PROYECCIONES

Revista de Matemática Vol. 8/9 Pág. 17-39

1989/90 - ISSN 0716-0917

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AN EQUIVALENT FORMULATION FOR VARIATIONAL
PROBLEMS WITH CONSTRAINTS

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A B S T R A C T

The main purpose of this paper is to reconsider the studying of existence, uniqueness and approximation of solution to a variational problem with constraints. Under certain assumptions on the bilinear forms involved it is proved that the original formulation is equivalent to a simplified version of it on smaller spaces. By applying the usual Brezzi's theory to this new formulation a modified Babuska-Brezzi condition is deduced.

1. INTRODUCTION

One of the most important topics in optimization theory is the classical constrained minimization model. It is well known that under certain assumptions these problems can be converted into an equivalent saddle-point formulation by using the method of Lagrange multipliers. Moreover, by taking variations of the corresponding Lagrangian, one obtains that the saddle-point is characterized by what we call a variational problem with constraints. A typical example of this is the minimization of the energy functional associated to a boundary value problem with non-homogeneous boundary conditions. In this case, the corresponding saddle-point problem is called a variational boundary value problem with constraints.

The firsts general approach for solving a constrained variational problem was given by Brezzi [6]. In there, the following formulation was

considered (irrespective of whether it is obtained from a saddle-point problem): Find a pair $(u, \lambda) \in H \times Q$ such that

$$\begin{aligned} a(u, v) + b(v, \lambda) &= f(v), & \forall v \in H \\ b(u, \mu) &= g(\mu), & \forall \mu \in Q \end{aligned} \quad (1.1)$$

where H, Q are Hilbert spaces, a, b are bounded bilinear forms on $H \times H$ and $H \times Q$, respectively, and f, g are given elements in the duals H^* and Q^* , respectively. Brezzi gave in [6] sufficient conditions on the bilinear forms a, b which insure that (1.1) has one and only one solution for all $(f, g) \in H^* \times Q^*$. Also, the discrete problem associated was considered in [6]: Find $(u_h, \lambda_h) \in H_h \times Q_h$ such that

$$\begin{aligned} a(u_h, v_h) + b(v_h, \lambda_h) &= f(v_h), & \forall v_h \in H_h \\ b(v_h, \mu_h) &= g(\mu_h), & \forall \mu_h \in Q_h \end{aligned} \quad (1.2)$$

where H_h and Q_h are finite dimensional subspaces of H and Q , respectively. Under analogous assumptions on the bilinear forms a, b , Brezzi obtained in [6] an abstract error estimate for the quantity

$$\|u - u_h\|_H + \|\lambda - \lambda_h\|_Q$$

It should also be mentioned that Nicolaides [9] extended the results of [6] to a more general class of saddle-point problems. This extension can be viewed as an analogous result to the generalized Lax - Milgram lemma provided by Babuska [2], [3].

On the other hand, it has been shown in many papers that the problems (1.1) and (1.2) are very closely related to the analysis of the so called mixed and hybrid finite element methods. Precisely, much of the studying of mixed and hybrid methods can be considered as an approximation theory for variational boundary value problems with constraints. The first abstract analyses of mixed finite element methods are due to Aubin and Burchard [1], and Babuska [3]. In particular, Babuska developed an abstract theory which resembles that of [6], and which is the basis of the work presented in [5].

In this paper we are interested in reconsidering the abstract formulations given by (1.1) and (1.2). More precisely, we assume slightly different conditions from those in [6] and then prove analogous existence - uniqueness results for the variational problems (1.1) and (1.2). The way of doing this is by means of an equivalent formulation to (1.1) on Hilbert spaces H_E and Q_E which are strictly contained in H and Q , respectively. Indeed, we show that the corresponding Babuska - Brezzi condition for this new formulation constitutes a modified Babuska - Brezzi condition for the original formulation on H and Q . The main difference with the analysis provided by Brezzi [6] is that we consider not only the kernel of the operator induced by b , but also the one induced by a . The paper is presented as follows. In section 2 we introduce some preliminary results and review the Brezzi's theory for the studying of (1.1) and (1.2).

We provide in section 3 our equivalent formulation and the corresponding existence-uniqueness result. The application of this approach to the discrete problem (1.2) is studied in section 4. Finally, in order to illustrate the ideas, an example is given in section 5.

2. PRELIMINARY RESULTS

Let us consider a real Hilbert H with scalar product $\langle \cdot, \cdot \rangle_H$ and corresponding norm $\|\cdot\|_H$. Given a bounded bilinear form $B : H \times H \rightarrow \mathbb{R}$ and a bounded linear functional $F \in H^*$, we are interested in finding $u \in H$ such that

$$B(u, v) = F(v), \quad \forall v \in H \quad (2.1)$$

The following well known Lax-Milgram lemma provides sufficient conditions for the existence and uniqueness of solution to (2.1).

Theorem 2.1.

Let B be a bilinear form on H . Suppose that there exist positive constants M, α such that

$$|B(v, w)| \leq M \|v\|_H \|w\|_H, \quad \forall v, w \in H$$

and

$$\alpha \|v\|_H^2 \leq B(v, v), \quad \forall v \in H \quad (2.2)$$

Then, for every bounded linear functional $F \in H^*$, there exists a unique $u \in H$ such that

$$B(u, v) = F(v), \quad \forall v \in H$$

Moreover, the following continuous - dependence result holds

$$\|u\|_H \leq \frac{1}{\alpha} \|F\|_{H^*}$$

Proof: See [7] : Theorem 1.1.3.

Now there are many situations in which the strong coerciveness condition (2.2) does not hold, but we may still have existence and uniqueness for (2.1). We refer specifically to those bilinear forms that are weakly coercive. More precisely, we give the following result which is known as the generalized Lax-Milgram lemma.

Theorem 2.2.

Let H_1, H_2 be real Hilbert spaces and let B denote a bilinear form on $H_1 \times H_2$ which has the following properties

i) B is bounded, i.e. there exists $M > 0$ such that

$$|B(v,w)| \leq M \|v\|_{H_1} \|w\|_{H_2}, \quad \forall (v,w) \in H_1 \times H_2 \quad (2.3)$$

ii) B is weakly-coercive, i.e. there exists $\alpha > 0$ such that

$$\sup_{\substack{w \in H_2 \\ w \neq 0}} \frac{B(v,w)}{\|w\|_{H_2}} \geq \alpha \|v\|_{H_1}, \quad \forall v \in H_1 \quad (2.4)$$

iii) for every $w \in H_2$, $w \neq 0$

$$\sup_{v \in H_1} B(v,w) > 0 \quad (2.5)$$

Then, given $F \in H_2^*$ there exists a unique $u \in H_1$ such that

$$B(u,w) = F(w), \quad \forall w \in H_2$$

Moreover,

$$\|u\|_{H_1} \leq \frac{1}{\alpha} \|F\|_{H_2^*} \quad (2.6)$$

Proof: See [2] : Theorem 5.2.1. [10] : Theorem 1.4.3.

