DOI: 10.4067/S0716-09172017000400753

Proyecciones Journal of Mathematics
Vol. 36, N^o 4, pp. 753-767, December 2017.
Universidad Católica del Norte
Antofagasta - Chile

Ostrowski type fractional integral inequalities for s-Godunova-Levin functions via k-fractional integrals

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Received : April 2017. Accepted : July 2017

Abstract

In this paper, we give some fractional integral inequalities of Ostrowski type for s-Godunova-Levin functions via Riemann-Liouville $k-$fractional integrals. We deduce some known Ostrowski type fractional integral inequalities for Riemann-Liouville fractional integrals and we also prove results for $p-$functions and Godunova-Levin functions by taking $s = 0$ ans $s = 1$ respectively.

Subjclass [2010]: *26A33, 26A51, 26D07, 26D10, 26D15.*

Keywords: *Ostrowski inequality, Riemann-Liouville fractional integrals, s-Godunova-Levin functions.*

1. Introduction

In 1938 Ostrowski [1] proved an inequality stated in the following result (see also [8, p.468]). Ostrowski inequality gives bounds of integral average of a function f over an interval $[a, b]$ to its value $f(x)$ at point $x \in [a, b]$.

Theorem 1.1. *Let $f : I \to \mathbf{R}$ where I is interval in $\mathbf{R}$, be a mapping differentiable in I° the interior of I and $a, b \in I^\circ$, $a < b$. If $\left|f'(t)\right| \leq M$, for all $t \in [a, b]$, then we have*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq \left[\frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2}\right](b-a)M, x \in [a, b].$$

Ostrowski and Ostrowski type inequalities have great importance in numerical analysis as they provide the error bounds of many quaderature rules. In recent years, such inequalities have been extended and generalized in various aspects (see [3, 4, 5, 6, 7] and references therein).
Fractional integration is the generalization of classical integration. In this research article we use Riemann-Liouville $k-$fractional integrals to generalize the Ostrowski type inequalities via Riemann-Liouville fractional integrals given in [7]. In the following we give some important definitions.

Definition 1.2. *[11] A non-negative function $f : I \to \mathbf{R}$ is said to be p-function, if for any two points $x, y \in I$ and $t \in [0, 1]$*

$$f(tx + (1-t)y) \leq f(x) + f(y).$$

Definition 1.3. *[12] A function $f : I \to \mathbf{R}$ is said to be Godunova-Levin function, if for any two points $x, y \in I$ and $t \in (0, 1)$*

$$f(tx + (1-t)y) \leq \frac{f(x)}{t} + \frac{f(y)}{1-t}.$$

Definition 1.4. *[13] A function $f : I \to \mathbf{R}$ is said to be s-Godunova-Levin function of first kind, if $s \in (0, 1]$ and $t \in (0, 1)$, then for all $x, y \in I$ we have*

$$f(tx + (1-t)y) \leq \frac{f(x)}{t^s} + \frac{f(y)}{1-t^s}.$$

Definition 1.5. *[14] A function $f : I \to \mathbf{R}$ is said to be s-Godunova-Levin function of second kind, if $s \in [0, 1]$ and $t \in (0, 1)$, then for all $x, y \in I$ we have*

$$f(tx + (1-t)y) \leq \frac{f(x)}{t^s} + \frac{f(y)}{(1-t)^s}.$$

Definition 1.6. *[10] Let $f \in L_1[a,b]$. Then Riemann-Liouville fractional integrals $J^{\alpha}_{a+}f$ and $J^{\alpha}_{b-}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, x > a$$

and

$$J^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, x < b,$$

respectively, where $\Gamma(\alpha) = \int_0^\infty e^{-u}u^{\alpha-1}du$. Here $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$, $J^0_{a+}f(x) = J^0_{b-}f(x) = f(x)$. In case of $\alpha = 1$, the fractional integral reduces to the classical integral.

Definition 1.7. *[9] Let $f \in L_1[a,b]$. Then k-fractional integrals of order $\alpha, k > 0$ with $a \geq 0$ are defined as*

$$I^{\alpha,k}_{a+}f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t)\, dt, x \geq a$$

and

$$I^{\alpha,k}_{b-}f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_x^b (t-x)^{\frac{\alpha}{k}-1} f(t)\, dt, x \leq b.$$

where $\Gamma_k(\alpha)$ is the k-gamma function. For $k = 1$, k-fractional integrals give Riemann-Liouville fractional integrals.

In the following we give definitions of k-gamma and k-beta functions as well as their relationship.

