CONVERGENCE OF BOUNDARY ELEMENT METHODS FOR NUMERICAL SOLUTIONS OF FOURIER PROBLEMS

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ABSTRACT

Convergence proofs are given for the projection based boundary element methods for the numerical solution of various Fourier problems in regions with smooth compact boundaries. Volterra integral equations of the 2nd kind are formulated with associated integral operators mapping the space of continuous functions on a compactum into itself. The compactness of these operators is shown, yielding the error estimates in supremum norms for a wide class of projection based BEMs. Extensions of the error analysis to the initial-boundary value problems of convective heat conduction are also discussed.

Key words and phrases. Boundary elements, heat equation, heat potentials, Volterra weakly singular integral equations, collocation methods.

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INTRODUCTION

The boundary element methods (BEMs) have become well established in numerical solution of boundary and initial-boundary value problems in the last 25 years (see, e.g., [4], [5]). Part of their success is due to the underlying idea of formulating the problem in terms of a boundary in-

tegral equation and discretizing it to obtain a system of algebraic equations. The resulting system is of much smaller dimension than the systems obtained from application of finite difference or finite element methods.

The mathematical analysis of BEMs focuses on two aspects:

- a. questions of existence, uniqueness, and regularity of the associated integral equations,
- b. questions of convergence and stability of discretizations.

The answer to the first group of questions is provided by the theory of integral equations. The theory for boundary integral equations arising in linear initial-boundary problems of classical mathematical physics is well understood in the case when the domain of the corresponding partial differential equation has a smooth boundary (see [13]). In the case of domains with corners much less is known about the solvability and the behaviour of the solutions of the integral equations and hence about the convergence of the corresponding BEMs. Most of the progress for the problems with non-smooth boundaries has been achieved in the analysis of BEMs for elliptic problems ([13], [2], [6], [9]).

The convergence analysis of the BEMs for problems with smooth boundaries is almost complete for elliptic equations [14]. In comparison, the convergence analysis of the BEMs for parabolic problems has obtained much less attention (see [8], [12]), although there exist many algorithms for solution of transient problems in diffusion and heat transfer ([4], [5], [7], [11], [12]).

In this paper we perform the asymptotic error analysis of a class of BEMs for the solution of various Fourier problems in smooth compact regions (or their complements). In the last section we show that the results can be extended to a conductive heat equation with constant coefficients. The proofs of the results rest on the formulations of the problems in terms of Volterra integral equations of the second kind, which can be viewed as operator equations on the space of continuous functions into itself. In case of the Fourier problem of the 3rd (or 2nd) type this requires a modification of the standard integral equation. The compactness of the associated operators implies the convergence of the projection based BEMs and yields the asymptotic error estimates.

Although there are many boundary integral representations for any linear initial-boundary value problem with different implications for the numerical schemes, the BEMs investigated in this work are based on the integral equations of the second kind derived from the potential formulation. Other types of formulations can also be used (for example, a direct approach) provided that they lead to integral equations of the second kind.

2. FORMULATION OF FOURIER PROBLEMS

Let $\Omega \subset \mathbb{R}^n$ denote an open bounded set, $n \geq 2$, with a sufficiently smooth boundary Γ . Consider the heat equation:

$$(2.1) \quad (\Delta - \partial/\partial t)u(x, t) = q(x, t), \quad t \in (0, T), \quad T < \infty, \quad x \in \Omega,$$

(or $x \in \Omega^C$ for external problems) subject to the initial condition $u(x,0) = \varphi(x)$ and to some specified boundary condition.

For simplicity we consider only a planar case, $n=2$, of the internal boundary problem, $x \in \Omega$. Noting that a particular solution of (2.1) is given by the volume potential

$$A(x,t) = \int_0^t \int_{\Omega} q(y,\tau) w(x-y, t-\tau) dy d\tau,$$

where

$$w(x,t) = \frac{1}{4\pi t} \exp \left\{ -\frac{|x|^2}{4t} \right\}$$

is the fundamental solution of the heat equation (2.1), it suffices, without loss of generality, to restrict ourselves to the homogeneous case, $q = 0$.

We examine two different types of Fourier problems with the following boundary conditions:

$$(2.2) \quad u(\xi, t) = \psi(\xi, t), \quad \xi \in \Gamma \quad (\text{Dirichlet or 1st type condition}) \quad \text{and}$$

$$(2.3) \quad \left(\frac{\partial}{\partial n} - \lambda(\xi, t) \right) u(\xi, t) = \gamma(\xi, t) \quad (\text{Robin or 3rd type condition}),$$

where n denotes the normal inward unit to Γ .