Remark 2.1.

If $H_1 = H_2 = H$ and B is symmetric on $H \times H$, then it is easy to see that the condition (2.5) is a consequence of (2.4).

In what follows, we review the main aspects of the Brezzi's theory for the studying of a variational problem with constraints.

We consider now two real Hilbert spaces H, Q with norms $\|\cdot\|_H$ and $\|\cdot\|_Q$, respectively.

Let $a : H \times H \rightarrow \mathbb{R}$ and $b : H \times Q \rightarrow \mathbb{R}$ be two bounded bilinear forms.

Also, let f and g be two linear bounded functionals on H and Q , respectively, i.e. $f \in H^*$ and $g \in Q^*$.

The variational problem associated to a, b, f and g consists of finding $(u, \lambda) \in H \times Q$ such that

$$\begin{aligned} a(u,v) + b(v,\lambda) &= f(v), & \forall v \in H \\ b(u,\mu) &= g(\mu), & \forall \mu \in Q \end{aligned} \quad (2.7)$$

At this point we also introduce the following spaces

$$N(b) := \{v \in H / b(v,\lambda) = 0, \quad \forall \lambda \in Q\}$$

$$N(b^*) := \{\lambda \in Q / b(v, \lambda) = 0, \quad \forall v \in H\}$$

and

$$Z := Q / N(b^*)$$

In order to study the existence of solutions to (2.7) we define the bilinear form

$$B : (H \times Q) \times (H \times Q) \rightarrow R$$

$$B((u, \lambda), (v, \mu)) := a(u, v) + b(v, \lambda) + b(u, \mu)$$

and the linear functional

$$F : H \times Q \rightarrow R$$

$$F(v, \mu) := f(v) + g(\mu)$$

Then, (2.7) can be written as follows : FIND $(u, \lambda) \in H \times Q$ such that

$$B((u, \lambda), (v, \mu)) = F(v, \mu), \quad \forall (v, \mu) \in H \times Q \quad (2.8)$$

Let the norm on the product space $H \times Q$ be given by

$$\|(v, \mu)\|_{H \times Q} := \|v\|_H + \|\mu\|_Q$$

Then, it follows easily from the boundedness of a , b , f and g that B is a bounded bilinear form and F is a bounded linear functional.

If we think of the generalized Lax-Milgram lemma then under the additional assumption that a is symmetric (and hence B) we obtain that a sufficient condition for the existence of a unique solution to (2.8) would be

$$\sup_{\substack{(v, \mu) \in H \times Q \\ (v, \mu) \neq 0}} \frac{|a(u, v) + b(v, \lambda) + b(u, \mu)|}{\|(v, \mu)\|_{H \times Q}} > \alpha \|(u, \lambda)\|_{H \times Q} \quad (2.9)$$

for all $(u, \lambda) \in H \times Q$, with $\alpha > 0$ independent of (u, λ) .

Unfortunately, the above inequality is sometimes a too strong condition to be of great value in studying (2.8). The problem is that $N(b)$ and $N(b^*)$ may not consist of only the zero element. Then, we could never expect (2.8) to have a unique solution. Indeed, to any $\lambda \in Q$ satisfying (2.8) we could always add an arbitrary element of $N(b^*)$ and obtain a different solution to (2.8).

A slightly different way of studying (2.8) was developed by Brezzi [6]. His main result can be summarized in the following theorem.

Theorem 2.3.

Let $a : H \times H \rightarrow \mathbb{R}$ be a symmetric and bounded bilinear form, and let $b : H \times Q \rightarrow \mathbb{R}$ be a bounded bilinear form. Suppose that there exist $\alpha_0, \beta > 0$ such that

$$\sup_{\substack{v \in N(b) \\ v \neq 0}} \frac{|a(u, v)|}{\|v\|_H} \geq \alpha_0 \|u\|_H, \quad \forall u \in N(b) \quad (2.10)$$

and

$$\sup_{\substack{v \in H \\ v \neq 0}} \frac{|b(v, \lambda)|}{\|v\|_H} \geq \beta \|\lambda\|_Z \quad \forall \lambda \in Q \quad (2.11)$$

where

$$[\lambda] := \{\mu \in Q : (\lambda - \mu) \in N(b^*)\}$$

and

$$\|[\lambda]\|_Z := \inf_{\mu \in [\lambda]} \|\mu\|_Q$$

Then, given $F \in (H \times Q)^*$ there exists a unique solution $(u, [\lambda]) \in H \times Z$ of problem (2.8). The Lagrange multiplier λ is then unique up to an arbitrary element of $N(b^*)$.

Proof : See [6], [10] : Theorem 4.2.1.

Remark 2.2.

Brezzi [6] actually proved that under (2.10), the hypothesis (2.11) is a necessary and sufficient condition for (2.8) to be well posed. The importance of (2.11) was also pointed out by Babuska [3], [4]. Because of this, the hypothesis (2.11) which also plays a very important role in mixed finite element methods is referred to as the Babuska-Brezzi condition.

Remark 2.3. A result analogous to theorem 2.3. is obtained for the discrete problem (1.2). In this case, the hypothesis corresponding to (2.11), which is called the discrete Babuska-Brezzi condition, is given by

$$\sup_{\substack{v_h \in H_h \\ v_h \neq 0}} \frac{|b(v_h, \lambda_h)|}{\|v_h\|_H} \geq \beta_h \|[\lambda_h]\|_{Z_h}, \quad \forall \lambda_h \in Q_h \quad (2.12)$$

where

$$Z_h := Q_h / N_h(b^*)$$

and

$$N_h(b^*) := \{\lambda_h \in Q_h : b(v_h, \lambda_h) = 0, \quad \forall v_h \in H_h\}$$

Moreover, in order to obtain stable error estimates the conditions $\beta_h \geq \beta_0 > 0$, and $\alpha_h \geq \alpha_0 > 0$, for all h , must be imposed (see [6], [9]).

