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EIGENVALUE LOCALIZATION FOR COMPLEX MATRICES*

OSCAR ROJO and RICARDO L. SOTO

Universidad Católica del Norte, Antofagasta, Chile.

HECTOR ROJO

Universidad de Antofagasta, Antofagasta, Chile.

TIN-YAU TAM

Auburn University, Alabama, U.S.A.

Abstract

In this paper, we prove that all the eigenvalues λ_i of a complex matrix A of order n lie in a disk with center at $\text{tr}A/n$ and radius

$$\left[\frac{n-1}{n} \left(\sum_{i=1}^n |\lambda_i|^2 - \frac{|\text{tr}A|^2}{n} \right) \right]^{1/2}$$

This radius can be bounded without knowing the eigenvalues of A .

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1. Introduction.

Tarazaga [3] has established that the eigenvalues λ_i of a real symmetric matrix A of order n satisfy

$$(1.1) \quad \left| \lambda_i - \frac{\text{tr}A}{n} \right| \leq \left[\frac{n-1}{n} \left(\|A\|_F^2 - \frac{(\text{tr}A)^2}{n} \right) \right]^{1/2},$$

where $\text{tr}A$ and $\|A\|_F$ denote the trace and the Frobenius norm of A , respectively. Basically, Tarazaga used a geometric approach to obtain his estimate.

In section 2 we improve and generalize the eigenvalue estimate (1.1) to any $n \times n$ complex matrix. In particular, we prove that if A is an $n \times n$ complex matrix with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ then

$$(1.2) \quad \left| \lambda_i - \frac{\text{tr}A}{n} \right| \leq \left[\frac{n-1}{n} \left(\sum_{i=1}^n |\lambda_i|^2 - \frac{|\text{tr}A|^2}{n} \right) \right]^{1/2}, \quad i = 1, 2, \dots, n.$$

We observe that if the eigenvalues of A are all real then

$$(1.3) \quad \left| \lambda_i - \frac{\text{tr}A}{n} \right| \leq \left[\frac{n-1}{n} \left(\text{tr}(A^2) - \frac{(\text{tr}A)^2}{n} \right) \right]^{1/2}, \quad i = 1, 2, \dots, n.$$

Since $\text{tr}(A^2) = \sum_{i=1}^n \lambda_i^2$ and $\sum_{i=1}^n |\lambda_i|^2 \leq \|A\|_F^2$, the Tarazaga inequality (1.1) follows from the inequality (1.3) immediately.

In section 3 we recall some inequalities which allow us to bound the radius of the disk (1.2) without knowing the eigenvalues of A . Some examples are also given.

2. The main result.

Lemma 2.1. If $d_1, \dots, d_n$ are complex numbers such that $d_1 + \dots + d_n = 0$ then

$$(2.1) \quad |d_k|^2 \leq \frac{n-1}{n} \sum_{i=1}^n |d_i|^2, \quad k = 1, \dots, n$$

Proof: It is sufficient to prove the lemma for real d_k , $k = 1, \dots, n$, because

$$\begin{aligned} |d_k|^2 &= |\operatorname{Re} d_k|^2 + |\operatorname{Im} d_k|^2 \\ &\leq \frac{n-1}{n} \sum_{i=1}^n |\operatorname{Re} d_i|^2 + \frac{n-1}{n} \sum_{i=1}^n |\operatorname{Im} d_i|^2 \\ &= \frac{n-1}{n} \sum_{i=1}^n |d_i|^2 \end{aligned}$$

Suppose that $d_1, \dots, d_n$ are real numbers such that $d_1 + \dots + d_n = 0$. If $d_k = 0$, then there is nothing to prove. Let $d_k \neq 0$ and $x_i = -d_i/d_k$ for $i \neq k$. Define $f: \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ by

$$f(x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n) = 1 + \sum_{i \neq k} x_i^2.$$