We note that if $\lambda = 0$, then (2.3) describes the boundary conditions of Neumann type (2nd type Fourier problem).

The solution to the Fourier problem of the 1st type can be written as a superposition of the heat potential of the double layer U and the Poisson-Weierstrass integral I :

$$(2.4) \quad u(x,t) = U(x,t) + I(x,t),$$

$$U(x,t) = \int_0^t \int_{\Gamma} f(\xi, \tau) \frac{\partial w}{\partial n_{\xi}}(x-\xi, t-\tau) ds_{\xi} d\tau, \quad I(x,t) = \int_{\Omega} w(x-y,t) \varphi(y) dy,$$

where f is the density of the double layer potential and n_{ξ} denotes the normal inward unit vector to Γ at the point $\xi \in \Gamma$.

The integral equation for the unknown potential density f is obtained using the jump condition for U , yielding

$$(2.5) \quad f(\xi, t) + \int_0^t \int_{\Gamma} f(\zeta, \tau) k(\xi-\zeta, t-\tau) ds_{\zeta} d\tau = g(\xi, t),$$

where $k(\xi-\zeta, t-\tau) = 2 \frac{\partial w}{\partial n_{\zeta}}(\xi-\zeta, t-\tau)$, $g(\xi, t) = 2 [\psi(\xi, t) - I(\xi, t)]$, $\xi, \zeta \in \Gamma$.

The above equation is a two-dimensional Volterra equation of the second kind with a singularity for $\zeta \rightarrow \xi$, $\tau \rightarrow t$. The iterated kernel method [13] can be used to show the existence of a unique continuous solution f of (2.5) given the sufficient smoothness of Γ (Γ ought to be a

Liapunov boundary) and the continuity of φ and ψ .

The solution of the Robin problem is sought as a superposition of the heat potential of a single layer V and the integral I :

$$(2.6) \quad u(x, t) = V(x, t) + I(x, t),$$

$$V(x, t) = \int_0^t \int_{\Gamma} h(\xi, \tau) w(x - \xi, t - \tau) ds_{\xi} d\tau.$$

In this case the integral equation for the density h is

$$(2.7) \quad h(\xi, t) + \int_0^t \int_{\Gamma} h(\zeta, \tau) \hat{v}(\xi - \zeta, t - \tau) ds_{\zeta} d\tau = \hat{k}(\xi, t),$$

where

$$\hat{v}(\xi - \zeta, t - \tau) = -2 \left[\frac{\partial w}{\partial n_{\xi}}(\xi - \zeta, t - \tau) + \lambda(\xi, t) w(\xi - \zeta, t - \tau) \right],$$

$$\hat{k}(\xi, t) = -2 \{ \gamma(\xi, t) - \left[\frac{\partial}{\partial n_{\xi}} + \lambda(\xi, t) \right] I(\xi, t) \}.$$

Remark 2.1.

The existence and uniqueness of the solution h can be proved via the iterated kernel method. The potential density is continuous for $\xi \in \Gamma$, $t > 0$ and, in general, unbounded as $t \rightarrow 0$. The singular behaviour of h is a result of the singularity of the normal derivative of the Poisson-Weierstrass integral I . More specifically it can be shown that

$$|h(\xi, t)| < \text{const. } t^{-\alpha}, \text{ for any } \alpha > 1/2.$$

The above fact complicates the numerical analysis of the boundary element methods for the Fourier problems with Robin (or Neumann) boundary conditions. To circumvent this difficulty we multiply (2.7) by t^{α} and set $f(\xi, t) = t^{\alpha} h(\xi, t)$ which yields the following integral equation for the unknown f

$$(2.8) \quad f(\xi, t) + \int_0^t \int_{\Gamma} f(\zeta, \tau) v(\xi - \zeta, t - \tau) ds_{\zeta} d\tau = k(\xi, t),$$

$$v(\xi - \zeta, t - \tau) = (t/\tau)^{\alpha} \hat{v}(\xi - \zeta, t - \tau), \quad k(\xi, t) = t^{\alpha} \hat{k}(\xi, t).$$

The equation (2.8) also belongs to the class of multidimensional, weakly singular Volterra equation of the second kind, but the singularity is stronger than that in the case of equation (2.7). The unique solution f exists and is continuous for $t \geq 0$ by virtue of the uniqueness of h in equation (2.7).