3. THE EQUIVALENT FORMULATION

Now on, we assume that $a : H \times H \rightarrow R$ is a symmetric bounded bilinear form and that $b : H \times Q \rightarrow R$ is a bounded bilinear form. Given $f \in H^*$ and $g \in Q^*$, we are interested in finding $(u, \lambda_0) \in H \times Q$ such that

$$\begin{aligned} a(u, v) + b(v, \lambda_0) + b(u, \mu) &= f(v) + g(\mu) \\ \forall (v, \mu) &\in H \times Q \end{aligned} \quad (P_0)$$

In addition to the notations and definitions given in section 2, we now introduce the space

$$N(a) := \{q \in H : a(q, v) = 0, \quad \forall v \in H\} \quad (3.1)$$

Since a is symmetric, it is clearly seen that

$$N(a) = N(a^*) := \{q \in H : a(v, q) = 0, \quad \forall v \in H\} \quad (3.2)$$

We assume that $N(a)$ is a nontrivial finite bidimensional subspace of H , and that there exists a bounded linear map $S : N(a) \rightarrow Q$ such that the following condition is satisfied

$$q \in N(a) \quad \text{and} \quad b(q, S(q)) = 0 \implies q = 0 \quad (3.3)$$

Thus, we have the following result.

Lemma 3.1.

Given any bounded linear functional $k : N(a) \rightarrow R$, there exist unique $\bar{p}, \bar{q} \in N(a)$ such that

$$i) \quad b(q, S(\bar{p})) = k(q), \quad \forall q \in N(a)$$

and

$$ii) \quad b(\bar{q}, S(q)) = k(q), \quad \forall q \in N(a)$$

Proof : Let $\{q_1, \dots, q_m\}$ be a basis of $N(a)$. Then, if we put $\bar{p} = \sum_{j=1}^m \alpha_j q_j$, the equation i) is equivalent to the set of linear equations

$$\sum_{j=1}^m \alpha_j b(q_i, S(q_j)) = k(q_i), \quad \forall i = 1, \dots, m$$

It follows from (3.3) that the only one solution of the homogeneous system associated is the trivial solution. This proves the existence and uniqueness of $\bar{p}$. The proof of ii) is identical.

Hence, by using lemma 3.1. we deduce that there exists a unique $p_0 \in N(a)$ such that

$$b(q, S(p_0)) = f(q) \quad \forall q \in N(a) \quad (3.4)$$

Therefore, by making the identification $\lambda := \lambda_0 - S(p_0)$, we can rewrite our problem (P_0) as follows: FIND $(u, \lambda) \in H := {}^0H \times Q$ such that

$$\begin{aligned} B((u, \lambda), (v, \mu)) &= F(v, \mu) \\ \forall (v, \mu) &\in H \end{aligned} \quad (P_1)$$

where

$$B((u, \lambda), (v, \mu)) := a(u, v) + b(v, \lambda) + b(u, \mu) \quad (3.5)$$

and

$$F(v, \mu) := f(v) - b(v, S(p_0)) + g(\mu) \quad (3.6)$$

Remark 3.1.

Note that if we take $v = q \in N(a)$ and $\mu = 0$ in (P_1) , then (3.4) implies that any solution (u, λ) of (P_1) must satisfy

$$b(q, \lambda) = 0, \quad \forall q \in N(a)$$

We now define the following subspaces of H and Q , respectively

$$H_E := \{ v \in H : b(v, S(q)) = 0, \quad \forall q \in N(a) \}$$

$$Q_E := \{ \lambda \in Q : b(q, \lambda) = 0, \quad \forall q \in N(a) \}$$

Then, instead of (P_1) we propose the following formulation :
FIND $(\bar{u}, \lambda, p) \in \bar{H} := H_E \times Q \times N(a)$ such that

$$\begin{aligned} B((\bar{u}, \lambda), (\bar{v}, \mu)) + b(p, \mu) &= F(\bar{v}, \mu) \\ b(q, \lambda) &= 0 \\ \forall (\bar{v}, \mu, q) &\in \bar{H} \end{aligned} \quad (P_2)$$

We have the following result relating (P_1) and (P_2)

Theorem 3.1.

The problems (P_1) and (P_2) are equivalent, i.e.

- i) If (u, λ) is a solution to (P_1) , then $(u - p_u, \lambda, p_u)$ is a solution to (P_2) , where p_u is the unique element of $N(a)$ that satisfies

$$b(p_u, S(q)) = b(u, S(q)) \quad , \quad \forall q \in N(a)$$

Conversely, if $(\bar{u}, \lambda, p)$ is a solution to (P_2) , then $(\bar{u} + p, \lambda)$ is a solution to (P_1) .

ii) (P_1) has unique solution $\iff (P_2)$ has unique solution.

Proof : Let (u, λ) be a solution to (P_1) . Then, from remark 3.1, it is clear that $b(q, \lambda) = 0, \forall q \in N(a)$. Now, from lemma 3.1. there exists a unique $p_u \in N(a)$ such that

$$b(p_u, S(q)) = b(u, S(q)), \quad \forall q \in N(a)$$

and hence $\bar{u} := (u - p_u) \in H_E$. So, by substituting $u = (\bar{u} + p_u)$ in (P_1) , we can write

$$\begin{aligned} B((\bar{u}, \lambda), (\bar{v}, \mu)) + b(p_u, \mu) &= F(\bar{v}, \mu) \\ b(q, \lambda) &= 0 \end{aligned}$$

for all $(\bar{v}, \mu, q) \in \bar{H}$, which proves that $(\bar{u}, \lambda, p_u)$ is a solution to (P_2) .

Conversely, let $(\bar{u}, \lambda, p) \in \bar{H}$ be a solution to (P_2) . Given $v \in H$, define $\bar{v} := v - q_v$, where q_v is the unique element of $N(a)$ that satisfies

$$b(q_v, S(q)) = b(v, S(q)), \quad \forall q \in N(a)$$

Then, since $\bar{v} \in H_E$, we obtain from (P_2) that

$$\begin{aligned} a(\bar{u}, v - q_v) + b(v - q_v, \lambda) + b(\bar{u}, \mu) + b(p, \mu) \\ = f(v - q_v) - b(v - q_v, S(p_0)) + g(\mu) \end{aligned}$$

for all $(v, \mu) \in H \times Q$. Thus, since $b(q, \lambda) = 0, \forall q \in N(a)$ we deduce from (3.4) and the above equation that

$$B((\bar{u} + p, \lambda), (v, \mu)) = F(v, \mu), \quad \forall (v, \mu) \in H \times Q$$

which proves that $(u, \lambda) := (\bar{u} + p, \lambda)$ is a solution to (P_1) .

Now, suppose that (P_2) has unique solution and let (u_i, λ_i) , $i = 1, 2$, be two solutions to (P_1) . Then, it follows that $(u_i - p_{u_i}, \lambda_i, p_{u_i})$, $i = 1, 2$ are two solutions to (P_2) . Thus, the uniqueness of (P_2) gives $(u_1 - p_{u_1}, \lambda_1, p_{u_1}) = (u_2 - p_{u_2}, \lambda_2, p_{u_2})$, from which $(u_1, \lambda_1) = (u_2, \lambda_2)$.