Definition 1.8. *[2] For $k \in \mathbf{R}^+$ and $x \in \mathbf{C}$, the k-gamma function is defined by*

$$\Gamma_k(x) = \lim_{n\to\infty} \frac{n!k^n nk^{\frac{x}{k}-1}}{(x)_{n,k}}.$$

Its integral representation is given by

$$\Gamma_k(\alpha) = \int_0^\infty t^{\alpha-1} e^{-\frac{t^k}{k}} dt. \tag{1.1}$$

One can note that

$$\Gamma_k(\alpha+k) = \alpha\Gamma_k(\alpha).$$

For $k = 1$,(1.1) gives integral representation of gamma function.

Definition 1.9. *[2] For $k \in \mathbf{R}^+$ and $x \in \mathbf{C}$, the k-beta function with two parameters x and y is defined as*

$$\beta_k(x,y) = \frac{1}{k}\int_0^1 t^{\frac{x}{k}-1}(1-t)^{\frac{y}{k}-1}dt. \tag{1.2}$$

For $k=1$, (1.2) gives integral representation of beta function.

Theorem 1.10. *[2] For $x, y > 0$, the following equality in $k-$gamma and $k-$beta functions holds*

$$\beta_k(x,y) = \frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)}. \tag{1.3}$$

We organize the paper in such a way that first we prove Ostrowski type fractional integral inequalities for s-Godunova-Levin functions of second kind via Riemann-Liouville $k-$fractional integrals. Then we obtain results for p-functions and Godunova-Levin functions, we also deduce some known results of [7].

2. Ostrowski type fractional integral inequalities for mappings whose derivatives are s-Godunova-Levin of second kind via Riemann-Liouville $k-$fractional integrals

The following lemma is very useful to obtain our results.

Lemma 2.1. *Let $f : [a,b] \to \mathbf{R}$ be a differentiable mapping on (a,b) with $a < b$ such that $f' \in L_1[a,b]$. Then for all $x \in [a,b]$ one has*

$$\begin{aligned}&\tfrac{(x-a)^{\frac{\alpha}{k}}+(b-x)^{\frac{\alpha}{k}}}{b-a}f(x) - \frac{\Gamma_k(\alpha+k)}{b-a}\left[J^{\alpha,k}_{x^-}f(a) + J^{\alpha,k}_{x^+}f(b)\right]\\ &= \tfrac{(x-a)^{\frac{\alpha}{k}+1}}{b-a}\int_0^1 t^{\frac{\alpha}{k}} f'(tx+(1-t)a)dt\\ &-\tfrac{(b-x)^{\frac{\alpha}{k}+1}}{b-a}\int_0^1 t^{\frac{\alpha}{k}} f'(tx+(1-t)b)dt.\end{aligned}$$

(2.1)

with $\alpha, k > 0$.

Theorem 2.2. *Let $f : I \to \mathbf{R}$ be a differentiable mapping on (a,b) with $a < b$ such that $f' \in L_1[a,b]$. If $\left|f'\right|$ is s-Godunova-Levin function of second kind on $[a,b]$ and $\left|f'(x)\right| \leq M$, $x \in [a,b]$, then the following inequality for Riemann-Liouville k-fractional integrals holds*

$$\left|\frac{(x-a)^{\frac{\alpha}{k}}+(b-x)^{\frac{\alpha}{k}}}{b-a}f(x)-\frac{\Gamma_k(\alpha+k)}{b-a}\left[J_{x^-}^{\alpha,k}f(a)+J_{x^+}^{\alpha,k}f(b)\right]\right|$$
$$\leq M\left[\frac{(x-a)^{\frac{\alpha}{k}+1}+(b-x)^{\frac{\alpha}{k}+1}}{b-a}\right]\left[\frac{1}{\frac{\alpha}{k}+1-s}+\frac{\Gamma_k[\alpha+k]\;\Gamma_k(k-sk)}{\Gamma_k(\alpha+2k-sk)}\right],$$

(2.2)

where $\alpha, k>0$.

Proof. *Using Lemma 2.1 and the fact that $\left|f'\right|$ is s-Godunova-Levin function of second kind, we have*