3. NUMERICAL ALGORITHMS

The numerical solution of presented integral equations constitutes the first and most difficult step in the boundary element method. Of course, once the density f is found, the BEM requires postprocessing for the evaluation of the originally sought solution u at (x, t) from representations (2.4) or (2.6). This is often costly and creates some numerical difficulties if the point x lies near the boundary Γ . However, we will not discuss those problems here.

There exist many ways to discretize equation (2.5) or (2.8). Some of them include Galerkin, quadrature or collocation techniques. In this paper we consider a class of collocation methods, postponing some remarks on the other methods until section 6.

Assuming that Γ is a smooth Jordan curve with a parametrization $\Gamma: \xi = \xi(s)$, $\xi(0) = \xi(1)$, the integral equations (2.5), (2.8) become the equations for 1-periodic function $f: f(\xi(s), t)$ for $s \in [0, 1]$ and $t \in [0, T]$ for some $T > 0$.

To facilitate the numerical analysis of the methods we rewrite equations (2.5), (2.8) in the operator notation as

$$(3.1) \quad (I + K)f = g, \quad \text{and}$$

$$(3.2) \quad (I + N)f = k.$$

Let X be the space of continuous functions on $D = [0, 1] \times [0, T]$, periodic in the first variable. Choosing a family of grid partitions on D with $s_i = i/m$ ($i=1, \dots, m$), $t_j = T/n$ ($j=1, \dots, n$), we let $\{P_{mn}\}$ be a set of projection operators on X defined by

$$P_{mn}g(s, t) = \sum_{i=1}^m \sum_{j=1}^n g(s_i, t_j) S_{ij}(s, t)$$

such that $P_{mn}g \rightarrow g$ in the uniform norm, where S_{ij} are (continuous) shape functions. This definition of P_{mn} includes, for example, most piecewise polynomial interpolation operators (but not piecewise constant interpolation), spline or Hermite interpolation operators, etc.

The equations (3.1), (3.2) are replaced by

$$(I + K) \sum_{i=1}^m \sum_{j=1}^n f_{ij} S_{ij}(s, t) = \sum_{i=1}^m \sum_{j=1}^n g(s_i, t_j) S_{ij}(s, t), \quad \text{and}$$

$$(I + N) \sum_{i=1}^m \sum_{j=1}^n f_{ij} S_{ij}(s, t) = \sum_{i=1}^m \sum_{j=1}^n k(s_i, t_j) S_{ij}(s, t),$$

which are required to be satisfied at the collocation points s_i, t_j . This leads to a system of linear equations for each of the above equations, for the unknown values f_{ij} . Such systems are completely equivalent to the operator equations

$$(3.3) \quad (I + K_{mn})f_{mn} = g_{mn} \quad \text{and}$$

$$(3.4) \quad (I + N_{mn})f_{mn} = k_{mn},$$

where $K_{mn} = P_{mn}K$, $g_{mn} = P_{mn}g$ and other symbols are similarly defined.

Once the integral equations and their discrete approximations are written as operator equations, the numerical analysis of the convergence of these approximations may follow a standard path of the following theorem on projection operators (cf., [1], [3]) :

Theorem 3.1.

Let X be a Banach space and $K : X \rightarrow X$ a linear, bounded, compact operator such that $(I + K)^{-1}$ exists and is bounded. Let $\{P_n\}_{n=1}^{\infty}$ be a family of projections and $P_n x \rightarrow x$ as $n \rightarrow \infty$ for all $x \in X$. Let $K_n = P_n K$. Then $\|K - K_n\| \rightarrow 0$.

Consequently, given $(I + K)f = g$ and $(I + K_n)f_n = P_n g$ we have

$$\|f - f_n\| \leq \|(I + K_n)^{-1}\| [\|K - K_n\| \|f\| + \|P_n g - g\|],$$

where $\|(I + K_n)^{-1}\|$ exist and are uniformly bounded for sufficiently large n .

To prove the compactness of integral operators the following lemma is often useful :

Lemma 3.2.

Let $D \in \mathbb{R}^n$ be a compact set, $X = C(D)$, and $K : X \rightarrow X$ an integral operator defined as $Kf(x) = \int_D k(x,y)f(y)dy$. If

$$\lim_{\delta \rightarrow 0} \sup_{|x-z| \leq \delta} \int_D |k(x,y) - k(z,y)| dy = 0, \text{ where } |x-z| = \text{dist}(x,z),$$

then K is compact.