Conversely, suppose that (P_1) has unique solution and let $(\bar{u}_i, \lambda_i, p_i)$, $i = 1, 2$ be two solutions to (P_2) . Then, since $(\bar{u}_i + p_i, \lambda_i)$, $i = 1, 2$ are solutions to (P_1) we obtain that $\lambda_1 = \lambda_2$ and $\bar{u}_1 + p_1 = \bar{u}_2 + p_2$. So, we have $p_2 - p_1 = \bar{u}_1 - \bar{u}_2$, and since $\bar{u}_1, \bar{u}_2 \in H_E$ we deduce that $p_2 - p_1 \in H_E$. This means that $b(p_2 - p_1, S(q)) = 0, \forall q \in N(a)$, which according to (3.3) gives $p_1 = p_2$ and therefore $\bar{u}_1 = \bar{u}_2$. This proves the uniqueness of (P_2) and completes the proof.

Remark 3.2.

In the above proof we use implicitly the fact that $H = H_E \oplus N(a)$. This result is a trivial corollary of lemma 3.1. Also, from lemma 3.1. it is very easy to prove that $Q = Q_E \oplus S(N(a))$.

We consider now the following formulation which is a simplified version of (P_2) .

$$\begin{aligned} \text{FIND } (\bar{u}, \bar{\lambda}) \in \hat{H} : &= H_E \times Q_E \text{ such that} \\ B((\bar{u}, \bar{\lambda}), (\bar{v}, \bar{\mu})) &= F(\bar{v}, \bar{\mu}) \\ \forall (\bar{v}, \bar{\mu}) \in \hat{H} \end{aligned} \quad (P_3)$$

Then, we have the following result.

Theorem 3.2.

The problems (P_2) and (P_3) are equivalent, i.e.

- i) If $(\bar{u}, \lambda, p)$ is a solution to (P_2) then $(\bar{u}, \bar{\lambda}) := (\bar{u}, \lambda)$ is a solution to (P_3) . Conversely, if $(\bar{u}, \bar{\lambda})$ is a solution to (P_3) , then $(\bar{u}, \bar{\lambda}, p_g)$ is a solution to (P_2) where $p_g \in N(a)$ is the unique element of $N(a)$ that satisfies

$$b(p_g, S(q)) = g(S(q)), \quad \forall q \in N(a)$$

- ii) (P_2) has unique solution $\iff (P_3)$ has unique solution.

Proof : Let $(\bar{u}, \lambda, p) \in \hat{H}$ be a solution to (P_2) . Then taking $\mu := \bar{\mu} \in Q_E$ in (P_2) , it follows easily that $(\bar{u}, \bar{\lambda})$ is a solution to (P_3) with $\bar{\lambda} := \lambda \in Q_E$.

Conversely, suppose that $(\bar{u}, \bar{\lambda}) \in \hat{H}$ is a solution to (P_3) . Let $\mu \in Q$ and let $p_\mu \in N(a)$ be the unique element of $N(a)$ that satisfies

$$b(q, S(p_\mu)) = b(q, \mu), \quad \forall q \in N(a) \quad (3.7)$$

It follows that $\bar{\mu} := \mu - S(p_\mu) \in Q_E$. So, from (P_3) we can write

$$B((\bar{u}, \bar{\lambda}), (\bar{v}, \mu)) - b(\bar{u}, S(p_\mu)) = F(\bar{v}, \mu) - g(S(p_\mu)) \quad (3.8)$$

for all $(\bar{v}, \mu) \in H \times Q$. But, since $\bar{u} \in H_E$ we have clearly that $b(\bar{u}, S(p_\mu)) = 0$. Thus, noting that $\bar{\lambda} \in Q_E$ we can rewrite (3.8) as follows

$$\begin{aligned} B((\bar{u}, \bar{\lambda}), (\bar{v}, \mu)) + g(S(p_\mu)) &= F(\bar{v}, \mu) \\ b(q, \bar{\lambda}) &= 0 \\ \forall (\bar{v}, \mu, q) \in \tilde{H} \end{aligned} \quad (3.9)$$

On the other hand, there exists a unique $p_g \in N(a)$ such that

$$b(p_g, S(q)) = g(S(q)), \quad \forall q \in N(a) \quad (3.10)$$

Thus, from (3.7) and (3.10) we deduce that

$$g(S(p_\mu)) = b(p_g, S(p_\mu)) = b(p_g, \mu)$$

In this way, (3.9) is the same as

$$B((\bar{u}, \bar{\lambda}), (\bar{v}, \bar{\mu})) + b(p_g, \bar{\mu}) = F(\bar{v}, \bar{\mu})$$

$$b(q, \bar{\lambda}) = 0$$

for all $(\bar{v}, \bar{\mu}, q) \in \tilde{H}$, which proves that $(\bar{u}, \bar{\lambda}, p_g)$ is a solution to (P_2) .

It is easily seen from the above analysis that the uniqueness of (P_3) follows from that of (P_2) . Conversely, suppose that (P_3) has unique solution and let $(\bar{u}_i, \lambda_i, p_i)$, $i = 1, 2$ be two solutions to (P_2) . Then, since $(\bar{u}_i, \lambda_i)$, $i = 1, 2$ are solutions to (P_3) we have $(\bar{u}_1, \lambda_1) = (\bar{u}_2, \lambda_2)$. Thus, from (P_2) we obtain that $b(p_2 - p_1, \mu) = 0$, $\forall \mu \in Q$. In particular $b(p_2 - p_1, S(p_2 - p_1)) = 0$ and hence (3.3) gives $p_1 = p_2$.

As a clear consequence of the above results we can state the following theorem.

Theorem 3.3.

The problems (P_1) and (P_3) are equivalent, i.e.

- i) If (u, λ) is a solution of (P_1) , then $(u - p_u, \lambda)$ is a solution of (P_3) where p_u is the unique element of $N(a)$ such that

$$b(p_u, S(q)) = b(u, S(q)), \quad \forall q \in N(a)$$

Conversely, if $(\bar{u}, \bar{\lambda})$ is a solution of (P_3) , then $(\bar{u} + p_g, \bar{\lambda})$ is a solution of (P_1) where p_g is the unique element of $N(a)$ that satisfies

$$b(p_g, S(q)) = g(S(q)), \quad \forall q \in N(a)$$

- ii) (P_1) has unique solution $\iff (P_3)$ has unique solution.

Proof : It follows directly from theorem 3.1. and theorem 3.2.

Therefore, in order to prove existence and uniqueness of solution to our original formulation (P_1) , it is necessary and sufficient to consider the simplified formulation given by (P_3) , i.e.