$$\left|\frac{(x-a)^{\frac{\alpha}{k}}+(b-x)^{\frac{\alpha}{k}}}{b-a}f(x)-\frac{\Gamma_k(\alpha+k)}{b-a}\left[J_{x^-}^{\alpha,k}f(a)+J_{x^+}^{\alpha,k}f(b)\right]\right|$$
$$\leq \frac{(x-a)^{\frac{\alpha}{k}+1}}{b-a}\int_0^1 t^{\frac{\alpha}{k}}\left|f'(tx+(1-t)a)\right|dt$$
$$+\frac{(b-x)^{\frac{\alpha}{k}+1}}{b-a}\int_0^1 t^{\frac{\alpha}{k}}\left|f'(tx+(1-t)b)\right|dt$$
$$\leq \frac{(x-a)^{\frac{\alpha}{k}+1}}{b-a}\int_0^1\left[\frac{t^{\frac{\alpha}{k}}}{t^s}\left|f'(x)\right|+\frac{t^{\frac{\alpha}{k}}}{(1-t)^s}\left|f'(a)\right|\right]dt$$
$$+\frac{(b-x)^{\frac{\alpha}{k}+1}}{b-a}\int_0^1\left[\frac{t^{\frac{\alpha}{k}}}{t^s}\left|f'(x)\right|+\frac{t^{\frac{\alpha}{k}}}{(1-t)^s}\left|f'(b)\right|\right]dt$$
$$\leq \frac{M(x-a)^{\frac{\alpha}{k}+1}}{b-a}\int_0^1\left[\frac{t^{\frac{\alpha}{k}}}{t^s}+\frac{t^{\frac{\alpha}{k}}}{(1-t)^s}\right]dt$$
$$+\frac{M(b-x)^{\frac{\alpha}{k}+1}}{b-a}\int_0^1\left[\frac{t^{\frac{\alpha}{k}}}{t^s}+\frac{t^{\frac{\alpha}{k}}}{(1-t)^s}\right]dt$$
$$=M\left[\frac{(x-a)^{\frac{\alpha}{k}+1}+(b-x)^{\frac{\alpha}{k}+1}}{b-a}\right]\int_0^1\left[t^{\frac{\alpha}{k}-s}+t^{\frac{\alpha}{k}}(1-t)^{-s}\right]dt$$
$$=M\left[\frac{(x-a)^{\frac{\alpha}{k}+1}+(b-x)^{\frac{\alpha}{k}+1}}{b-a}\right]\left[\frac{1}{\frac{\alpha}{k}+1-s}+\frac{\Gamma_k(\alpha+k)\;\Gamma_k(k-sk)}{\Gamma_k(\alpha+2k-sk)}\right].$$

The proof is completed. □

Remark 2.3. *(i) If we put $k=1$ in (2.2) we get [7, Theorem 3.1].*
(ii) If we put $k=1$ and $\alpha=1$ in (2.2) we get [7, Corollary 3.1].

Corollary 2.4. *In Theorem 2.2, if we take $s=0$, which means that $\left|f'\right|$ is p-function, then (2.2) becomes the following inequality*

$$\left|\frac{(x-a)^{\frac{\alpha}{k}}+(b-x)^{\frac{\alpha}{k}}}{b-a}f(x)-\frac{\Gamma_k(\alpha+k)}{b-a}\left[J_{x^-}^{\alpha,k}f(a)+J_{x^+}^{\alpha,k}f(b)\right]\right|$$
$$\leq M\left[\frac{(x-a)^{\frac{\alpha}{k}+1}+(b-x)^{\frac{\alpha}{k}+1}}{b-a}\right]\left[\frac{1}{\frac{\alpha}{k}+1}+\frac{k\Gamma_k[k]}{\alpha+k}\right],$$

where $\alpha, k>0$.

Corollary 2.5. *In Theorem 2.2, if we take $s = 1$, which means that $\left|f'\right|$ is Godunova-Levin function, then (2.2) becomes the following inequality*

$$\left|\frac{(x-a)^{\frac{\alpha}{k}}+(b-x)^{\frac{\alpha}{k}}}{b-a}f(x) - \frac{\Gamma_k(\alpha+k)}{b-a}\left[J^{\alpha,k}_{x^-}f(a) + J^{\alpha,k}_{x^+}f(b)\right]\right|$$
$$\leq M\left[\frac{(x-a)^{\frac{\alpha}{k}+1}+(b-x)^{\frac{\alpha}{k}+1}}{b-a}\right]\left[\frac{k(\alpha+1)}{\alpha}\right],$$

where $\alpha, k > 0$.