Proof. An application of the Arzela-Ascoli theorem. $\square$

The next 2 sections will demonstrate that K (and N) satisfies the hypotheses of Theorem 3.1, proving convergence of the collocation based BEMs for the Fourier problem of the 1st (3rd) type, together with the associated rates of convergence.

4. ERROR ANALYSIS FOR THE FIRST FOURIER PROBLEM

We need the following lemmas to show that the projection theorem can be applied to equations (3.1), (3.3).

Lemma 4.1.

The integral operator K in equation (3.1) is a linear, bounded, compact operator mapping a Banach space X into itself.

Proof. It is clear that X is a B-space and K is linear. The boundedness of K follows from the fact that the kernel of K can be written as

$$k(s, \sigma, t, \tau) = (t - \tau)^{-q} R_{s\sigma}^{-2+2q-\beta} L(s, \sigma, t, \tau),$$

where $1/2 < q < 1$, $0 < \beta < 2q-1$, $R_{s\sigma} = \|\xi(s) - \xi(\sigma)\|_2$, $B = \frac{R_{s\sigma}^2}{4(t-\tau)}$, $\|\cdot\|_2$ a euclidean norm, and L given by

$$L(s, \sigma, t, \tau) = 2^{2-2q/\pi} R^{\beta} \|\xi'\|_2 \frac{\cos \xi(s) - \xi(\sigma), \eta}{R_{s\sigma}} B^{2-q} \exp(-B).$$

One can show that function L is uniformly continuous in $D \times D$, so that the kernel $k(s, \sigma, t, \tau)$ is weakly singular. Then

$$\|K\| \leq \|L\| \int_0^t (t-\tau)^{-q} d\tau \int_0^1 R_{s\sigma}^{-2+2q-\beta} d\sigma \leq \text{const.} < \infty, \quad 0 \leq t \leq T.$$

To show that K is compact we use the Ascoli-Arzelà theorem. The set $S = \{Kf : f \in X, \|f\| \leq 1\}$ is bounded by the previous argument. Its equicontinuity can be seen from

$$\begin{aligned} & |Kf(s_1, t_1) - Kf(s_2, t_2)| \leq \\ & \leq \left| \int_{t_2}^{t_1} \int_0^1 k(s_1, \sigma, t_1, \tau) f(\sigma, \tau) d\sigma d\tau \right| \\ & + \left| \int_0^{t_2} \int_0^1 \frac{f(\sigma, \tau)}{(t_1 - \tau)^q R_{s_1 \sigma}^{2-2q+\beta}} [L(s_1, \sigma, t_1, \tau) - L(s_2, \sigma, t_2, \tau)] d\sigma d\tau \right| \\ & + \left| \int_0^{t_2} \int_0^1 \left[\frac{f(\sigma, \tau)}{(t_1 - \tau)^q R_{s_1 \sigma}^{2-2q+\beta}} - \frac{f(\sigma, \tau)}{(t_2 - \tau)^q R_{s_2 \sigma}^{2-2q+\beta}} \right] L(s_2, \sigma, t_2, \tau) d\sigma d\tau \right| \\ & \leq \|f\| \left| \int_{t_2}^{t_1} \int_0^1 k(s_1, \sigma, t_1, \tau) d\sigma d\tau \right| \\ & + \|f\| \sup_{\sigma, \tau} |L(s_1, \cdot, t_1, \cdot) - L(s_2, \cdot, t_2, \cdot)| \sup_{\sigma, \tau} \int_0^{t_2} \int_0^1 \frac{d\sigma d\tau}{(t_1 - \tau)^q R_{s_1 \sigma}^{2-2q+\beta}} \\ & + \|L\| \|f\| \left| \int_0^{t_2} \int_0^1 \left[\frac{1}{(t_1 - \tau)^q R_{s_1 \sigma}^{2-2q+\beta}} - \frac{1}{(t_2 - \tau)^q R_{s_2 \sigma}^{2-2q+\beta}} \right] d\sigma d\tau \right|. \end{aligned}$$

All the above bounds can be made arbitrarily small irrespectively of $f \in S$, for sufficiently small $|t_1 - t_2|$, $|s_1 - s_2|$ thanks to the uniform continuity of L and

$$\int_0^t \int_0^1 \frac{d\sigma d\tau}{(t-\tau)^{q_R 2-2q+\beta}} \quad s \in \sigma$$

The above remarks complete the arguments to establish the equicontinuity of S , which proves the compactness of K . $\square$

Lemma 4.2.