It is clear that f attains its minimum under the constraint $\sum_{i \neq k} x_i = 1$ at the point of coordinates $(1/(n-1), \dots, 1/(n-1))$. Hence

$$\begin{aligned} f\left(\frac{1}{n-1}, \dots, \frac{1}{n-1}\right) &= \frac{n}{n-1} \\ &\leq 1 + \sum_{i \neq k} x_i^2 \\ &= \frac{\sum_{i=1}^n d_i^2}{d_k^2} \end{aligned}$$

which yields the desired result. We remark that if $d_k = 1$ and $d_i = -1/(n-1)$ for $i \neq k$, then we have an equality in (2.1), which shows that the inequality is sharp.

The proof for real d_i was formerly given in [2].

Theorem 2.2. Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of an $n \times n$ complex matrix A . Then each λ_i , $i = 1, \dots, n$ satisfies

$$(2.2) \quad \left| \lambda_i - \frac{\operatorname{tr} A}{n} \right| \leq \left[\frac{n-1}{n} \left(\sum_{i=1}^n |\lambda_i|^2 - \frac{|\operatorname{tr} A|^2}{n} \right) \right]^{1/2}.$$

Proof: From lemma 2.1 we have the following inequality

$$\begin{aligned}
 \left| \lambda_i - \frac{1}{n} \sum_{i=1}^n \lambda_i \right|^2 &\leq \frac{n-1}{n} \sum_{i=1}^n \left| \lambda_i - \frac{1}{n} \sum_{i=1}^n \lambda_i \right|^2 \\
 &= \frac{n-1}{n} \sum_{i=1}^n \left[|\lambda_i|^2 + \frac{1}{n^2} \left| \sum_{i=1}^n \lambda_i \right|^2 - 2 \operatorname{Re} \left(\frac{1}{n} \sum_{i=1}^n \lambda_i \right) \bar{\lambda}_i \right] \\
 &= \frac{n-1}{n} \left[\sum_{i=1}^n |\lambda_i|^2 + \frac{1}{n} \left| \sum_{i=1}^n \lambda_i \right|^2 - \frac{2}{n} \operatorname{Re} \left(\sum_{i=1}^n \lambda_i \right) \left(\sum_{i=1}^n \bar{\lambda}_i \right) \right] \\
 &= \frac{n-1}{n} \left[\sum_{i=1}^n |\lambda_i|^2 + \frac{1}{n} \left| \sum_{i=1}^n \lambda_i \right|^2 - \frac{2}{n} \left| \sum_{i=1}^n \lambda_i \right|^2 \right] \\
 &= \frac{n-1}{n} \left[\sum_{i=1}^n |\lambda_i|^2 - \frac{1}{n} \left| \sum_{i=1}^n \lambda_i \right|^2 \right].
 \end{aligned}$$

whence (2.2) is obtained.

Corollary 2.3. Let $\lambda_1, \dots, \lambda_n$ be eigenvalues of an $n \times n$ complex matrix A . Then each $\lambda_i, i = 1, \dots, n$, satisfies

$$(2.3) \quad \left| \lambda_i - \frac{\operatorname{tr} A}{n} \right| \leq \left[\frac{n-1}{n} \left(\|A\|_F^2 - \frac{|\operatorname{tr} A|^2}{n} \right) \right]^{1/2}$$

Proof: It is because $\sum_{i=1}^n |\lambda_i|^2 \leq \|A\|_F^2$. Then (2.3) follows from (2.2) immediately. If A is not a normal matrix and has real eigenvalues, the bound in (1.3) is better than the bound in (2.3).

We note that the estimate in (2.3) is easy to compute in practice and Tarazaga's eigenvalue estimate (1.1) follows from (2.3) immediately.