FIND $(\bar{u}, \bar{\lambda}) \in H_E \times Q_E$ such that

$$B((\bar{u}, \bar{\lambda}), (\bar{v}, \bar{\mu})) = F(\bar{v}, \bar{\mu}), \quad \forall (\bar{v}, \bar{\mu}) \in H_E \wedge Q_E \quad (P_3)$$

Now, we introduce the following subspaces of H_E and Q_E , respectively

$$N_E(b) := \{v \in H_E : b(v, \lambda) = 0, \quad \forall \lambda \in Q_E\}$$

$$N_E(b^*) := \{\lambda \in Q_E : b(v, \lambda) = 0, \quad \forall v \in H_E\}$$

The following result is crucial in our subsequent analysis.

Lemma 3.2.

Let $N(b)$ and $N(b^*)$ be the subspaces defined in section 2. Then, we have

$$N(b) = N_E(b) \quad \text{and} \quad N(b^*) = N_E(b^*)$$

Proof : The inclusions $N(b) \subset N_E(b)$ and $N(b^*) \subset N_E(b^*)$ follow directly from the definitions of $N(b)$ and $N(b^*)$, respectively.

Now, let $v \in N_E(b)$. Then, $b(v, S(q)) = 0$, $\forall q \in N(a)$, and $b(v, \bar{\lambda}) = 0$, $\forall \bar{\lambda} \in Q_E$. So, since $Q = Q_E \oplus S(N(a))$, given any $\lambda \in Q$ we can write $\lambda = \bar{\lambda} + S(p_\lambda)$ where $\bar{\lambda} \in Q_E$ and p_λ is the unique element of $N(a)$ such that $b(q, S(p_\lambda)) = b(q, \lambda)$, $\forall q \in N(a)$. Hence, we obtain

$$b(v, \lambda) = b(v, \bar{\lambda}) + b(v, S(p_\lambda)) = 0$$

which proves that $v \in N(b)$.

On the other hand, let $\lambda \in N_E(b^*)$. Then, we have $b(q, \lambda) = 0$, $\forall q \in N(a)$ and $b(\bar{v}, \lambda) = 0$, $\forall \bar{v} \in H_E$. Since $H = H_E \oplus N(a)$, given any $v \in H$ we can write $v = \bar{v} + p_v$, where $\bar{v} \in H_E$ and p_v is the unique element of $N(a)$ that satisfies $b(p_v, S(q)) = b(v, S(q))$, $\forall q \in N(a)$. Thus, we conclude that

$$b(v, \lambda) = b(\bar{v}, \lambda) + b(p_v, \lambda) = 0$$

which proves that $\lambda \in N(b^*)$. This completes the proof.

Therefore, if we apply the Brezzi's result given by theorem 2.3 to the formulation (P_3) we conclude that the following two conditions are needed

$$\sup_{\substack{v \in N(b) \\ v \neq 0}} \frac{|a(u, v)|}{\|v\|_H} \geq \alpha_0 \|u\|_H, \quad \forall u \in N(b) \quad (3.11)$$

and

$$\sup_{\substack{v \in H_E \\ v \neq 0}} \frac{|b(v, \lambda)|}{\|v\|_H} \geq \beta \|[\lambda]\|_{Z_E} \quad \forall \lambda \in Q_E \quad (3.12)$$

where

$$Z_E := Q_E / N(b^*) \quad (3.13)$$

$$[\lambda] := \{\mu \in Q_E : (\lambda - \mu) \in N(b^*)\} \quad (3.14)$$

In this way, we have proved the following theorem.

Theorem 3.4.

Suppose that there exist positive constants α_0, β such that (3.11) and (3.12) are satisfied. Then, given $F \in (H \times Q)^*$, there exists a unique solution $(\bar{u}, [\bar{\lambda}]) \in H_E \times Z_E$ of problem (P_3) . The Lagrange multiplier $\bar{\lambda}$ is then unique up to an arbitrary element of $N(b^*)$.

Proof: It follows directly from theorem 2.3. and lemma 3.2.

Consequently, because of the complete equivalence between the formulations (P_1) and (P_3) we can state the following result.

Theorem 3.5.

Suppose that there exist positive constants α_0, β such that (3.11) and (3.12) are satisfied. Then, given $F \in (H \times Q)^*$, there exists a unique solution $(u, [\lambda]) \in H \times Z$ of problem (P_1) . More precisely, if $(\bar{u}, [\bar{\lambda}]) \in H_E \times Z_E$ is the unique solution of (P_3) , then $(u, [\lambda]) := (\bar{u} + p_0, [\bar{\lambda}])$. Moreover, under (3.11), the hypothesis (3.12) is a necessary and sufficient condition for (P_1) to be well posed on $H \times Z$.

Proof: It follows directly from theorem 3.3, theorem 3.4 and remark 2.2.

Remark 3.3.

Under the extra assumption given by (3.3) we have shown that theorem 3.5. provides an alternative to the usual Brezzi's result given by theorem 2.3. We will refer to (3.12) as the modified Babuska-Brezzi condition. As a matter of fact, since (2.10) and (3.11) coincide, we deduce that (2.11) and (3.12) are equivalent. However, for practical purposes it might be easier to prove one condition than the other. In fact, on one hand the "sup" in (2.11) is taken on H which contains H_E , but on the other hand the condition (3.12) is required to be satisfied on Q_E which is contained in Q .

Now, since $H = H_E \oplus N(a)$, we can define the projection-type linear mapping

$$\begin{aligned} P : H &\rightarrow H_E \\ v &\rightarrow P(v) = \bar{v} \end{aligned} \quad (3.15)$$

where v is written uniquely as $v := \bar{v} + p_v$, with $\bar{v} \in H_E$ and $p_v \in N(a)$.

Let us now assume that P is bounded, i.e. there exists $C > 0$ such that

$$\|\bar{v}\|_H \leq C \|v\|_H, \quad \forall v \in H \quad (3.16)$$

Then, for $\lambda \in Q_E$ we obtain that

$$\begin{aligned}
\sup_{\substack{v \in H \\ v \neq 0}} \frac{|b(v, \lambda)|}{\|v\|_H} &= \sup_{\substack{v \in H \\ v \neq 0}} \frac{|b(\bar{v}, \lambda)|}{\|\bar{v}\|_H} \\
&\leq C \sup_{\substack{v \in H \\ v \neq 0}} \frac{|b(\bar{v}, \lambda)|}{\|\bar{v}\|_H} \\
&\leq C \sup_{\substack{v \in H_E \\ v \neq 0}} \frac{|b(v, \lambda)|}{\|v\|_H}
\end{aligned}$$

It follows clearly that under the assumption (3.16), the modified Babuska-Brezzi condition (3.12) is equivalent to

$$\sup_{\substack{v \in H \\ v \neq 0}} \frac{|b(v, \lambda)|}{\|v\|_H} \geq \beta \|\lambda\|_{Z_E}, \quad \forall \lambda \in Q_E \quad (3.17)$$

In particular, if $N(b^*) = \{0\}$, then the modified Babuska-Brezzi condition is given by

$$\sup_{\substack{v \in H \\ v \neq 0}} \frac{|b(v, \lambda)|}{\|v\|_H} \geq \beta \|\lambda\|_Q, \quad \forall \lambda \in Q_E$$

which is, obviously, simpler than the usual Babuska-Brezzi conditions (2.11).