(See [13]). The inverse of $I+K : X \rightarrow X$ exists and is bounded.

Proof. The inverse of the operator $I+K$ is given by

$$(I+K)^{-1}g(s,t) = g(s,t) + \int_0^t \int_0^1 \mathcal{K}(s,\sigma,t,\tau)g(\sigma,\tau)d\sigma d\tau, \quad s \in \sigma$$

where $g \in X$, $\mathcal{K}(s,\sigma,t,\tau) = \sum_{i=0}^{\infty} K_i(s,\sigma,t,\tau)$,

$$K_i(s,\sigma,t,\tau) = \int_{\tau}^t \int_0^1 K(s,r,t,u)K_{i-1}(r,\sigma,u,\tau)drdu$$

and $\int_0^t \int_0^1 \mathcal{K}(s,\sigma,t,\tau)d\sigma d\tau$ is uniformly convergent for all $(s,t) \in D$. $\square$

The above two lemmas combined with Theorem 3.1 give the following result on the convergence of the projection based BEMs for a Fourier problem of the first type :

Theorem 4.3.

Let $f, g \in C^1(D)$, $D = \Omega \times [0, T]$, Ω a planar domain with a differentiable boundary and $\{P_{mn}\}$ a family of projections such that

$\|P_{mn}g - g\| = O(m^{-r} + n^{-r})$. Then f_{mn} from equation (3.3) approximates f in (3.1) and

$$\|f - f_{mn}\| = O(m^{-r} + n^{-r}).$$

Remark 4.1.

In the implementation of the BEMs from Theorem 4.3 one needs to evaluate the elements of discretization matrices corresponding to K_{mn} .

These elements are double integrals of the form $\int_0^t \int_0^1 K(s,\sigma,t,\tau)S_{ij}(\sigma,\tau)d\sigma d\tau$,

which can not be evaluated exactly even for the simplest shape functions S_{ij} (sometimes integration with respect to τ can be performed, see [4] or [5]). A numerical scheme must be employed to assemble the required matrices. We neglect here the influence of the errors of numerical integration, commenting however, that they ought to be of the same order as the error of the method.

5. ERROR ANALYSIS FOR THE THIRD FOURIER PROBLEM

Now we turn to the investigation of the BEM for the Fourier problem of the third type. We will show that the operator N in equation (3.2) satisfies the hypotheses of Theorem 3.1. Note that the kernel of the operator N can be written in the form

$$(5.1) \quad v(s, \sigma, t, \tau) = (t/\tau)^\alpha (t-\tau)^{-q} R_{s\sigma}^{-2+2q-\beta} L(s, \sigma, t, \tau)$$

where $R_{s\sigma} = \|\xi(s) - \xi(\sigma)\|_2$, $1/2 < q < 1$, $0 < \beta < 2q-1$, $B = \frac{R_{s\sigma}^2}{4(t-\tau)}$,

$$L(s, \sigma, t, \tau) = 2^{1-2q/\pi} R^B \|\xi'\|_2 \left[2B \frac{\cos(\xi(s) - \xi(\sigma), n)}{R_{s\sigma}} + \lambda(\sigma, \tau) \right] B^{1-q} \exp(-B).$$

The singularity of v in (5.1) is weak, but slightly stronger than the one discussed in section 4. We begin its investigation with the following lemma:

Lemma 5.1.

Given $T > 0$, let Q be an operator defined as

$$Qf(t) = \int_0^t (t/\tau)^\alpha (t-\tau)^{-q} f(\tau) d\tau, \quad 0 < \alpha, q < 1.$$

Then Q is a linear, bounded, compact operator from Y , the space of continuous functions on $[0, T]$ into itself.

Proof. The boundedness of Q follows from

$$|Qf(t)| \leq t^\alpha \|f\| \int_0^t \tau^{-\alpha} (t-\tau)^{-q} d\tau = t^{1-q} \|f\| B(1-\alpha, 1-q)$$

where $B(1-\alpha, 1-q)$ is the value of the Beta function.

