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**SPECTRA OF RELATED GRAPHS AND
SELF-REPRODUCING POLYHEDRA***

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Abstract

If G is a d -valent graph, the eigenvalues of the adjacency matrix of G determine those of the line graph, the subdivision graph and the graph made by replacing vertices with complete graphs. A property of the eigenvectors of the graphs of certain regular polyhedra is then seen to carry over to some truncations of the polyhedra.

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1. Spectra of related graphs.

Let G be a connected graphs (loops and multiple edges not allowed) with p vertices and q edges. (Definitions of graph-theoretic terms may be found in [2].) We will be particularly interested in forming other graphs from G . The first is the line graph G_L of G , whose vertices are the edges of G . Two vertices of G_L are adjacent if and only if the corresponding edges of G have a common endpoint. The second is the subdivision G_S of G , which is formed by inserting a new vertex in each edge of G . (See Figure 1.)

A graph G with p vertices and q edges can be described by a $p \times q$ incidence matrix R , which has a 1 in the i, k -position if vertex i is an endpoint of edge k , or a 0 if not. The same graph can also be described by a $p \times p$ adjacency matrix A , which has a 1 in the i, j -position if vertices i and j are the endpoints of an edge, or a 0 if not.

The relations among incidence and adjacency matrices of a graph G , its line graph G_L and its subdivision graph G_S are as follows [4,p.16]

$$(1.1) \quad RR^T = A_G + D$$

$$(1.2) \quad R^T R = A_L + 2I$$

$$(1.3) \quad \begin{bmatrix} 0 & R \\ R^T & 0 \end{bmatrix} = A_S$$

In Eq. (1.1), D is the degree matrix, $D = \text{diag}\{d_1, \dots, d_p\}$, where d_i is the degree or valency of vertex i .

Generally, there is no connection between the eigenvalues of A_G and A_L . However, if G is d -valent, then $D = dI$, and we will assume this to be true. Now, let x, y be a pair of singular vectors of R corresponding to singular value $\rho > 0$; that is $R^T x = \rho y$ and $Ry = \rho x$. Then

$$RR^T x = \rho^2 x = (A_G + dI)x$$

$$R^T R y = \rho^2 y = (A_L + 2I)y$$

$$\begin{bmatrix} 0 & R \\ R^T & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \pm \rho \begin{bmatrix} x \\ y \end{bmatrix}$$

From these equations we see that the eigenvalues of G , G_L and G_S are related by

$$(1.4) \quad \lambda_L = \lambda_G + d - 2$$

$$(1.5) \quad \lambda_S = \pm \sqrt{\lambda_G + d}$$

and that the multiplicity of any eigenvalue $\lambda_G > -d$ carries over to the related λ_L or λ_S .

It is known [4, p. 114] that the rank of R is $p - 1$ if G is bipartite or p otherwise. In the bipartite case, there are $q - p + 1$ independent solutions of $Ry = 0$ and one of $R^T x = 0$. Thus, for a bipartite G , we have eigenvalues $\lambda_G = -d$ (simple), $\lambda_L = -2$ (multiplicity $q - p + 1$), and $\lambda_S = 0$ (multiplicity $q - p + 2$). For a nonbipartite graph G , $-d$ is not an eigenvalue of A_G , but $\lambda_L = -2$ (multiplicity $q - p$) and $\lambda_S = 0$ (multiplicity $q - p$). (Note that with $d \geq 2$, $q = pd/2 \geq p$.)

Figure 1

We are interested also in the graph G_K (See Figure 1) made from G by replacing each vertex v with a complete graph K_d , so that each of the d edges of G that were incident with v is made incident to exactly one vertex of K_d . (This is a special case of a graph operation called substitution. See [7].) If we start from a d -valent graph G with p vertices, then G_K is d -valent with pd vertices. The key to finding the spectrum of G_K lies in recognizing that G_K is the line graph of the subdivision graph of G .

