PROYECCIONES

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A MESH INDEPENDENCE PRINCIPLE FOR NONLINEAR EQUATIONS USING NEWTON'S METHOD AND NONLINEAR PROJECTIONS.

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A B S T R A C T

We consider the nonlinear operator equation in a Banach space. We make use of nonlinear projections on finite dimensional spaces to produce the finite dimensional discretization of the nonlinear equation. Using Newton's method we then prove the mesh-independence principle for this problem. Our results cover and extend previous results involving linear projections on finite dimensional spaces.

INTRODUCTION

In this paper we extend the validity of the mesh independence principle for discretizations of operators. We consider the equation

$$F(z) = 0 \quad (1)$$

where F is a nonlinear operator on a Banach space X . Newton's iteration for (1) is given by

$$z_{n+1} = z_n - F'(z_n)^{-1} F(z_n) \quad (2)$$

The above iteration, under certain assumptions, converges quadratically to

a solution z^* of (1). However, it is not at all easy in general to compute the iterates in (2). That is why we consider the discretized family of equations

$$T_h(x) = 0 \quad (3)$$

indexed by some $h > 0$ to solve (1), where T_h is a nonlinear operator on a finite dimensional space X_h . Under certain assumptions we show that the equations (3) have solutions

$$x_h^* = P_h(z^*) + O(h^p)$$

which are the limit of the Newton sequence applied to (3) such that:

$$x_0^h = P_h(z_0), \quad x_{n+1}^h = x_n^h - T_h'(x_n^h)^{-1} T_h(x_n^h), \quad n = 0, 1, 2, \dots \quad (4)$$

To achieve this, we define the discretization on X by the bounded nonlinear operators $P_h : X \rightarrow X_h$.

It has been observed in many computations that for sufficiently small h there is at most a difference of one between the number of steps required by the iterations (2) and (4) to converge to within a given tolerance $\epsilon > 0$. This is one aspect of the mesh-independence principle for Newton's method. Here we actually show that the number of steps in both iterations is the same.

The above results have already been proved in [3] but for linear projections from X to X_h . For special classes of boundary value problems for mesh-independence principle was proved in [1], [5]. It was used to construct certain mesh-refinement strategies.

Most of our results carry over immediately to Newton like methods [7].

I. PRELIMINARIES

The norms in all spaces will be denoted by the same symbol $\|\cdot\|$

Let $F : D \subset X \rightarrow X$ be a nonlinear operator with Lipschitz continuous Fréchet derivative on the open domain, that is

$$\|F'(x) - F'(y)\| \leq \gamma \|x - y\|, \quad x, y \in D, \quad \gamma > 0. \quad (5)$$

Let us assume that (1) has a simple solution and set $d = \|F'(z^*)^{-1}\|$.

We will need the following theorem [8].

Theorem 1. Let $F : \mathbb{D} \rightarrow X$ satisfy the stated hypotheses and with

$$r^* = \frac{2}{3d_Y}, \quad (6)$$

suppose that $U = U(z^*, r^*) = \{z \in X \mid \|z - z^*\| < r^*\}$. Then for any $z_0 \in U$, iteration (2) converges to z^* and the iterates satisfy

$$\|z_{n+1} - z^*\| \leq \frac{d_Y \|z_n - z^*\|^2}{2(1 - d_Y \|z_n - z^*\|)}, \quad n = 0, 1, 2, \dots \quad (7)$$

Consider the family

$$\{T_h, P_h\}, \quad h > 0 \quad (8)$$

where

$$T_h : D_h \subset X_h \rightarrow X_h, \quad h > 0$$

are nonlinear operators and

$$P_h : X \rightarrow X_h, \quad h > 0$$

are nonlinear operators which are twice continuously Fréchet - differentiable at least in some ball $U(z^*, \rho)$ for some $\rho > 0$ and $P_h(U) \subset D_h$.