The above inequality also shows that given $f \in Y$, Qf is continuous at $t = 0$ and $Qf(0) = 0$. To show the continuity of Qf for $t > 0$ we must consider the differences $|Qf(t+\delta) - Qf(t)|$ and $|Qf(t) - Qf(t-\delta)|$, $\delta > 0$, separately. The first expression can be seen to converge to zero as $\delta \rightarrow 0$ by the dominated convergence theorem. For the second one we have

$$|Qf(t) - Qf(t-\delta)| \leq \|f\| \left[\int_0^{t-\delta} s^{-\alpha} \left[\frac{t^\alpha}{(t-s)^q} - \frac{(t-\delta)^\alpha}{(t-\delta-s)^q} \right] ds + \int_{t-\delta}^t (t/s)^\alpha (t-s)^{-q} ds \right].$$

It is easy to show that the last integral can be made arbitrarily small. To see that this is also true for the first one we write it as

$$\int_0^{t-\delta} s^{-\alpha} \left[\frac{t^\alpha}{(t-s)^q} - \frac{(t-\delta)^\alpha}{(t-s)^q} \right] ds + \int_0^{t-\alpha} s^{-\alpha} \left[\frac{(t-\delta)^\alpha}{(t-s)^q} - \frac{(t-\delta)^\alpha}{(t-\delta-s)^q} \right] ds.$$

Another application of the dominated convergence theorem for the second integral and the continuity of the function t^α in the first one show that these expressions decrease to 0 as $\delta \rightarrow 0$. This shows that Qf is continuous for all $t \in (0, T]$, so $Q : Y \rightarrow Y$.

The above arguments also show that the lemma 3.2 can be used for operator Q implying its compactness. $\square$

Using the above lemma one can easily obtain:

Lemma 5.2.

The integral operator N in equation (3.2) is linear, bounded and compact and $N : X \rightarrow X$.

Proof. The boundedness of N follows from representation (5.1) the fact that L is (uniformly) continuous and the boundedness of Q .

Noting that N can be viewed as a composition of operators, $N = QT$, where

$$Tf(s, t, \tau) = \int_0^1 R_{s\sigma}^{-2+2q-\beta} L(s, \sigma, t, \tau) f(\sigma, \tau) d\sigma,$$

and T is a linear, bounded operator mapping X into the space of continuous functions on $D \times [0, T]$, we conclude that $N : X \rightarrow X$ and N is compact. $\square$

Lemma 5.3.

The operator $(I + N)^{-1} : X \rightarrow X$ exists and is bounded.

Proof. $(I + N)^{-1}$ is given by the uniformly convergent resolvent series (see [13]). $\square$

The lemmas 5.1 - 5.3 give the following.

Theorem 5.4.

Let $k \in C^r(D)$, $D = \Omega \times [0, T]$, Ω a planar domain with a differentiable boundary and $\{P_{mn}\}$ a family of projections such that $\|P_{mn}k - k\| = O(m^{-r} + n^{-r})$. Then f_{mn} from equation (3.4) approximates f in (3.2) and

$$\|f - f_{mn}\| = O(m^{-r} + n^{-r}).$$

6. CONCLUDING REMARKS

We would like to point out that the presented approach to numerical analysis of convergence of BEMs for Fourier problems on irregular, smooth domains in the plane can be easily extended to various other problems. First, the restriction to planar problems is not essential;

similar proofs for convergence apply in higher dimensions with some small modifications. The error analysis does not change for exterior problems provided that the boundary of the region is compact.

More importantly, one can consider more general parabolic equations, for which a closed formula for the fundamental solution is known. The convergence of the numerical solutions of corresponding Volterra equations (of the second kind) will follow, provided that the kernels of such integral equations can be written as products of their continuous and weakly singular parts. As an example, we consider the following parabolic equation

$$(6.1) \quad \frac{\partial u}{\partial t} - \nabla \cdot (k \nabla u) + v \cdot \nabla u + ru = q,$$

where $u = u(x, t)$, $x \in \Omega \subset \mathbb{R}^n$, $t \in [0, T]$, subject to some initial and boundary conditions. In the case when the coefficients k , r , $v = (v_1, \dots, v_n)$, are constant, the fundamental solution of equation (6.1) is explicitly known [10] and is equal to

$$w(x, t) = \frac{1}{(4k\pi t)^{n/2}} \exp \left[-\frac{v \cdot x}{2k} - \frac{v \cdot v}{4k} t - \frac{|x|^2}{4kt} \right].$$

It is clear that the boundary integral equations formed with the use of the fundamental solution w have kernels with exactly the same kind of singularity as the kernels discussed in sections 4 and 5. Thus the convergence analysis in this case is essentially identical.

In the end we remark that the analysis of the BEMs for the Fourier (or more generally, parabolic) problems on domains with corners is more complicated due to the lack of compactness of corresponding integral operators (see a paper by Costabel et al., [8]).

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