Theorem 2.6. *Let $f : I \to \mathbf{R}$ be a differentiable mapping on (a, b) with $a < b$ such that $f' \in L_1[a, b]$. If $\left|f'\right|^q, q > 1$, is s-Godunova-Levin function of second kind on $[a, b]$ and $\left|f'(x)\right| \leq M$, $x \in [a, b]$, then the following inequality for Riemann-Liouville k-fractional integrals holds*

$$\left|\frac{(x-a)^{\frac{\alpha}{k}}+(b-x)^{\frac{\alpha}{k}}}{b-a}f(x) - \frac{\Gamma_k(\alpha+k)}{b-a}\left[J^{\alpha,k}_{x^-}f(a) + J^{\alpha,k}_{x^+}f(b)\right]\right|$$
$$\leq M\left[\frac{(x-a)^{\frac{\alpha}{k}+1}+(b-x)^{\frac{\alpha}{k}+1}}{(b-a)(1+p\frac{\alpha}{k})^{\frac{1}{p}}}\right]\left[\frac{2}{1-s}\right]^{\frac{1}{q}}$$

(2.3)

with $\alpha, k > 0$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. *Using Lemma 2.1 and then Holder's inequality, we have*

$$\left|\frac{(x-a)^{\frac{\alpha}{k}}+(b-x)^{\frac{\alpha}{k}}}{b-a}f(x) - \frac{\Gamma_k(\alpha+k)}{b-a}\left[J^{\alpha,k}_{x^-}f(a) + J^{\alpha,k}_{x^+}f(b)\right]\right|$$
$$\leq \frac{(x-a)^{\frac{\alpha}{k}+1}}{b-a}\int_0^1 t^{\frac{\alpha}{k}}\left|f'(tx+(1-t)a)\right|dt$$
$$+\frac{(b-x)^{\frac{\alpha}{k}+1}}{b-a}\int_0^1 t^{\frac{\alpha}{k}}\left|f'(tx+(1-t)b)\right|dt$$
$$\leq \frac{(x-a)^{\frac{\alpha}{k}+1}}{b-a}\left(\int_0^1 t^{p\frac{\alpha}{k}}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'(tx+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}$$
$$+\frac{(b-x)^{\frac{\alpha}{k}+1}}{b-a}\left(\int_0^1 t^{p\frac{\alpha}{k}}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'(tx+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}.$$

(2.4)

Since $\left|f'\right|^q$ is s-Godunova-Levin function of second kind and $\left|f'(x)\right| \leq M$, therefore we have

$$\int_0^1\left|f'(tx+(1-t)a)\right|^q dt \leq \int_0^1\left[\frac{1}{t^s}\left|f'(x)\right|^q + \frac{1}{(1-t)^s}\left|f'(a)\right|^q\right]dt$$
$$\leq M^q\int_0^1\left[\frac{1}{t^s}+\frac{1}{(1-t)^s}\right]dt = \frac{2M^q}{1-s}$$

(2.5)

similarly

$$\int_0^1 \left| f'(tx+(1-t)b) \right|^q dt \le \frac{2M^q}{1-s}. \tag{2.6}$$

We also have

$$\int_0^1 t^{p\frac{\alpha}{k}} dt = \frac{1}{1+p\frac{\alpha}{k}}. \tag{2.7}$$

Using (2.5), (2.6) and (2.7) in (2.4) we can get (2.3). $\square$

Remark 2.7. *(i) If we put $k=1$ in (2.3) we get [7, Theorem 3.2].*
(ii) If we put $k=1$ and $\alpha=1$ in (2.3) we get [7, Corollary 3.2].

Corollary 2.8. *In Theorem 2.6, if we take $s=0$, which means that $\left|f'\right|$ is p-function, then (2.3) becomes the following inequality*

$$\left| \frac{(x-a)^{\frac{\alpha}{k}}+(b-x)^{\frac{\alpha}{k}}}{b-a} f(x) - \frac{\Gamma_k(\alpha+k)}{b-a} \left[J^{\alpha,k}_{x^-} f(a) + J^{\alpha,k}_{x^+} f(b) \right] \right| \le \frac{M(2)^{\frac{1}{q}}}{(1+p\frac{\alpha}{k})^{\frac{1}{p}}} \left[\frac{(x-a)^{\frac{\alpha}{k}+1}+(b-x)^{\frac{\alpha}{k}+1}}{b-a} \right],$$

where $\alpha, k > 0$.

Theorem 2.9. *Let $f : I \to \mathbf{R}$ be a differentiable mapping on I° such that $f' \in L_1[a,b]$, where $a,b \in I$, with $a < b$. If $\left|f'\right|^q, q > 1$ is s-Godunova-Levin function of second kind on $[a,b]$ and $\left|f'(x)\right| \le M$, $x \in [a,b]$, then the following inequality for Riemann-Liouville k-fractional integrals holds*

$$\left| \frac{(x-a)^{\frac{\alpha}{k}}+(b-x)^{\frac{\alpha}{k}}}{b-a} f(x) - \frac{\Gamma_k(\alpha+k)}{b-a} \left[J^{\alpha,k}_{x^-} f(a) + J^{\alpha,k}_{x^+} f(b) \right] \right| \le \frac{M}{\left[1+\frac{\alpha}{k}\right]^{1-\frac{1}{q}}} \left[\frac{(x-a)^{\frac{\alpha}{k}+1}+(b-x)^{\frac{\alpha}{k}+1}}{b-a} \right] \left[\frac{1}{\frac{\alpha}{k}-s+1} + \frac{\Gamma_k[\alpha+k]\ \Gamma_k(k-sk)}{\Gamma_k(\alpha+2k-sk)} \right]^{\frac{1}{q}}, \tag{2.8}$$

with $\alpha, k > 0$.