The discretization (8) is called Lipschitz uniform if there exist scalars $\rho > 0, L > 0$ such that

$$\bar{U}(P_h(z^*), \rho) \subset D_h, \quad h > 0 \quad (9)$$

and

$$\|T'_h(w) - T'_h(v)\| \leq L \|w - v\|, \quad h > 0, w, v \in \bar{U}(P_h(z^*), \rho) \quad (10)$$

Moreover, the family (8) is called "bounded" if there exist constants $q_1, q_2 > 0$ such that

$$\|P_h(u) - P_h(z^*)\| \leq q_1 \|u - z^*\| + q_2 \|u - z^*\|^2, \quad u \in U \quad (11)$$

The motivation for the above condition is due to the identity

$$P_h(u) = P_h(z^*) + P'_h(z^*)(u - z^*) + \frac{1}{2} P''_h(\bar{z}^*)(u - z^*)^2, \quad (12)$$

where P'_h, P''_h represent the first and second Fréchet derivatives of P_h and $\bar{z}^* \in U$.

Note that the identity (12) is not true, in general, for any nonlinear operator P_h that is twice Fréchet-differentiable on X . However, it is certainly true for P_h being a polynomial operator [4], [7].

From now on we restrict ourselves only to that subclass of twice Fréchet-differentiable nonlinear operators on X that can satisfy the identity (12).

Our proofs will indicate that the same results can be obtained for any nonlinear sufficiently Fréchet-differentiable operator P_h , under similar assumptions. However, due to clarity and to the fact that the main results obtained here can be achieved using the above mentioned subclass of nonlinear operators P_h , we do not pursue this goal here.

We will use the following well known estimate [7]:

$$\|P_h(u) - P_h(z^*) - P'_h(z^*)(u - z^*)\| \leq q_2 \|u - z^*\|^2 \quad (13)$$

Let us assume from now on that: The operators P_h are continuously Fréchet differentiable at a closed ball U^* such that

$$0 \in U^* \subset U \quad (14)$$

and

$$P_h(F(z^*)) = P_h(0) = 0, \quad h > 0 \quad (15)$$

The family (8) is called stable if there is $\sigma > 0$ such that

$$\|T'_h(P_h(u))^{-1}\| \leq \sigma, \quad u \in U, \quad h > 0 \quad (16)$$

and consistent of order p if there are two constants $c_0, c_1 > 0$ such that

$$\|P_h(F(z)) - T_h(P_h(z))\| \leq c_0 h^p, \quad z \in U, \quad h > 0 \quad (17)$$

and

$$\|P_h(F'(u))(v) - T'_h(P_h(u))P_h(v)\| \leq c_1 h^p, \quad u, v \in U, \quad h > 0 \quad (18)$$

II. MAIN RESULTS

We will need the following definition.

Definition

Let us define the numbers r, c and the real functions f_1 and f_2 by:

$$r = \|z_0 - z^*\| \quad (19)$$

$$c = \max (c_0, c_1) \quad (20)$$

$$f_1(r) = 3Lq_2\sigma r^2 + 3Lq_1\sigma r + 16L\sigma^2 c \cdot h^p - 2 \quad (21)$$

$$f_2(r) = q_2r^2 + q_1r + 4\sigma ch^p - \rho \quad (22)$$

Let, h_1, r_1 be such that:

$$h_1 = \min \left(\frac{h^p}{4\sigma c}, \frac{1}{8L\sigma^2 c} \right)^{\frac{1}{p}} \quad (23)$$

$$r_1 = \min (r_1^+, r_2^+, r^*, \rho) \quad (24)$$

where, r_1^+, r_2^+ are the positive solutions of (20) and (21) respectively, provided that :

$$h \in [0, h_1) \quad (25)$$

It is easy to check that if

$$r \in [0, r_1) \quad (26)$$

then

$$f_1(r) \leq 0 \quad (27)$$

and

$$f_2(r) \leq 0 \quad (28)$$

Define the real function f_3 and the number b by

$$f_3(s) = As^2 + Bs + C \quad (29)$$

$$b = \|z^*\| \quad (30)$$

where,

$$A = \frac{3}{2} L\sigma$$

$$B = B(r) = B_3r^3 + B_2r^2 + B_1r + B_0$$

$$C = C(r) = C_2r^2 + C_1r + C_0$$

$$B_3 = 16 L\sigma q_2$$

$$B_2 = L\sigma (4q_1 + 8bq_2 - 6q_2)$$

$$B_1 = 2 L\sigma q_1b + 2L\sigma q_2b^2 - 4q_1 - 4bq_2$$

$$B_0 = -(L\sigma q_1b + Lq_2\sigma + 1)$$

and

$$\begin{aligned}C_2 &= 6q_2 \\C_1 &= 4(q_1 + bq_2) \\C_0 &= 2ch^p\sigma + q_1b + q_2b^2\end{aligned}$$