Corollary 2.4. Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of an $n \times n$ complex matrix A and suppose that $\lambda_1, \dots, \lambda_n$ are all lying on a line segment passing through the origin. Then each $\lambda_i, i = 1, \dots, n$, satisfies

$$(2.4) \quad \left| \lambda_i - \frac{\operatorname{tr} A}{n} \right| \leq \left[\frac{n-1}{n} \left| \operatorname{tr}(A^2) - \frac{(\operatorname{tr} A)^2}{n} \right| \right]^{1/2}.$$

Proof: If $\lambda_1, \dots, \lambda_n$ are all lying on a line segment passing through the origin, then

$$\begin{aligned} \left| \text{tr}(A^2) - \frac{1}{n}(\text{tr}A)^2 \right| &= \left| \sum_{i=1}^n \lambda_i^2 - \frac{1}{n} \left(\sum_{i=1}^n \lambda_i \right)^2 \right| \\ &= \sum_{i=1}^n |\lambda_i|^2 - \frac{1}{n} |\text{tr}A|^2 \end{aligned}$$

Hence, (2.4) follows from (2.2).

Under the hypothesis of corollary (2.4), we observe that the spectral radius $\rho(A - hI)$ of the matrix $A - hI$, with $h = \text{tr}A/n$, is bounded by the right side of (2.4). In particular, if $\text{tr}A = 0$, then

$$\rho(A) \leq \left[\frac{n-1}{n} |\text{tr}(A^2)| \right]^{1/2}$$

Similary, from (2.3), we have

$$\rho(A) \leq \left[\frac{n-1}{n} \right]^{1/2} \|A\|_F$$

for any complex matrix A with trace equal to zero.

Corollary 2.5. Let A an $n \times n$ matrix with real eigenvalues $\lambda_1, \dots, \lambda_n$. Then (1.3) holds and Tarazaga's inequality (1.1) follows from (1.3) immediately.

Theorem 2.6. Let A be a complex matrix of order n with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$. Then at least one eigenvalue of A lies in the annulus

$$(2.5) \quad S \leq \left| \lambda - \frac{\text{tr}A}{n} \right| \leq R,$$

where

$$R = \left[\frac{n-1}{n} \left(\sum_{i=1}^n |\lambda_i|^2 - \frac{|\text{tr}A|^2}{n} \right) \right]^{1/2}$$

is the radius of the disk in (2.2) and $S = R/\sqrt{n-1}$.

Proof: Let $M = \max_j |\lambda_j - \text{tr}A/n|$. Then from the proof of theorem 2.2 we have

$$\begin{aligned} S &= \left(\frac{1}{n} \left(\sum_{k=1}^n |\lambda_k|^2 - \frac{|\text{tr}A|^2}{n} \right) \right)^{1/2} \\ &= \left(\frac{1}{n} \sum_{k=1}^n \left| \lambda_k - \frac{\text{tr}A}{n} \right|^2 \right)^{1/2} \\ &\leq \left[\frac{1}{n} n M^2 \right]^{1/2} \\ &= M \end{aligned}$$

If $|\lambda_j - \text{tr}A/n| < S$ for all j , then $M < S$, which contradicts the above inequality.

3. Bounds for the radius of the disk.

In practice we do not know the eigenvalues λ_i of A . This impede us to use (2.2) directly. Therefore we need bounds for the radius of the disk in (2.2). One of these bounds is given by corollary 2.3. For real eigenvalues λ_i we have the results given by corollaries 2.4 and 2.5.

Theorem 3.1. Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the eigenvalues of an $n \times n$ complex matrix A . Then each λ_j $i = 1, 2, \dots, n$, satisfies

$$(3.1) \quad \left| \lambda_j - \frac{\text{tr}A}{n} \right| \leq \left[\frac{n-1}{n} \left(\|A\|_F^2 - \frac{(v(A))^2}{6\|A\|_F^2} - \frac{|\text{tr}A|^2}{n} \right) \right]^{1/2},$$

where $v(A) = \|A^*A - AA^*\|_F$.