4. THE GALERKIN APPROXIMATION

Here, we are interested in the Galerkin approximation of the abstract saddle-point formulation studied in section 3.

Let H_h and Q_h be finite dimensional subspaces of H and Q , respectively. Then, we define the spaces

$$H_{E,h} := \{v_h \in H_h : b(v_h, S(q)) = 0, \quad \forall q \in N(a)\} \quad (4.1)$$

$$Q_{E,h} := \{\lambda_h \in Q_h : b(q, \lambda_h) = 0, \quad \forall q \in N(a)\} \quad (4.2)$$

It follows that the discrete formulations associated to (P_1) , (P_2) and (P_3) are given by

FIND $(u_h, \lambda_h) \in H_h := H_h \times Q_h$ such that

$$\begin{aligned}
B((u_h, \lambda_h), (v_h, \mu_h)) &= F(v_h, \mu_h) \\
\forall (v_h, \mu_h) &\in H_h
\end{aligned} \quad (P_1^h)$$

FIND $(\bar{u}_h, \lambda_h, p_h) \in \tilde{H}_h := H_{E,h} \times Q_h \times N(a)$ such that

$$\begin{aligned} B((\bar{u}_h, \lambda_h), (\bar{v}_h, \mu_h)) + b(p_h, \mu_h) &= F(\bar{v}_h, \mu_h) \\ b(q_h, \lambda_h) &= 0 \\ \forall (\bar{v}_h, \mu_h, q_h) &\in \tilde{H}_h \end{aligned} \quad (P_2^h)$$

FIND $(\bar{u}_h, \bar{\lambda}_h) \in \hat{H}_h := H_{E,h} \times Q_{E,h}$ such that

$$\begin{aligned} B((\bar{u}_h, \bar{\lambda}_h), (\bar{v}_h, \bar{\mu}_h)) &= F(\bar{v}_h, \bar{\mu}_h) \\ \forall (\bar{v}_h, \bar{\mu}_h) &\in \hat{H}_h \end{aligned} \quad (P_3^h)$$

If we assume that $N(a) \subset H_h$ and that $S(N(a)) \subset Q_h$, then by following the same procedure described in the above section, we obtain that the discrete formulations (P_1^h) , (P_2^h) and (P_3^h) are completely equivalent. In particular, the result analogous to theorem 3.3. is stated as follows.

Theorem 4.1.

Suppose that $N(a) \subset H_h$ and that $S(N(a)) \subset Q_h$. Then, the problems (P_1^h) and (P_3^h) are equivalent, i.e.

- i) If (u_h, λ_h) is a solution of (P_1^h) , then $(u_h - p_{u_h}, \lambda_h)$ is a solution of (P_3^h) where p_{u_h} is the unique element of $N(a)$ such that

$$b(p_{u_h}, S(q)) = b(u_h, S(q)), \quad \forall q \in N(a)$$

Conversely, if $(\bar{u}_h, \bar{\lambda}_h)$ is a solution of (P_3^h) , then $(\bar{u}_h + p_g, \bar{\lambda}_h)$ is a solution of (P_1^h) , where p_g is the unique element of $N(a)$ that satisfies

$$b(p_g, S(q)) = g(S(q)), \quad \forall q \in N(a)$$

- ii) (P_1^h) has unique solution $\iff (P_3^h)$ has unique solution.

Proof : It is a corollary of theorem 3.3.

Remark 4.1.

As in section 3, it is not difficult to see that H_h and Q_h can be written as

$$H_h := H_{E,h} \oplus N(a) \quad \text{and} \quad Q_h := Q_{E,h} \oplus S(N(a)) \quad (4.3)$$

We now introduce the subspaces

$$N_h(b) := \{v_h \in H_h : b(v_h, \lambda_h) = 0, \quad \forall \lambda_h \in Q_h\} \quad (4.4)$$

$$N_h(b^*) := \{\lambda_h \in Q_h : b(v_h, \lambda_h) = 0, \forall v_h \in H_h\} \quad (4.5)$$

with similar definitions for $N_{E,h}(b)$ and $N_{E,h}(b^*)$.

Then, by using (4.3) and repeating the same arguments given in the proof of lemma 3.2, we can show that $N_h(b) = N_{E,h}(b)$ and $N_h(b^*) = N_{E,h}(b^*)$. So, the quotient spaces associated are given by

$$Z_h := Q_h / N_h(b^*)$$

and

$$Z_{E,h} := Q_{E,h} / N_{E,h}(b^*)$$

Consequently, the result analogous to theorem 3.5. is as follows

Theorem 4.2.

Suppose that there exist positive constants α_h, β_h such that

$$\sup_{\substack{v_h \in N_h(b) \\ v_h \neq 0}} \frac{|a(u_h, v_h)|}{\|v_h\|_H} > \alpha_h \|u_h\|_H, \quad \forall u_h \in N_h(b) \quad (4.6)$$

and

$$\sup_{\substack{v_h \in H_{E,h} \\ v_h \neq 0}} \frac{|b(v_h, \lambda_h)|}{\|v_h\|_H} > \beta_h \|[\lambda_h]\|_{Z_{E,h}}, \quad \forall \lambda_h \in Q_{E,h} \quad (4.7)$$

Then, given $F \in (H \times Q)^*$, there exists a unique solution $(u_h, [\lambda_h]) \in H_h \times Z_h$ of problem (P_1^h) . More precisely, if $(\bar{u}_h, [\bar{\lambda}_h]) \in H_{E,h} \times Z_{E,h}$ is the unique solution of (P_3^h) , then $(u_h, [\lambda_h]) := (\bar{u}_h + p_g, [\bar{\lambda}_h])$. Furthermore, under (4.6), the hypothesis (4.7) is a necessary and sufficient condition for (P_1^h) to be well posed on $H_h \times Z_h$.

Proof : It follows from theorem 3.5.

Remark 4.2.

The hypothesis (4.7) is referred to as the modified discrete Babuska-Brezzi condition.