Theorem 1. Let G be a d -valent graph and G_K the made by replacing each vertex of G by the complete graph K_d . The eigenvalues λ_K of the adjacency matrix of G_K are related to those of G by

$$(1.6) \quad \lambda_K = \frac{1}{2} (d - 2 \pm \sqrt{d^2 + 4 + 4\lambda_G})$$

Proof: The adjacency matrix of A_S is given in Eq.(1.3). Suppose that Q is an incidence matrix of the subdivision graph G_S . Then

$$(1.7) \quad QQ^T = A_S + D_S = \begin{bmatrix} dI & R \\ R^T & 2I \end{bmatrix}$$

Note that G_S is not generally d -valent: each of the new vertices introduced in the subdivision process has degree 2. The adjacency matrix of the line graph of G_S is $A_K = QQ^T - 2I$. The eigenvalues other than -2 of A_K coincide (including multiplicity) with those of

$$(1.8) \quad M = QQ^T - 2I = \begin{bmatrix} (d-2)I & R \\ R^T & 0 \end{bmatrix}.$$

Now, suppose that x and y are singular vectors of R corresponding to singular value $\rho > 0$. Consider the partitioned matrix product

$$M \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} = \begin{bmatrix} (d-2)x & \rho x \\ \rho y & 0 \end{bmatrix} = \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \begin{bmatrix} d-2 & \rho \\ \rho & 0 \end{bmatrix}$$

Since both x and y are nonzero, the eigenvalues of the last 2×2 matrix are eigenvalues of M and hence of A_K . It may be verified easily that they satisfy the relation given in the statement of the theorem.

The graph G_S is always bipartite, and so its line graph G_K has -2 as an eigenvalue of multiplicity $q_s - p_s + 1 = pd/2 - p + 1$. One occurrence of the eigenvalue $\lambda_K = -2$ is included in Eq.(1.6), for $\lambda_G = d$.

Example. Let G be the complete graph on 4 vertices as shown in Figure 1. The eigenvalues, with multiplicities in brackets, are shown in the Table. Note that $0[2]$ was added to the spectrum of G_S , $-2[2]$ to that of G_L , and $-2[4]$ to that of G_K .

TABLE

graph	#vertices	Eigenvalues
$G \cong K_4$	4	$3[1], -1[3]$
G_S	10	$\sqrt{6}[1], 2[3], 0[2], -2[3], -\sqrt{6}[1]$
$G_L \cong K_{2,2,2}$	6	$4[1], 0[3], -2[2]$
G_K	12	$3[1], 2[3], -1[3], -2[5]$

2. Truncations of polyhedra.

Let P be an n -dimensional polytope with p vertices. The skeleton of P is the graph formed by its vertices and edges. For example, Figure 1 shows the skeleton of a tetrahedron. Godsil [5] suggested that a graph G on p vertices could produce a polytope in the following way. Let λ be an eigenvalue of the adjacency matrix of A with multiplicity m , and let U be a $p \times m$, matrix of orthonormal eigenvectors:

$$(2.1) \quad AU = \lambda U, \quad U^T U = 1$$

The distinct rows of U can be interpreted as the coordinates of points in m -dimensional euclidean space. The convex hull of these points, $C(\lambda)$, is the polytope associated with the eigenvalue λ .

If a polytope P is isomorphic to a polytope $C(\lambda)$ associated with an eigenvalue of its skeleton, we say that P is *self-reproducing* [6]. This means that the mere adjacency information in the skeleton can generate the metric information in the coordinates of the vertices of the polytope. To prove that an n -dimensional polytope is self-reproducing, one must show (1.1) that the adjacency matrix of the skeleton has an eigenvalue λ of multiplicity n , and (1.2) that the sum of the vectors of position of the neighbors of vertex i is λ times the vector of position of vertex i , for each i . This second requirement is equivalent to the i th row of the eigenvector equation (2.1).

For example, the skeleton of the cube has eigenvalues $3[1], 1[3], -1[3], -3[1]$. We may take the vertices to be $(\pm 1, \pm 1, \pm 1)$. The three neighbors of vertex (a, b, c) are $(-a, b, c)$, $(a, -b, c)$, $(a, b, -c)$, and $(-a, b, c) + (a, -b, c) + (a, b, -c) = 1(a, b, c)$. Thus, we have shown that the cube, being isomorphic to $C(1)$, is self-reproducing.

In [6] it was shown that many regular polytopes are self-reproducing, including all polygons and the platonic solids. In all cases, the polytope of the second largest eigenvalue is isomorphic to the original polytope. We will show that the same properties apply to some truncations of these polytopes.

For simplicity we consider only 3-dimensional polytopes (polyhedra). A simple truncation [3, p. 148] of a polyhedron P is formed by cutting off each vertex with a plane through the midpoints of the edges issuing from it. For example the octahedron is the simple truncation of a tetrahedron, and the cuboctahedron is the simple truncation of both the octahedron and the cube.

