ON THE COERCIVENESS PROPERTY OF THE BIHARMONIC OPERATOR

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A B S T R A C T

We consider the weak formulation of the biharmonic equation under two different kinds of boundary conditions. It is shown, in one case, that the coerciveness of the bilinear form associated can be easily deduced by using the continuous- dependence result for the Laplace equation with Dirichlet data. In the second case, a generalized Poincaré inequality is readily employed.

INTRODUCTION

Let Ω be a bounded domain in the plane with smooth boundary $\partial\Omega$. For any integer m , $H^m(\Omega)$ and $H_0^m(\Omega)$ denote the usual Sobolev spaces of order m [4], [5]. Similarly, for $s \in \mathbb{R}$, $H^s(\partial\Omega)$ denotes the corresponding Sobolev space on the boundary [5]. Now, given $f \in L^2(\Omega)$, $g \in H^{3/2}(\partial\Omega)$, $h_1 \in H^{-1/2}(\partial\Omega)$ and $h_2 \in H^{-3/2}(\partial\Omega)$, we define the following BVPs

$$\begin{aligned} \Delta^2 u &= f, \text{ in } \Omega \\ u &= g, \text{ on } \partial\Omega \\ M[u] &= h_1, \text{ on } \partial\Omega \end{aligned} \quad (1)$$

and

$$\begin{aligned} \Delta^2 u &= f, \text{ in } \Omega \\ M[u] &= h_1, \text{ on } \partial\Omega \\ N[u] &= h_2, \text{ on } \partial\Omega \end{aligned} \quad (2)$$

where M and N are the boundary operators given by

$$M[u] := \sigma \Delta u + (1 - \sigma) \frac{\partial^2 u}{\partial n^2}$$

and

$$N[u] := - \frac{\partial \Delta u}{\partial n} + (1 - \sigma) \frac{\partial}{\partial T} \left[\frac{\partial^2 u}{\partial x^2} n_x n_y - \frac{\partial^2 u}{\partial x \partial y} (n_x^2 - n_y^2) - \frac{\partial^2 u}{\partial y^2} n_x n_y \right]$$

Here, $n := (n_x, n_y)$ is the outward normal to $\partial\Omega$, $x := (x, y)$ is a generic vector in $\mathbb{R}^2$, $\frac{\partial}{\partial T}$ denotes differentiation with respect to the arc length, and σ is the Poisson coefficient which is assumed to be $0 < \sigma < 1$.

The following version of the Lax-Milgram lemma will be utilized [2], [6].

Lemma 1.

Let B be a bilinear form in a real Hilbert H with norm $\|\cdot\|_H$. Suppose that there exist positive constants α, β such that

$$|B(v, w)| \leq \alpha \|v\|_H \|w\|_H \quad (3)$$

for all $v, w \in H$, and

$$\beta \|v\|_H^2 \leq B(v, v) \quad (4)$$

for all $v \in H$.

Then, for every bounded linear functional $F \in H'$, there exists a unique $u \in H$ such that

$$B(u, v) = F(v)$$

for all $v \in H$.

Moreover, the following estimate holds

$$\|u\|_H \leq \frac{1}{\beta} \|F\|_{H'} \quad (5)$$

Proof:

See [2] : Theorem 1.1.3.

The inequality (4) is called the coerciveness property or H -ellipticity of B . Also, (5) is commonly referred as the continuous-dependence result. It is worth remarking that a more general lemma than the above one can be seen in ([4] : Theorem 14.1) and also in ([1] : Theorem 5.2.1).

THE WEAK FORMULATIONS

First, we turn our interest to the BVP (1) which physically represents the deflection of a horizontal vertically loaded plate with prescribed deformation and moments along the boundary.

By writing

$$\begin{aligned} \Delta^2 u = & \frac{\partial^2}{\partial x^2} \left[\frac{\partial^2 u}{\partial x^2} \right] + 2(1-\sigma) \frac{\partial^2}{\partial x \partial y} \left[\frac{\partial^2 u}{\partial x \partial y} \right] + \sigma \frac{\partial^2}{\partial x^2} \left[\frac{\partial^2 u}{\partial y^2} \right] + \\ & \sigma \frac{\partial^2}{\partial y^2} \left[\frac{\partial^2 u}{\partial x^2} \right] + \frac{\partial^2}{\partial y^2} \left[\frac{\partial^2 u}{\partial y^2} \right] \end{aligned}$$

and then, applying integration by parts we obtain after a few computations (see [6] : chapter 23)

$$\int_{\Omega} v \Delta^2 u dx = B(u, v) - \int_{\partial\Omega} v N[u] dT - \int_{\partial\Omega} \frac{\partial v}{\partial n} M[u] dT \quad (6)$$

for all $u \in H^4(\Omega)$, $v \in H^2(\Omega)$, where B is the bilinear form given by

$$B(u, v) := \int_{\Omega} \{ \sigma \Delta u \Delta v + (1-\sigma) \left[\frac{\partial^2 u}{\partial x^2} \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \frac{\partial^2 v}{\partial y^2} + 2 \frac{\partial^2 u}{\partial x \partial y} \frac{\partial^2 v}{\partial x \partial y} \right] \} dx \quad (7)$$