Let us consider again the linear mapping P defined by (3.15). Then, it is easily seen that

$$Pv_h := \bar{v}_h = (v_h - p_{v_h}) \in H_{E,h}, \quad \forall v_h \in V_h$$

So, if we assume that (3.16) holds, then it is not difficult to prove that the modified discrete Babuska-Brezzi condition (4.7) is equivalent to

$$\sup_{\substack{v_h \in H_h \\ v_h \neq 0}} \frac{|b(v_h, \lambda_h)|}{\|v_h\|_H} \geq \beta_h \|[\lambda_h]\|_{Z_{E,h}}, \quad \forall \lambda_h \in Q_{E,h} \quad (4.8)$$

which is obviously simpler than (2.12). For instance, in the particular case in which $N_h(b^*) = \{0\}$, (2.12) and (4.8) are reduced to

$$\sup_{\substack{v_h \in H_h \\ v_h \neq 0}} \frac{|b(v_h, \lambda_h)|}{\|v_h\|_H} \geq \beta_h \|\lambda_h\|_Q, \quad \forall \lambda_h \in Q_h \quad (2.12)',$$

and

$$\sup_{\substack{v_h \in H_h \\ v_h \neq 0}} \frac{|b(v_h, \lambda_h)|}{\|v_h\|_H} \geq \beta_h \|\lambda_h\|_Q, \quad \forall \lambda_h \in Q_{E,h} \quad (4.8)',$$

In practice, this means that in the process of constructing the finite element spaces H_h, Q_h for solving (P_h^1) , we need not to check the discrete Babuska-Brezzi condition for all $\lambda_h \in Q_h$, but only for $\lambda_h \in Q_{E,h}$.

5. AN EXAMPLE

In order to illustrate the ideas, we consider here a variational formulation for which our approach is easily applied.

The following lemma which is a corollary of the generalized Poincaré's inequality (cf. [8] : Theorem 5.11.2) will be needed.

Lemma 5.1.

Let Ω be a bounded domain in R^n with Lipschitzian boundary Γ . Let $P_{m-1}(\Omega)$ be the set of polynomials defined on Ω of degree less than or equal to $m-1$. Denote by $H^m(\Omega)$ the Sobolev space of integer order m with norm $\|\cdot\|_{H^m(\Omega)}$ and corresponding semi-norm $|\cdot|_{H^m(\Omega)}$. Assume that there exists a finite family F of bounded linear functionals on $H^m(\Omega)$ with the property

$$v \in P_{m-1}(\Omega) \quad \text{and} \quad f(v) = 0, \quad \forall f \in F \implies v = 0 \quad (5.1)$$

Then, the map

$$||| \cdot ||| : H^m(\Omega) \rightarrow \mathbb{R}$$

$$||| v ||| := \{ |v|^2_{H^m(\Omega)} + \sum_{f \in F} |f(v)|^2 \}^{1/2} \quad (5.2)$$

is a norm which is equivalent to the usual norm $\|\cdot\|_{H^m(\Omega)}$. That is, there exist $C_1, C_2 > 0$ such that

$$C_1 ||| v ||| \leq \|v\|_{H^m(\Omega)} \leq C_2 ||| v |||, \quad \forall v \in H^m(\Omega) \quad (5.3)$$

Proof See [8] : Theorem 5.11.2.

Now, let Ω be a bounded domain in $\mathbb{R}^2$ with Lipschitzian boundary Γ . Let H, Q be two real Hilbert spaces with $H := H^1(\Omega)$. Let $b : H^1(\Omega) \times Q \rightarrow \mathbb{R}$ be a bounded bilinear form and let $a : H^1(\Omega) \times H^1(\Omega) \rightarrow \mathbb{R}$ be the bounded bilinear form defined by

$$a(u, v) := \int_{\Omega} \nabla u \cdot \nabla v \, dx, \quad \forall u, v \in H^1(\Omega) \quad (5.4)$$

Given two bounded linear functionals $f \in (H^1(\Omega))^*$ and $g \in Q^*$, we consider the abstract saddle-point problem : FIND $(u, \lambda_0) \in H^1(\Omega) \times Q$ such that

$$\begin{aligned} a(u, v) + b(v, \lambda_0) + b(u, \mu) &= f(v) + g(\mu) \\ \forall (v, \mu) &\in H^1(\Omega) \times Q \end{aligned} \quad (5.5)$$

It follows easily that the subspace $N(a)$ is given by the constant functions defined on Ω , i.e.

$$N(a) := \{v : \Omega \rightarrow \mathbb{R} / v \equiv \text{constant}\} \quad (5.6)$$

or equivalently, $N(a) := P_0(\Omega)$. It is clear that $N(a)$ is a non-trivial finite dimensional subspace of $H^1(\Omega)$. Further, we assume that there exists a bounded linear map $S : N(a) \rightarrow Q$ such that (3.3) is satisfied. In this case, this condition is equivalent to put

$$b(1, S(1)) \neq 0 \quad (5.7)$$

So, if p_0 is the unique element of $N(a)$ that satisfies (3.4) then it is easy to see that this constant is given by

$$p_0 = f(1)/b(1, S(1))$$

Hence, if $\lambda := \lambda_0 - p_0 S(1)$, the problem (5.5) is rewritten as follows : FIND $(u, \lambda) \in H^1(\Omega) \times Q$ such that

$$a(u, v) + b(v, \lambda) + b(u, \mu) = f(v) - p_0 b(v, S(1)) + g(\mu) \quad (5.8)$$

$$\forall (v, \mu) \in H^1(\Omega) \times Q$$

Let us now define the subspaces

$$H_E := \{v \in H^1(\Omega) : b(v, S(1)) = 0\} \quad (5.9)$$

$$Q_E := \{\lambda \in Q : b(1, \lambda) = 0\} \quad (5.10)$$

and

$$N(b) := \{v \in H^1(\Omega) : b(v, \lambda) = 0, \quad \forall \lambda \in Q\}$$

$$N(b^*) := \{\lambda \in Q : b(v, \lambda) = 0, \quad \forall v \in H^1(\Omega)\}$$

We also consider the following map on $H^1(\Omega)$

$$\| \cdot \| : H^1(\Omega) \rightarrow \mathbb{R}^+$$

$$\| \| v \| \| := \{ |v|_{H^1(\Omega)}^2 + |f_1(v)|^2 \}^{1/2} \quad (5.11)$$

where $f_1 \in (H^1(\Omega))^*$ is defined by

$$f_1(v) := b(v, S(1)), \quad \forall v \in H^1(\Omega)$$

It follows from (5.7) that f_1 satisfies the property (5.1) of lemma 5.1. Consequently, $\| \cdot \|$ defines a norm on $H^1(\Omega)$ which is equivalent to the usual norm $\| \cdot \|_{H^1(\Omega)}$, i.e. there exist $C_1, C_2 > 0$ such that

$$C_1 \| \| v \| \| \leq \| v \|_{H^1(\Omega)} \leq C_2 \| \| v \| \|, \quad \forall v \in H^1(\Omega) \quad (5.12)$$

Thus, from (5.9), (5.11) and (5.12) we deduce that

$$\| \bar{v} \|_{H^1(\Omega)} \leq C_2 | \bar{v} |_{H^1(\Omega)}, \quad \forall \bar{v} \in H_E$$

Hence, if P is the projection-type mapping defined in (3.15) we obtain that for any $v \in H^1(\Omega)$

$$\| P_v \|_{H^1(\Omega)} := \| \bar{v} \|_{H^1(\Omega)} \leq C_2 | \bar{v} |_{H^1(\Omega)} = C_2 | v |_{H^1(\Omega)} \leq C_2 \| v \|_{H^1(\Omega)}$$

which shows that P satisfies (3.16).

