PROYECCIONES

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**A FIXED POINT THEOREM FOR ORBITALLY CONTINUOUS
FUNCTIONS.**

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ABSTRACT

A fixed point theorem for orbitally continuous functions is presented which extends well known results.

INTRODUCTION

We extend a result of Jaggi [4] for orbitally continuous functions. Note that Ćirić [2] has shown that orbitally continuous functions need not be continuous.

Definition 1.

Let X be a normed linear space and P be a mapping of X into itself. Then P is said to be asymptotically regular at a point $q \in X$ if

$$\|P^n q - P^{n+1} q\| \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Denote by, $P/\bar{A}(q,P)$ the restriction of P to $\bar{A}(q,P)$, the closure of the orbit of P at q [4] .

Definition 2.

A mapping of P defined on X into itself is said to be q-orbitally continuous if

$$P/\bar{A}(q,P) : \bar{A}(q,P) \rightarrow X$$

is continuous.

We now prove the theorem.

Theorem

Let S be a convex subset of a normed linear space X such that $S \neq \emptyset$, let $P : S \rightarrow S$ be a mapping satisfying

$$\begin{aligned} \|Px - Py\| \leq & c_1(x,y) \|x - y\| + c_2(x,y) [\|x - Px\| + \|y - Py\|] \\ & + c_3(x,y) [\|x - Py\| + \|y - Px\|] + \\ & + c_4(x,y) \|y - Px\| \frac{[1 + \|y - Py\| + \|x - Py\|]}{1 + \|x - y\|} + \\ & + c_5(x,y) \|x - Py\| \frac{[1 + \|x - Px\| + \|y - Px\|]}{1 + \|x - y\|}, \end{aligned} \quad (1)$$

for all $x, y \in S$ and $c_i : S \times S \rightarrow [0,1[$ with

$$\sup_{x,y \in S} \{c_1(x,y) + 2c_2(x,y) + 2c_3(x,y) + c_4(x,y) + 3c_5(x,y)\} \leq 1 \quad (2)$$

$$\inf_{x,y \in S} c_2(x,y) > 0. \quad (3)$$

Let us choose λ_i , $i = 0, 1, 2, \dots, k$ for some $k \in \mathbb{N}$, $\lambda_i \in]0,1[$, such that $\sum_{i=0}^k \lambda_i = 1$. Define the operator B on S such that

$$B = \sum_{i=0}^k \lambda_i P^i, \quad P^0 \text{ being the identity mapping on } S.$$

Assume:

- (i) there exists a point $q \in S$ such that $w \in S$ is a cluster point of the sequence $\{B^n q\}$, $n = 1, 2, \dots$,
- (ii) B is q -orbitally continuous and asymptotically regular at q and
- (iii) the function c_5 is such that

$$c_5(q, y) = 0 \quad \text{for all } y \in S.$$

Then the sequence $\{B^n q\}$, $n = 1, 2, \dots$ converges to w and w is the unique fixed point of P on S .

Proof. Step 1. Existence of the fixed point

By hypothesis (i), there exists a subsequence $\{B^{n_i} q\}$, $i = 1, 2, \dots$ of the sequence $\{B^n q\}$ converging to w . By hypothesis (ii), we have

$$(I - B) B^{n_i} q \rightarrow (I - B) w$$

and

$$\|(I - B) B^{n_i} q\| = \|B^{n_i} q - B^{n_i+1} q\| \rightarrow 0 \text{ as } i \rightarrow \infty.$$

$$\text{So, } \|(I - B)w\| = 0 \rightarrow Bw = w \quad \text{and} \quad Pw = w.$$

Step 2. Uniqueness

Let v be such that $v \in S$, $v \neq w$ and $Pv = v$. By hypothesis (i), we have

$$\begin{aligned} \|w - v\| &= \|Pw - Pv\| \leq c_1(w, v) \|w - v\| + c_2(w, v) [\|w - Pw\| + \|v - Pv\|] + \\ &+ c_3(w, v) [\|w - Pv\| + \|v - Pw\|] + c_4(w, v) \|v - Pw\| \frac{[1 + \|v - Pv\| + \|w - Pv\|]}{1 + \|w - v\|} \\ &+ c_5(w, v) \|w - Pv\| \frac{[1 + \|w - Pw\| + \|v - Pw\|]}{1 + \|w - v\|} \\ &\leq \{c_1(w, v) + 2c_3(w, v) + c_4(w, v) + c_5(w, v)\} \|w - v\| \\ &< \|w - v\| \end{aligned}$$

which is a contradiction so $w = v$.

Step 3. The convergence of the sequence $\{B^n q\}$, $n = 1, 2, \dots$.

(i) Let v be an arbitrary element of S . We have using hypothesis

$$\begin{aligned} \|Pw - Pv\| &\leq c_1(w, v) \|w - v\| + c_2(w, v) \|v - Pv\| \\ &+ c_3(w, v) (\|w - v\| + \|w - Pv\|) + c_4(w, v) \|w - Pv\| + \\ &+ c_5(w, v) \|v - Pw\| \frac{[1 + \|v - Pv\| + \|w - Pv\|]}{1 + \|v - w\|} \end{aligned}$$

which gives (using the triangle inequality and the fact that

$$\|v - w\| < 1 + \|v - w\|$$

$$\|Pv - Pw\| \leq \frac{c_1(w, v) + c_2(w, v) + c_3(w, v)}{1 - c_2(w, v) - c_3(w, v) - c_4(w, v) - 2c_5(w, v)} (\|w - v\| + c_5(w, v)).$$

We can prove similarly, for $i = 1, 2, \dots, k$,

$$\|P^i v - P^i w\| \leq \|w - v\| + \frac{c_5(w, v)}{1 - c_2(w, v) - c_3(w, v) - c_4(w, v) - 2c_5(w, v)}$$

We now have that

$$\begin{aligned} \|Bv - w\| &= \left\| \sum_{i=0}^k \lambda_i P^i (v - w) \right\| \leq \sum_{i=0}^k \lambda_i \|P^i v - P^i w\| \\ &\leq \sum_{i=0}^k \lambda_i (\|v - w\| + f(w, v)) \\ &\leq \|v - w\| + f(w, v) \end{aligned}$$

where,

$$f(w, v) = \frac{c_5(w, v)}{1 - c_2(w, v) - c_3(w, v) - c_4(w, v) - 2c_5(w, v)}$$

In particular, the above inequality for $v = q$ gives,

$$\|B^{n+1} q - w\| \leq \|B^n q - w\|.$$

Since $B^{n_i}q \rightarrow w$ as $i \rightarrow \infty$ this implies that $B^n q \rightarrow w$ as $n \rightarrow \infty$. The case $v = w$ trivially gives us $B^n v \rightarrow w$ and that completes the proof.

Note that if we take $c_1 = c_3 = c_4 = c_5 = 0$, $c_2 = \frac{1}{2}$ and $k = 1$ we obtain the result in [4].

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