We now observe the following:

$$(a) \quad B_0 < 0, \quad B_3 > 0 \quad (31)$$

(b) The function $B^2 - 4AC$ is a sixth degree polynomial with the coefficient of the highest power being positive and the constant given by

$$B_0^2 - 4AC_0 = E_1q_1^2 + E_2q_2^2 + E_3q_1q_2 + E_4q_1 + E_5q_2 + E_6 \quad (32)$$

where,

$$\begin{aligned}E_1 &= (L\sigma b)^2 \\E_2 &= (L\sigma)^2 \\E_3 &= 2(L\sigma)^2b \\E_4 &= -4L\sigma b \\E_5 &= 2L\sigma(1 - 4b^2) \\E_6 &= 1 - 12L\sigma^2 h^p\end{aligned}$$

(c) Define the function g by

$$g(k) = A(4AB^2 - C')k^2 + 4AB\sigma ck + (\sigma c)^2, \quad C' = C - 2\sigma ch^p \quad (33)$$

The function $4AB^2 - C'$ is a sixth degree polynomial with the coefficient of the highest power being positive and the constant given by

$$4AB_0^2 - q_1b - q_2b^2 = E_1^1q_1^2 + E_2^1q_2^2 + E_3^1q_1q_2 + E_4^1q_1 + E_5^1q_2 + E_6^1 \quad (34)$$

where,

$$\begin{aligned}E_1^1 &= 6(L\sigma)^3b^2 \\E_2^1 &= 6(L\sigma)^3 \\E_3^1 &= 12(L\sigma)^3b\end{aligned}$$

$$E_q^1 = [12(L\sigma)^2 - 1] b$$

$$E_5 = 12 (L\sigma)^2 - b^2$$

$$E_6 = 6 L\sigma$$

$$\text{Let } q_1^* = \max \left(-\frac{E_4}{E_1}, -\frac{E_4^1}{E_1^1} \right), \quad q_2^* = \max \left(-\frac{E_5}{E_2}, -\frac{E_5^1}{E_2^1} \right) \quad (35)$$

and

$$k^* = \max \left(-\frac{\sigma c}{2AB}, k_\ell \right) \quad (36)$$

where k_ℓ is the large solution of the equation

$$g(k) = 0$$

Choose q_1, q_2, k, h and h_2 such that:

$$q_1 \geq q_1^* \quad (37)$$

$$q_2 \geq q_2^* \quad (38)$$

$$k \geq k^* \quad (39)$$

and

$$h < h_2 = \min \left(\left(\frac{1}{12L\sigma^2} \right)^{\frac{1}{p}}, \left(\frac{1}{Lk\sigma} \right)^{\frac{1}{p}}, h_1 \right) \quad (40)$$

It is easy to check that we can find $r_2 > 0$ such that if $r \in [0, r_3)$ where

$$r_3 = \min (r_1, r_2) \quad (41)$$

then the following are true:

$$B < 0$$

$$B^2 - 4AC \geq 0$$

$$\sigma c + 2AkB \leq 0$$

$$g(k) \geq 0$$

and

$$0 < s \leq kh^p < \frac{1}{L\sigma} \quad (42)$$

where s is the small solution of the equation

$$f_3(s) = 0 \quad (43)$$

Let $\varepsilon > 0$, since $z_n \rightarrow z^*$ as $n \rightarrow \infty$ there exists $N = N(\varepsilon)$ such that

$$\|z_n - z^*\| \leq \left[\frac{1}{q_2} k_1 h^p \right]^{\frac{1}{2}} \text{ for some } k_1 > 0 \quad (44)$$