Now on, we denote by H the Hilbert $H^2(\cdot) \cap H_0^1(\Omega)$ provided with the norm $\|\cdot\|_{H^2(\Omega)}$. Then, we might define the weak formulation of (1) as follows : FIND $u \in H^2(\Omega)$ with $\gamma_0 u := \text{tr } u = g$, on $\partial\Omega$, such that

$$B(u, v) = \int_{\Omega} f v dx + \int_{\partial\Omega} h_1 \frac{\partial v}{\partial n} dT \quad (8)$$

for all $v \in H$. Here, $\gamma_0 := \text{tr}$ stands for the trace operator of order 0 [5].

Obviously, we can not apply lemma 1 to the above formulation. Nevertheless, since $g \in H^{3/2}(\partial\Omega)$, the trace theorem implies the existence of a function $\tilde{w} \in H^2(\Omega)$ such that $\gamma_0 \tilde{w} := \text{tr } \tilde{w} = g$, on $\partial\Omega$. So, if we set $\tilde{u} = u - \tilde{w}$, then it is clear that $\tilde{u} \in H$. Consequently, we consider instead of (8) the following equivalent formulation : FIND $\tilde{u} \in H$ such that

$$B(\tilde{u}, v) = \int_{\Omega} f v dx + \int_{\partial\Omega} h_1 \frac{\partial v}{\partial n} dT - B(w, v) \quad (9)$$

for all $v \in H$.

Our aim here is to prove the H - ellipticity of B in a simpler way than that usually given in the literature (see, e.g., [6] : Pag. 395). For this, we need the following result.

Lemma 2

Let $|\cdot|_{H^2(\Omega)}$ be the usual seminorm in $H^2(\Omega)$, i.e.

$$|v|_{H^2(\Omega)}^2 = \sum_{|\alpha|=2} \|D^\alpha v\|_{L^2(\Omega)}^2$$

for all $v \in H^2(\Omega)$. Then, there exists a positive constant C so that

$$|v|_{H^2(\Omega)}^2 \geq C \|v\|_{H^2(\Omega)}^2 \quad (10)$$

for all $v \in H$.

Proof:

Given $v \in H := H^2(\Omega) \cap H_0^1(\Omega)$, we define $f_v = -\Delta v$ and consider the following Dirichlet problem with homogeneous data: FIND $u \in H_0^1(\Omega)$ such that

$$\int_{\Omega} \nabla u \cdot \nabla w \, dx = \int_{\Omega} f_v w \, dx \quad (11)$$

for all $w \in H_0^1(\Omega)$.

Since $f_v \in L^2(\Omega)$, it is very well known that (11) has a unique solution u which according to (5) (lemma 1) satisfies

$$\|u\|_{H^1(\Omega)} \leq \frac{1}{\beta} \|f_v\|_{L^2(\Omega)} \quad (12)$$

for some positive constant β . But, since $v \in H^2(\Omega)$, the Green's second identity provides that, in fact, the unique solution of (11) is v itself. Therefore, instead of (12) we can write

$$\|v\|_{H^1(\Omega)} \leq \frac{1}{\beta} \|\Delta v\|_{L^2(\Omega)}$$

Hence, we deduce that

$$\|v\|_{H^2(\Omega)}^2 \leq \frac{1}{\beta^2} \|\Delta v\|_{L^2(\Omega)}^2 + |v|_{H^2(\Omega)}^2$$

and since $\|\Delta v\|_{L^2(\Omega)}^2 \leq 2 |v|_{H^2(\Omega)}^2$, we conclude that

$$C \|v\|_{H^2(\Omega)}^2 \leq |v|_{H^2(\Omega)}^2$$

for all $v \in H$, with $C = (1 + \frac{2}{\beta^2})^{-1}$.

Finally, the H -ellipticity of B is a direct consequence of the above lemma. In fact, from (7) and (10) we obtain

$$\begin{aligned}
B(v, v) &= \sigma \|w\|_{L^2(\Omega)}^2 + (1-\sigma) \|v\|_{H^2(\Omega)}^2 \\
&\geq (1-\sigma) \|v\|_{H^2(\Omega)}^2 \\
&\geq C(1-\sigma) \|v\|_{H^2(\Omega)}^2
\end{aligned}$$

for all $v \in H$.

Now, we turn our attention to the Neumann problem (2) which also arises in the theory of thin plates [3]. It follows easily from (2) and (6) that the weak formulation of (2) is given by: FIND $u \in H^2(\Omega)$ such that

$$B(u, v) = F(v) \quad (13)$$

for all $v \in H^2(\Omega)$, where F is the bounded linear functional given by

$$F(v) := \int_{\Omega} f v dx + \int_{\partial\Omega} v h_2 d\Gamma + \int_{\partial\Omega} \frac{\partial v}{\partial n} h_1 d\Gamma$$

On the other hand, as indicated in [6], it is clearly seen that any linear combination of the polynomials 1, x and y is a solution of the homogeneous case for (13). So, in order to get uniqueness certain normalization conditions are required.