Therefore, the modified Babuska-Brezzi condition (3.17) is given by

$$\sup_{\substack{v \in H^1(\Omega) \\ v \neq 0}} \frac{|b(v, \lambda)|}{\|v\|_{H^1(\Omega)}} \geq \beta \|\lambda\|_{Z_E}, \quad \forall \lambda \in Q_E \quad (5.13)$$

where, as usual, $Z_E := Q_E / N(b^*)$.

In the same way, if we consider the discrete problem associated to (5.8), then the modified discrete Babuska-Brezzi condition (4.8) is given by

$$\sup_{\substack{v_h \in H_h \\ v_h \neq 0}} \frac{|b(v_h, \lambda_h)|}{\|v_h\|_{H^1(\Omega)}} \geq \beta_h \|\lambda_h\|_{Z_{E,h}}, \quad \forall \lambda_h \in Q_{E,h} \quad (5.14)$$

where

$$Q_{E,h} := \{\lambda_h \in Q_h : b(1, \lambda_h) = 0\}$$

$$Z_{E,h} := Q_{E,h} / N_h(b^*)$$

A particular case of the saddle-point formulation (5.5) is obtained when we consider the nonhomogeneous Dirichlet problem

$$\begin{aligned} -\Delta u &= f \in L^2(\Omega), & \text{in } \Omega \\ u &= g \in H^{1/2}(\Gamma), & \text{on } \Gamma \end{aligned} \quad (5.15)$$

In fact, by applying the Green's identity and introducing a Lagrange multiplier

$$\lambda := -\frac{\partial u}{\partial n} \in H^{-1/2}(\Gamma)$$

we arrive to the following weak formulation for (5.15) : FIND $(u, \lambda) \in H^1(\Omega) \times H^{-1/2}(\Gamma)$ such that

$$\begin{aligned} \int_{\Omega} \nabla u \cdot \nabla v \, dx + \langle \gamma v, \lambda \rangle + \langle \gamma u, \mu \rangle &= \int_{\Omega} f v \, dx + \langle g, \mu \rangle \\ \forall (v, \mu) &\in H^1(\Omega) \times H^{-1/2}(\Gamma) \end{aligned} \quad (5.16)$$

Here, γ stands for the trace operator from $H^1(\Omega)$ to $H^{1/2}(\Gamma)$, and $\langle \cdot, \cdot \rangle$ denotes the dual pairing of $H^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$ with respect to the L^2 -scalar product.

So, we have $Q := H^{-1/2}(\Gamma)$

$$b : H^1(\Omega) \times H^{-1/2}(\Gamma) \rightarrow \mathbb{R}$$

$$b(v, \lambda) := \langle \gamma v, \lambda \rangle$$

and the definitions of the functionals f and g are obvious.

In order to satisfy (5.7) it suffices to take

$$S := \gamma : N(a) + H^{1/2}(\Gamma) \subset H^{-1/2}(\Gamma)$$

Then, we obtain

$$b(1, S(1)) = \langle 1, 1 \rangle = \int_{\Gamma} ds = |\Gamma|$$

which is clearly non zero.

On the other hand, the subspaces H_E , Q_E are given by

$$H_E := \{v \in H^1(\Omega) : \langle \gamma v, 1 \rangle = 0\}$$

$$Q_E := \{\lambda \in H^{-1/2}(\Gamma) : \langle 1, \lambda \rangle = 0\}$$

and it is easily seen that

$$N(b) := \{v \in H^1(\Omega) : \langle \gamma v, \lambda \rangle = 0, \forall \lambda \in H^{-1/2}(\Gamma)\} = H_0^1(\Omega)$$

$$N(b^*) := \{\lambda \in H^{-1/2}(\Gamma) : \langle \gamma v, \lambda \rangle = 0, \forall v \in H^1(\Omega)\} = \{0\}$$

Hence, the modified Babuska-Brezzi condition (5.13) is trivially satisfied, and the corresponding discrete condition (5.14) reduces to

$$\sup_{\substack{v_h \in H_h \\ v_h \neq 0}} \frac{|\langle \gamma v_h, \lambda_h \rangle|}{\|v_h\|_{H^1(\Omega)}} \geq \beta_h \|\lambda_h\|_{H^{-1/2}(\Gamma)}, \quad \forall \lambda_h \in Q_{E,h}$$

where

$$Q_{E,h} := \{\lambda_h \in Q_h : \langle 1, \lambda_h \rangle = 0\}$$

Remark 5.1.

We should emphasize that the finite dimensional subspaces H_h and Q_h must contain $N(a)$ and $S(N(a))$, respectively. For instance, in the particular case given by (5.16), the constant functions need to be included in both, H_h and Q_h .

Remark 5.2.

A very similar analysis to the one described above is obtained when we substitute (5.4) by the following bounded bilinear form

$$a : H^2(\Omega) \times H^2(\Omega) \rightarrow \mathbb{R}$$

$$a(u,v) := \sigma \int_{\Omega} \Delta u \Delta v \, dx + (1 - \sigma) \sum_{|\alpha|=2} \int_{\Omega} D^{\alpha} u D^{\alpha} v \, dx, \quad \forall u, v \in H^2(\Omega)$$

which arises from the weak formulation of the biharmonic equation. Here, σ is a constant such that $0 \leq \sigma < 1$. In this situation, $N(a)$ is given by $P_1(\Omega)$, i.e. the set of polynomials of degree ≤ 1 defined on Ω . So, instead of (5.11) we consider

$$\|\cdot\| : H^2(\Omega) \rightarrow \mathbb{R}^+$$

$$\|v\| := \left\{ |v|_{H^2(\Omega)}^2 + \sum_{i=1}^3 |f_i(v)|^2 \right\}^{1/2}$$

where $f_i \in (H^2(\Omega))^*$, $i = 1, 2, 3$ are defined by

$$f_1(v) := b(v, S(1))$$

$$f_2(v) := b(v, S(x))$$

$$f_3(v) := b(v, S(y))$$

Thus, lemma 5.1. and condition (3.3) will provide again the boundness of the projection-type mapping P .

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