Finally, define the numbers k_2 , β , r_4 , h_4 and the function f_4 by

$$k_2 = k + 2\sigma c + k_1 \quad (45)$$

$$\beta = \min \left(2 [\bar{q}_1 - (q_1 + q_2)], \frac{1}{q_1 + q_2} \right) \text{ for some fixed number } \bar{q}_1 \text{ with}$$

$$\bar{q}_1 > q_1 + q_2 \quad (46)$$

$$h_3 = \min \left\{ h_2, \left(\frac{\beta}{(1+2\beta)L\sigma^2 c} \right)^{\frac{1}{p}}, \left(\frac{(q_1 + q_2)\varepsilon}{k_2} \right)^{\frac{1}{p}} \right\} \quad (47)$$

$$f_4(r) = L\sigma q_2 (1+\beta)r^2 + L\sigma q_1 (1+\beta)r + (1+2\beta)h^p - \beta \quad (48)$$

and r_4 to be the positive solution of the equation

$$f_4(r) = 0 \quad (49)$$

guaranteed to exist by the choice of h_3 and β

Let h , r be such that:

$$h \in (0, h_3) \quad (50)$$

and

$$r \in [0, r_4) \quad (51)$$

then it can easily be checked that

$$f_4(r) \leq 0 \quad (51)$$

We can now prove the first result.

Theorem 2

Let $F : \mathbb{C}X \rightarrow X$ be a nonlinear operator satisfying the hypotheses of theorem 1 and consider a Lipschitz uniform discretization (8) which is "bounded", stable and consistent of order p .

Assume:

$$q_1 \geq q_1^*$$

$$q_2 \geq q_2^*$$

$$k \geq k^*$$

and that (14) and (15) are satisfied.

Then equation (3) has a locally unique solution x_h^* such that:

$$x_h^* = P_h(z^*) + O(h^p) \quad (53)$$

for all $h \in (0, h_1]$.

There exist constants $h \in (0, h_2]$ and $r \in (0, r_3)$ such that the iteration (4) converges to x_h^* .

Moreover, the following are true:

$$x_n^h = P_h(z_n) + O(h^p) \quad (54)$$

$$T_h(x_n^h) = P_h(F(z_n)) + O(h^p) \quad (55)$$

and

$$x_n^h - x_h^* = P_h(z_n - z^*) + O(h^p) \quad (56)$$

for $n = 0, 1, 2, \dots$ and all $z_0 \in U(z^*, r_3)$.

Proof. Using the Newton-Kantorovich theorem [7], (15) we show exactly as in [3] that (3) has a solution $x_h^* \in \bar{U}(P_h(z^*), r(h))$ which is unique in $U(P_h(z^*), \tilde{r}(h))$ where

$$\alpha(h) = 2\sigma_L \|T_h'(P_h(z^*))^{-1} T_h(P_h(z^*))\| \quad (57)$$

$$r(h) = \frac{1}{\sigma_L} (1 - \sqrt{1 - \alpha(h)}) \leq \rho \quad (58)$$

$$\tilde{r}(h) = \frac{1}{\sigma_L} (1 + \sqrt{1 - \alpha(h)}) \quad (59)$$

Moreover,

$$\|x_h^* - P_h(z^*)\| \leq r(h) \leq 2\sigma_C h^p \quad (60)$$

which proves (53).

As in [3] by applying theorem 1 to (3) we see that (4) converges to x_h^* if

$$\|P_h(z_0) - x_h^*\| < \frac{2}{3 \|T_h'(x_h^*)^{-1}\|} \quad (61)$$

and

$$\bar{U}(x_h^*, \|P_h(z_0) - x_h^*\|) \subset \bar{U}(P_h(z^*), \rho) \quad (62)$$

that is if

$$q_2 \|z_0 - z^*\|^2 + q_1 \|z_0 - z^*\| + 4\alpha c_0 h^p < \frac{2 - 4\sigma^2 L c_0 h^p}{3 L \sigma} \quad (63)$$

and

$$q_2 \|z_0 - z^*\|^2 + q_1 \|z_0 - z^*\| + 4\alpha c_0 h^p \leq \rho \quad (64)$$

hold respectively.

But (63) and (64) can now be written as

$$f_1(r) \leq 0$$

and

$$f_2(r) \leq 0$$

which are true by the choice of h , r , h_1 and r_1 .