Proof: From the Schur Theorem, there exists a unitary matrix U such that $U^*AU = D + N$, where U^* denotes the conjugate transpose of U , N is strictly upper triangular matrix and $D = \text{diag}\{\lambda_1, \lambda_2, \dots, \lambda_n\}$. Since the Frobenius matrix norm is invariant under unitary transformations, it follows that

$$(3.2) \quad \|A\|_F^2 = \sum_{k=1}^n |\lambda_k|^2 + \|N\|_F^2.$$

A lower bound for $\|N\|_F^2$ has been derived by Eberlein [1]. It is

$$(3.3) \quad \frac{(v(A))^2}{6\|A\|_F^2} \leq \|N\|_F^2.$$

From (3.2) and (3.3) we have

$$(3.4) \quad \sum_{k=1}^n |\lambda_k|^2 \leq \|A\|_F^2 - \frac{(v(A))^2}{6\|A\|_F^2}$$

and the result follows from (2.2).

Thus, the radius of the disk (2.2) may be approximate if the eigenvalues of A are unknown. If A is a non-normal matrix, then $v(A) \neq 0$ and the estimate in (3.1) strictly improves the estimate in (2.3).

Theorem 3.2. Let A be an $n \times n$ matrix with real eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$. Then

$$(3.5) \quad \left| \lambda_j - \frac{trA}{n} \right| \leq \left(\frac{n-1}{n^2} [(n-1)(trA)^2 - 2n(tr_2A)] \right)^{1/2},$$

where tr_2A denotes the sum of all principal 2-rowed minors of A .

Proof: Since the λ_i 's are real, $\sum_{1 \leq i \leq j \leq n} \lambda_i \lambda_j = tr_2A$. Then,

$$\begin{aligned} \sum_{1 \leq i \leq j \leq n} (\lambda_i - \lambda_j)^2 &= (n-1) \sum_{i=1}^n \lambda_i^2 - 2 \sum_{1 \leq i \leq j \leq n} \lambda_i \lambda_j \\ &= (n-1) \left(\sum_{i=1}^n \lambda_i^2 + 2 \sum_{1 \leq i \leq j \leq n} \lambda_i \lambda_j \right) - 2n(tr_2A) \\ &= (n-1)(trA)^2 - 2n(tr_2A). \end{aligned}$$

Now, the result follows from (2.2) and

$$\sum_{i=1}^n \lambda_i^2 - \frac{(trA)^2}{n} = \frac{1}{n} \sum_{1 \leq i \leq j \leq n} (\lambda_i - \lambda_j)^2.$$

We now show some examples :

1) Let

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \\ 3 & 4 & 1 & 2 \\ 2 & 3 & 4 & 1 \end{pmatrix}$$

The eigenvalues of A are $-2, 10$ and $-2 \pm 2i$. The disk in (2.3) is $|\lambda_j - 1| \leq 9.3274$. From theorem 2.6 we know that there exists at least one eigenvalue lying in the annulus

$$5.385 < |\lambda - 1| < 9.328.$$

2) Let

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 10 & 2 & 1 \\ 10 & 1 & 2 \end{pmatrix}$$

A has real eigenvalues, so that we may use theorem 3.2 to obtain

$$|\lambda_i - 2| \leq (4/3)^{1/2} = 1.1547$$

or $\lambda_i \in [0.8453, 3.1547]$. From the Gerschgorin Theorem we have $\lambda_i \in [-9, 13]$. The true eigenvalues of A are 1, 2 and 3.

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Oscar Rojo and Ricardo L. Soto
Departamento de Matemáticas
Universidad Católica del Norte
Casilla 1280, Antofagasta, Chile.

Héctor Rojo
Departamento de Matemáticas
Universidad de Antofagasta
Casilla 170, Antofagasta, Chile.

Tin-Yau Tam
Department of Algebra, Combinatorics and Analysis
Auburn University, AL 36849-5307, U.S.A.