The skeleton of the truncation of P has as vertices the midpoints of the edges. The midpoints surrounding a vertex of P are joined in a cycle, which is the vertex figure of the vertices of P . Thus, if the skeleton of P is a 3-valent graph, the skeleton of the truncation of P is isomorphic to the line graph of the skeleton of P . Hence, we can use Eq. (1.4) to find the spectrum of these truncations.

Theorem 2. The simple truncations of the platonic solids are self-reproducing.

Proof. Let P be a platonic solid with a 3-valent skeleton G , and let λ_G be the second largest eigenvalue of G . Then, if U is the $p \times 3$ matrix of coordinates of vertices of G , we know [6] that $A_G U = \lambda_G U$. Let R be an incidence matrix so that $RR^T = A_G + 3I$, and set $V = \frac{1}{2}R^T U$. Then each row of V contains the coordinates of the midpoints of an edge of P , and

$$\begin{aligned} A_L V &= (R^T R - 2I)V \\ &= (\lambda_G + 1)V \end{aligned}$$

Since $\lambda_G + 1$ is an eigenvalue of A_L with the same multiplicity, 3, that λ_G has in the spectrum of A_G , we have shown that the truncation of P is self-reproducing.

Finally, we note that cube and octahedron have the same truncation (cuboctahedron), and that icosahedron and dodecahedron also have the same truncation (icosidodecahedron).

An intermediate truncation [3, p. 148; 1, p. 76] of a polyhedron is made by cutting off each vertex with a plane that intercepts each of the edges issuing the vertex at a point between the vertex and the midpoint of the edge. For a polyhedron with 3-valent skeleton G , the skeleton of an intermediate truncation is isomorphic to G_K

Theorem 3. Let P be a polyhedron whose skeleton is 3-valent. If P is self-reproducing, then there is an intermediate truncation of P that is also self-reproducing.

Proof: Let the coordinates of a vertex of P and its three neighbors be row matrices u_0, u_1, u_2, u_3 . On each line segment from u_0 to u_i , we introduce two points with coordinates

$$(2.2) \quad w_{0i} = (1-t)u_0 + tu_i,$$

$$(2.3) \quad w_{i0} = (1-t)u_i + tu_0,$$

with $t < \frac{1}{2}$. These are vertices of an intermediate truncation of P . We are given that

$$(2.4) \quad u_1 + u_2 + u_3 = \lambda u_0$$

and we wish to show that

$$(2.5) \quad w_{i,0} + w_{0,i+1} + w_{0,i+2} = \alpha w_{0i}$$

for some t and α , and for each i . (Subscripts $i, i+1, i+2$ are interpreted modulo 3, in the set $\{1, 2, 3\}$.)

Substitution of Eqs. (2.2) and (2.3) into Eq. (2.5) leads to the equation

$$(t + (1-t)(2-\alpha))u_0 + (1-t-\alpha t)u_i + tu_{i+1} + tu_{i+2} = 0$$

Eq. (2.4) permits us to eliminate u_{i+1} and u_{i+2} leaving

$$(t(1+\lambda) + 1-t)(2-\alpha)u_0 + (1-2t-\alpha t)u_1 = 0.$$

Since u_0 and u_1 are independent row matrices, their coefficients in the equation above must both be 0. These can be reduced to

$$(2.6) \quad \alpha^2 - \alpha - (3+\lambda) = 0$$

$$(2.7) \quad t = \frac{1}{\alpha + 2}$$

Note that the roots of Eq. (2.6) satisfy Eq. (1.6), confirming the fact that α is an eigenvalue of G_K . Only the larger of the roots of Eq. (2.6) gives a value of $t < \frac{1}{2}$.

Since we showed in Theorem 1 that eigenvalue multiplicities carry over from G to G_K , and since P is self-reproducing, both λ and α must have multiplicity 3.

It is interesting to note we have not used any symmetry of the polyhedron P in the proof, except the symmetry in Eq. (2.4). Since this symmetry is inherited by the intermediate truncation, it seems that the process could be continued indefinitely.

3. Conclusion.

We have shown that the eigenvalue of G_K can be obtained from those G , if G is d -valent. As an application, we proved that the simple and intermediate truncations of some self-reproducing polyhedra are also self-reproducing.

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