We now show, using induction as in [3] that for $h \in (0, h_3)$, $z_0 \in U(z^*, r_3)$

$$\|x_n^h - P_h(z_n)\| \leq s, \quad n = 0, 1, 2, \dots \quad (65)$$

By rearranging the sequences in (65) we can start the induction from $n = 0$ if necessary. The above inequality is true for $n = 0$. Let us assume that it is true for $n = 0, 1, 2, \dots, i$.

Then

$$\begin{aligned} x_{i+1}^h - P_h(z_{i+1}) &= T_h'(x_i^h)^{-1} \{ [T_h'(x_i^h) (x_i^h - P_h(z_i)) - T_h(x_i^h) + T_h(P_h(z_i))] \\ &+ [(T_h'(x_i^h) - T_h'(P_h(z_i))) P_h(F'(z_i)^{-1}(z_i))] + [T_h'(P_h(z_i)) P_h(F'(z_i)^{-1}F(z_i) \\ &- P_h(F(z_i))) + [P_h(F(z_i)) - T_h(P_h(z_i))] \} + [P_h(z^*) - P_h(z_i) + P_h'(z^*)(z_i - z^*)] \\ &- P_h'(z^*) (F'(z_i)^{-1} F(z_i)) + P_h''(\bar{z}^*)(z_{i+1} - z^*)^2 + P_h(F'(z_i)^{-1} F(z_i)) \end{aligned} \quad (66)$$

We now have, using (10), (13).

$$\|P_h(z^*) - P_h(z_i) + P_h'(z^*)(z_i - z^*)\| \leq q_2 \|z_i - z^*\|^2$$

$$\|P_h'(z^*)(z_i - z_{i+1})\| \leq q_1 \|z_i - z_{i+1}\|$$

$$\begin{aligned} \|P_h''(\bar{z}^*)(z_{i+1} - z^*)^2\| &\leq q_2 \|z_{i+1} - z^*\|^2 \\ \|P_h(z_i - z_{i+1})\| &\leq \|P_h(z^*)\| + q_1 \|z_i - z_{i+1} - z^*\| + \\ &+ q_2 \|z_i - z_{i+1} - z^*\|^2 \end{aligned}$$

By theorem 1, $\|z_{i+1} - z^*\| \leq \|z_i - z^*\| \leq \dots \leq \|z_0 - z^*\|$ and the result on the norm of the braces above in [3] we have that the right hand side of (66) is bounded by

$$\begin{aligned} \frac{\sigma}{1 - L\sigma} \left[\frac{1}{2} Ls^2 + 2L(q_1(2r + b) + q_2(2r + b)^2)rs + 2ch^p \right] \\ + q_2r^2 + 2q_1r + q_2r^2 + 2q_1r + q_1b + 4q_2r^2 + q_2b^2 + 4bq_2r = s \end{aligned}$$

That is,

$$\|x_h^h - P(z_h)\| \leq s \leq kh^p \quad (67)$$

which proves (54).

Equation (55) can be proved exactly as in [3].

Finally, for some $\bar{z} \in U^*$

$$\begin{aligned} \|(x_n^h - x_h^*) - P_h(z_n - z^*)\| &\leq \|x_n^h - P_h(z_n)\| + \|x_h^* - P_h(z^*)\| \\ &+ \|[P_h(0) + (P_h(z^*) - P_h(z_n) + P_h'(\bar{z})(z_n - z^*))]\| \\ &\leq kh^p + 2uch^p + q_2 \|z_n - z^*\|^2 \leq k_2h^p \end{aligned}$$

by (13), (14), (15), (44), (45), (66), (67) and (68) which proves (56) and completes the proof of the theorem.

We now prove the last result.