We define the functionals

$$\begin{aligned}
f_1, f_2, f_3 &: H^2(\Omega) \rightarrow \mathbb{R} \\
f_1(v) &:= \int_{\Omega} v dx, \quad f_2(v) := \int_{\Omega} x v dx \\
f_3(v) &:= \int_{\Omega} y v dx
\end{aligned}$$

Since Ω is bounded, it is clear that f_i is a bounded linear functional for all $i = 1, 2, 3$.

Thus, we consider instead of (2) the following BVP

$$\begin{aligned}
\Delta^2 u &= f, \text{ in } \Omega \\
M[u] &= h_1, \quad N[u] = h_2, \text{ on } \partial\Omega \\
f_i(u) &= A_i, \quad i = 1, 2, 3
\end{aligned} \quad (14)$$

where A_1, A_2, A_3 are given real numbers. This means that (13) is replaced by:

FIND $u \in H^2(\Omega)$ with $f_i(u) = A_i, i = 1, 2, 3$ such that

$$B(u, v) = F(v) \quad (15)$$

for all $v \in H^2(\Omega)$.

Now, for a given $v \in H^2(\Omega)$, there exists a unique polynomial $p(x,y) := a + bx + cy$ such that

$$f_i(v) = f_i(p), \quad i = 1, 2, 3 \quad (16)$$

In fact, if we write out the equation (16), then we obtain the Gramm system whose determinant is always different from zero.

So, if $H_E^2(\Omega)$ denotes the space

$$H_E^2(\Omega) := \{ w \in H^2(\Omega) : f_i(w) = 0, \quad i = 1, 2, 3 \}$$

then, it is obvious that $\tilde{v} := v - p$ belongs to $H_E^2(\Omega)$ for all $v \in H^2(\Omega)$.

Therefore, by replacing $v = \tilde{v} + p$ in (15), noting that $B(p,v) = B(v,p) = 0$ for all p, v , and assuming that f, h_1, h_2 satisfy the compatibility conditions

$$F(v) = 0, \quad \text{for } v = 1, x, y$$

we obtain the following weak formulation, equivalent to (15) :

FIND $\tilde{u} \in H_E^2(\Omega)$ such that

$$B(\tilde{u}, v) = F(v) \quad (17)$$

for all $v \in H_E^2(\Omega)$.

It is important to note that if $\tilde{p}$ is the unique polynomial satisfying $f_i(\tilde{p}) = A_i$, $i = 1, 2, 3$, then $u := \tilde{u} + \tilde{p}$ is the corresponding solution of (15).

Our next aim is to show that the $H_E^2(\Omega)$ - ellipticity of the bilinear form B is a simple corollary of the following generalized Poincaré inequality [5].

Theorem 1

Denote by P_{k-1} the set of all polynomials in $\mathbb{R}^2$ of degree $\leq k-1$. Let F be a finite family of bounded linear functionals on the Sobolev space $H^{k,p}(\Omega)$, with the property.

$$v \in P_{k-1} \text{ and } f(v) = 0, \quad \text{for all } f \in F \text{ implies } v \equiv 0 \quad (18)$$

Suppose Ω is a $C^{0,1}$ domain. Then

$$\|v\|_F := \left\{ \sum_{|\alpha|=k} \|D^\alpha v\|_{L^p(\Omega)}^p + \sum_{f \in F} |f(v)|^p \right\}^{\frac{1}{p}} \quad (19)$$

is a norm in $H^{k,p}(\Omega)$ which is equivalent to the usual norm $\|v\|_{H^{k,p}(\Omega)}$, i.e. there exist positive constants C_1, C_2 so that

$$C_1 \|v\|_{H^{k,p}(\Omega)} \leq \|v\|_F \leq C_2 \|v\|_{H^{k,p}(\Omega)}$$

for all $v \in H^{k,p}(\Omega)$.

Proof:

See [5] : Theorem 5.4.2.

In the particular case of $k = p = 2$, we already noted that our family $F := \{f_1, f_2, f_3\}$ satisfies clearly the property (18). Therefore, there exists $C_1 > 0$ such that

$$\|v\|_F \geq C_1 \|v\|_{H^2(\Omega)} \quad (20)$$

for all $v \in H^2(\Omega)$, where

$$\|v\|_F^2 := |v|_{H^2(\Omega)}^2 + \sum_{i=1}^3 |f_i(v)|^2 \quad (21)$$

It follows from (21) that $\|v\|_F = |v|_{H^2(\Omega)}$, for all $v \in H_E^2(\Omega)$. Hence, we conclude that

$$\begin{aligned} B(v, v) &= \sigma \|Av\|_{L^2(\Omega)}^2 + (1-\sigma) |v|_{H^2(\Omega)}^2 \\ &\geq (1-\sigma) \|v\|_F^2 \\ &\geq C_1 (1-\sigma) \|v\|_{H^2(\Omega)}^2 \end{aligned}$$

for all $v \in H_E^2(\Omega)$.

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