambda))^n$ for some $\tau(\lambda) > 0$. A similar fact holds with $r(t) = t - \sqrt[n]{\lambda t^{n-1} + o(t^{n-1})}$, where $\lambda > 0$ and $n > 1$.

Example 4.

The equation

$$x'(t) = 2e^2 x(t-2) - x(t)$$

has the solution $x = e^t$. Thus Theorem 3 does not hold if $\sum^{\infty} \xi_i^{-1}$ does not converge.

Example 5.

The equation

$$x'(t) = 2e^{-t} x(2t) - x(t)$$

has the solution $x = e^t$. So, even if $a(t) \to 0$ as $t \to \infty$, the conclusion of Theorem 3 does not necessarily hold for $r(t)$ such that $r(t) > t$.

Theorem 5.

Assume the hypotheses of Theorem 3 and that $\|a\|_{[T,\infty)} < -b$. Then any solution x of (4.1) is defined, bounded and (uniformly) stable on $[r(T),\infty)$, namely there exists a constant $K \geq 1$ such that

$$|x(t)| \leq K \sup \{|x(s)|/s \in [r(T), T]\} \tag{4.4}$$

for $t \geq T$. If further $a \in L_1([0,\infty))$ then x is asymptotically stable.

Proof. Since $|a(t)| \leq -b$ for any $t \in [T,\infty)$, the hypothesis of Theorem 3 holds for $\alpha = 0$. Then any solution x of (4.1) is $O(1)$ and hence x is bounded on $[T,\infty)$. Moreover, by the proof of Theorem 3 we have $|z(s)| \leq K\|z\|_{[\ell n\, \xi_0,\ \ell n\, \xi_1]}$ for $s \geq \ell n\, \xi_0$ and as $x(t) = z(s) = z(\ell n t)$, this implies $|x(t)| \leq K\|x\|_{[\xi_0,\ \xi_1]}$ for $t \geq \xi_0$. Since $r(t) < t$ on $[T, \xi_0]$ the stability on $[r(T), \xi_0]$ and (4.4) follows at once from Theorem 1. The proof of the first assertion is now complete. If $a \in L_1([0,\infty))$ then $x(t) \to 0$ as $t \to \infty$ by Proposition 1.

Corollary 5.

Under the assumptions of Theorem 4 the solution zero is asymptotically stable on $[r(T),\infty)$ if $a_0 = \|a\|_{[T,\infty)} < -b$; more precisely

$$|x(t)| \leq K\, t^{\alpha} \sup_{r(T) \leq s \leq T} |x(s)|, \qquad t \geq T \tag{4.5}$$

where $K \geq 1$ is a constant and $\alpha < 0$ is given by (4.3). In particular, this is the case if $a(t) \to 0$ as $t \to \infty$. Moreover, for r a 0-contraction at infinity and $a = a(t)$ a bounded function such that $\|a\|_{[T,\infty)} > -b$ we have $x = O(t^{\alpha})$ for any $\alpha > 0$.

Proof. If $a_0 = \|a\|_{[T,\infty)} < -b$ then $\alpha < 0$ in (4.3). Furthermore, (4.5) follows in a similar way as (4.4) was obtained in Theorem 5. Thus the first conclusion follows from Theorems 4 and 5.

On the other hand, if r is a 0-contraction then for any $\alpha > 0$, there exists $T_{\alpha} \geq T$ such that

$$a_0 \left[\frac{r(t)}{t}\right]^{\alpha} < -b \qquad \text{for} \quad t \geq T_{\alpha} .$$

Hence, by Theorem 3 we have $x = O(t^{\alpha})$.

Our methods do not apply, in general, on delay r satisfying $r(t)t^{-1} \to 1$ as $t \to \infty$, even if r verifies the hypotheses of Theorem 3, but using Liappunov functions, at least the stability in the last Corollary is true for $r(t) = t - \tau(t)$, $0 \leq \tau(t) \leq \tau$ (see [5,12]). However, we do not know if Corollary 5 and in particular (4.5) hold, e.g. for $r(t) = t - \sqrt{t}$. This will be considered in other paper ([16]), where we will study functional equations with differents delays.

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Manuel Pinto
Departamento de Matemáticas
Facultad de Ciencias
Universidad de Chile.
Casilla 653
Santiago - CHILE