Theorem 3

Assume:

- (1) The hypotheses of theorem 2 hold;
- (2) There exist constants $\bar{q}_1, \bar{q}_2$ such that

$$\liminf_{n \rightarrow 0} \|P_h(z)\| \geq \bar{q}_1 \|z\| + \bar{q}_2 \|z\|^2 \quad \text{for } z \in \bar{U} \text{ and } \bar{q}_1 > q_1 + q_2 \quad (69)$$

Then for $r \in (0, r_4)$, and for any fixed $0 < \epsilon \leq 1$ and $z_0 \in \bar{U}(z^*, r)$ there exists $h_4 = h_4(\epsilon, z_0) \in (0, h_6)$ where,

$$h_6 = \min(h_3, h_5)$$

and $h_5 = h(z_0)$ is such that

$$\|P_h(z_i - z^*)\| \geq \bar{q}_1 \|z_i - z^*\| + \bar{q}_2 \|z_i - z^*\|^2 \quad h \in (0, h_5) \quad (70)$$

and i is the unique integer defined by

$$\|z_{i+1} - z^*\| < \epsilon \leq \|z_i - z^*\| \quad (71)$$

such that

$$|\min\{n \geq 0, \|z_n - z^*\| < \epsilon\} - \min\{n \geq 0, \|x_n^h - x_h^*\| < \epsilon\}| \leq 1 \quad (72)$$

for all $h \in (0, h_4]$.

Proof. By (68) and the choice of h, β we have

$$\begin{aligned} \|x_{i+1}^h - x_h^*\| &\leq \|P_h(z_{i+1} - z^*)\| + k_2 h^p \leq q_1 \|z_{i+1} - z^*\| + q_2 \|z_{i+1} - z^*\|^2 \\ &\quad + k_2 h^p \leq (q_1 + q_2) \epsilon + k_2 h^p < 2(q_1 + q_2) \epsilon \end{aligned} \quad (73)$$

Using theorem 1, and the choice of h, β, r we get

$$\begin{aligned} \|x_{i+2}^h - x_h^*\| &\leq \frac{L\sigma \|x_{i+1}^h - x_h^*\|^2}{2(1 - L\sigma \|x_{i+1}^h - x_h^*\|)} \\ &\leq \frac{L\sigma \|x_0^h - P_h(z^*) + P_h(z^*) - x_h^*\|}{2(1 - L\sigma \|x_0^h - P_h(z^*) + P_h(z^*) - x_h^*\|)} \|x_{i+1} - x_h^*\| \\ &< \frac{L\sigma [q_1 \|z_0 - z^*\| + q_2 \|z_0 - z^*\|^2 + 2\sigma ch^p]}{[1 - L\sigma (q_1 \|z_0 - z^*\| + q_2 \|z_0 - z^*\|^2 + 2\sigma ch^p)]} (q_1 + q_2) \epsilon \\ &\leq \beta (q_1 + q_2) \epsilon < \epsilon \end{aligned} \quad (74)$$

Moreover, by (68), (69) and (70),

$$\begin{aligned} \bar{q}_1 \epsilon + \bar{q}_2 \epsilon^2 &\leq q_1 \|z_i - z^*\| + q_2 \|z_i - z^*\|^2 \\ &\leq \|P_h(z_i - z^*)\| \leq (\|x_i^h - x_h^*\| + k_2 h^p) \end{aligned}$$

or

$$\|x_i^h - x_h^*\| \geq \bar{q}_1 \epsilon + \bar{q}_2 \epsilon^2 - c_2 h^p \quad (75)$$

If $\|x_{i-1}^h - x_h^*\| < \epsilon$ then as in (74) we get

$$\|x_i^h - x_h^*\| < \frac{1}{2} \beta \epsilon \quad (76)$$

By, (75) and (76) we must also have

$$\bar{q}_1 \epsilon + \bar{q}_2 \epsilon^2 - k_2 h^p < \frac{1}{2} \beta \epsilon$$

or

$$\bar{q}_1 \epsilon + \bar{q}_2 \epsilon^2 - (q_1 + q_2) \epsilon < \frac{1}{2} \beta \epsilon$$

or

$$\bar{q}_2 < \frac{1}{\epsilon} \left(\frac{1}{2} \beta + q_1 + q_2 - \bar{q}_1 \right) < 0 ,$$

that is

$$\bar{q}_2 < 0 ,$$

which is a contradiction. Therefore, we have

$$\|x_{i-1}^h - x_h^*\| \geq \epsilon . \quad (77)$$

The result is now obtained by (71), (74) and (77).

We can do better sometimes. Let us assume that:

$$0 < q_1 < 1 \quad (78)$$

and

$$\bar{q}_1 + \bar{q}_2 > \max \left(\frac{1}{2} , \frac{q_1 + 2q_2}{4} , \frac{q_2}{2(1-q_1)} \right) \quad (79)$$

Define $\beta_1 > 0$ by

$$\beta_1 = \min \left(\frac{1}{R}, \beta, 2\bar{q}_2, 2(\bar{q}_1 + \bar{q}_2) - 1, \frac{4(\bar{q}_1 + \bar{q}_2) - (q_1 + 2q_2)}{2} , \frac{2(1-q_1)(\bar{q}_1 + \bar{q}_2) - q_2}{1-q_1} \right)$$

Choosing $R, h_7 > 0$ such that:

$$\max \left(\frac{1}{2} , \frac{q_1 + 2q_2}{4} , \frac{q_2}{2(1-q_1)} \right) < R \leq \bar{q}_1 + \bar{q}_2 - \frac{1}{2} \beta^* , \text{ for some } \beta^* \in (0, \beta_1) ,$$

$$h_7 = \min \left(h_6 , \left(\frac{R\epsilon^2}{k_2} \right)^{\frac{1}{p}} , \left(\frac{8R - rq_2 - q_1}{k_2} \right)^{\frac{1}{p}} , \left(\frac{(2R - q_2)(2R - q_2 - 2Rq_1)}{4R^2(2R - q_2)k_2} \right)^{\frac{1}{p}} \right) .$$

Here, $\epsilon \in [\epsilon^+, \frac{1}{2R}]$ and ϵ^+ is the positive solution of the equation

$$(2R - q_2)\epsilon^2 - q_1\epsilon - k_2h^p = 0,$$

guaranteed to exist by the choice of h and R .

Define r_5 as the positive solution of the equation

$$f_5(r) = \text{Log}_2(1 + \beta^*)r^2 + \text{Log}_1(1 + \beta^*)r + (1 + 2\beta^*)h^p - \beta^* = 0,$$

guaranteed to exist by the choice of h , R and β^* .

Set

$$r_6 = \min(r_4, r_5).$$

We can now easily check that by (78), (79) and the choice of h , R , β^*

$$0 < \epsilon^+ < \frac{1}{2R}.$$

We can show exactly as in theorem 3 that:

$$\|x_{i+1}^h - x_h^*\| \leq q_2\epsilon^2 + q_1\epsilon + k_2h^p < 2R\epsilon^2$$

$$\|x_{i+2}^h - x_h^*\| \leq \beta^*R\epsilon^2 < \epsilon^2$$

$$\|x_{i-1}^h - x_h^*\| \geq \epsilon^2.$$

Then by theorem 3

$$|\min\{n \geq 0, \|z_n - z^*\| < \epsilon\} - \min\{n \geq 0, \|x_n^h - x_h^*\| < \epsilon^2 < \epsilon\}| \leq 1. \quad (80)$$

Moreover,

$$\|x_{i+1}^h - x_h^*\| \leq 2R\epsilon^2 < \epsilon. \quad (81)$$

Therefore (80), because of (81), becomes

$$I = \min\{n \geq 0, \|x_n^h - x_h^*\| < \epsilon\} - \min\{n \geq 0, \|z_n - z^*\| < \epsilon\}, \quad I = -1, 0.$$

We have now proved the following:

Corollary. Assume that the hypotheses of theorem 3, (78) and (79) hold.

Then for $r \in (0, r_6)$, and for any $\epsilon \in [\epsilon^+, \frac{1}{2R}]$ and $z_0 \in \bar{U}(z^*, r)$ there exists $h_8 = h_8(\epsilon, z_0) \in (0, h_7)$ such that:

$$I = \min \{n \geq 0, \|x_n^h - x_h^*\| < \epsilon\} - \min \{n \geq 0, \|z_n - z^*\| < \epsilon\}, \text{ with} \\ I = -1, 0 \quad (82)$$

for all $h \in (0, h_0]$.

The above result improves and extends the corresponding one in [3].

Note that (82) implies that:

$$\min \{n \geq 0, \|x_n^h - x_h^*\| < \epsilon\} \leq \min \{n \geq 0, \|z_n - z^*\| < \epsilon\}. \quad (83)$$

Inequality (83) arises the question of the actual equality being achieved in (83). The proof of (83) was based primarily on (7), (56) and (69).

We note:

$$\begin{aligned} x_n^h - x_h^* &= P_h(z_n - z^*) + O(h^p) \\ &= P_h(0) + P_h'(\bar{z})(z_n - z^*) + O(h^p), \text{ for some} \\ &\quad \bar{z} \in U^* \\ &= P_h'(\bar{z})(z_n - z^*) + O(h^p). \end{aligned} \quad (84)$$

Assume that the inverse of $P_h'(\bar{z})$ exists and is bounded for all sufficiently small h and $\bar{z} \in U^*$. Set $L_h = (P_h'(\bar{z}))^{-1}$ and $\bar{q}_1^* \geq \|L_h\|$.

Then the equation (84) can be written as

$$z_n - z^* = L_h (x_n^h - x_h^*) + O(h^p), \quad n = 0, 1, 2, \dots, h \text{ sufficiently small.} \quad (85)$$

By theorem 1 we can have,

$$\|x_{n+1}^h - x_h^*\| \leq \frac{L\alpha \|x_n^h - x_h^*\|^2}{2(1-L\alpha \|x_n^h - x_h^*\|)}, \quad n = 0, 1, 2, \dots \quad (86)$$

Moreover, note that the assumption on P_h given by (11), (13), (14), and (15) hold for the linear operator L_h with $q_1 = \bar{q}_1^*$, $q_2 = 0$.

If the rest of the assumptions made for P_h hold for L_h then the proofs of theorem 2 will go through for L_h .

Finally, assume that:

$$\lim_{h \rightarrow 0} \inf \|L_h(x)\| \geq \bar{q}_1^* \|x - x_h^*\| + \bar{q}_2^* \|x - x_h^*\|^2, \quad x \in \bar{U}(P_h(-*), \mu) \\ (\bar{q}_2^* = 0). \quad (87)$$

If we now interchange the role of the z 's with the x 's in theorems 1, 2, 3 and the Corollary, under similar assumptions on h , $\rho_h^* = \|x_0^h - x_h^*\|$, $\bar{q}_1^*$, $\bar{q}_2^*$, q_1^* the proofs of both theorem 3 and the Corollary will go through. Therefore, we can show that:

$$\min \{n \geq 0, \|x_n^h - x_h^*\| < \epsilon\} \geq \min \{n \geq 0, \|z_n - z^*\| < \epsilon\}, \quad (88)$$

for sufficiently small h and ρ_h^* .

By (83) and (88) we finally obtain

$$\min \{n \geq 0, \|x_n^h - x_h^*\| < \epsilon\} = \min \{n \geq 0, \|z_n - z^*\| < \epsilon\}, \quad (89)$$

where $h \in (0, h_9]$, $z_0 \in \bar{U}(z^*, r)$, $x_0^h \in \bar{U}(P_h(z^*), \rho)$ and any $\epsilon \in [\epsilon_1^+, \frac{1}{2R_1}] \cap [\epsilon_1, \frac{1}{2R}]$ provided that the intersection is nonempty.

Here, $h_9 = \min(h_8, h_8^*)$, $r \in (0, r_6)$, $\rho \in (0, \rho_6)$. Finally, the h_8^* , ρ_6^* , ρ_1^+ and R_1 were found as the h_8 , r_6 , ϵ^+ and R respectively.

The result given by equation (89) improves and extends the corresponding result in [3].

III. REMARKS - EXAMPLES

The condition (15) does not hold in general for any nonlinear projections P_h , $h > 0$. However, it holds for linear projections [3] and in fact for multilinear projections [4].

In particular, it has been shown in [4] that when F is a quadratic operator and the P_h are quadratic projections on $\mathbb{R}^n$ (or $\mathbb{C}^n$) then (3) can be replaced with a quadratic system on $\mathbb{R}^n$ (or $\mathbb{C}^n$).

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