Proof. Using Lemma 2.1 and power mean inequality, we have

$$\begin{aligned}
&\left|\frac{(x-a)^{\frac{\alpha}{k}}+(b-x)^{\frac{\alpha}{k}}}{b-a}f(x)-\frac{\Gamma_k(\alpha+k)}{b-a}\left[J_{x^-}^{\alpha,k}f(a)+J_{x^+}^{\alpha,k}f(b)\right]\right|\\
&\leq \frac{(x-a)^{\frac{\alpha}{k}+1}}{b-a}\int_0^1 t^{\frac{\alpha}{k}}\left|f'(tx+(1-t)a)\right|dt\\
&+\frac{(b-x)^{\frac{\alpha}{k}+1}}{b-a}\int_0^1 t^{\frac{\alpha}{k}}\left|f'(tx+(1-t)b)\right|dt\\
&\leq \frac{(x-a)^{\frac{\alpha}{k}+1}}{b-a}\left(\int_0^1 t^{\frac{\alpha}{k}}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{\frac{\alpha}{k}}\left|f'(tx+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}\\
&+\frac{(b-x)^{\frac{\alpha}{k}+1}}{b-a}\left(\int_0^1 t^{\frac{\alpha}{k}}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{\frac{\alpha}{k}}\left|f'(tx+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}.
\end{aligned}$$

(2.9)

Since $\left|f'\right|^q$ is s-Godunova-Levin function of second kind and $\left|f'(x)\right| \leq M$, therefore we have

$$\begin{aligned}
&\int_0^1 t^{\frac{\alpha}{k}}\left|f'(tx+(1-t)a)\right|^q dt\\
&\leq \int_0^1\left[\frac{t^{\frac{\alpha}{k}}}{t^s}\left|f'(x)\right|^q+\frac{t^{\frac{\alpha}{k}}}{(1-t)^s}\left|f'(a)\right|^q\right]dt\\
&\leq M^q\int_0^1\left[\frac{t^{\frac{\alpha}{k}}}{t^s}+\frac{t^{\frac{\alpha}{k}}}{(1-t)^s}\right]dt\\
&= M^q\left[\frac{1}{\frac{\alpha}{k}-s+1}+\frac{\Gamma_k[\alpha+k]\;\Gamma_k(k-sk)}{\Gamma_k(\alpha+2k-sk)}\right]
\end{aligned}$$

(2.10)

similarly

$$\int_0^1 t^{\frac{\alpha}{k}}\left|f'(tx+(1-t)b)\right|^q dt \leq M^q\left[\frac{1}{\frac{\alpha}{k}-s+1}+\frac{\Gamma_k[\alpha+k]\;\Gamma_k(k-sk)}{\Gamma_k(\alpha+2k-sk)}\right].$$

(2.11)

Using (2.10) and (2.11) in (2.9) one can attain (2.8). □

Remark 2.10. *(i) If we put $k=1$ in (2.8), then we get [7, Theorem 3.3]. (ii) If we put $k=1$ and $\alpha=1$ in (2.8), then we get [7, Corollary 3.3].*

Corollary 2.11. *In Theorem 2.9, if we take $s=0$, which means that $\left|f'\right|$ is p-function, then (2.8) becomes the following inequality*

$$\left|\tfrac{(x-a)^{\frac{\alpha}{k}}+(b-x)^{\frac{\alpha}{k}}}{b-a}f(x)-\frac{\Gamma_k(\alpha+k)}{b-a}\left[J_{x^-}^{\alpha,k}f(a)+J_{x^+}^{\alpha,k}f(b)\right]\right|$$
$$\leq \frac{M}{\left[1+\frac{\alpha}{k}\right]^{1-\frac{1}{q}}}\left[\tfrac{(x-a)^{\frac{\alpha}{k}+1}+(b-x)^{\frac{\alpha}{k}+1}}{b-a}\right]\left[\frac{1}{\frac{\alpha}{k}+1}+\frac{k\Gamma_k[k]}{\alpha+k}\right]^{\frac{1}{q}},$$

where $\alpha,k>0$.

Corollary 2.12. *In Theorem* 2.9*, if we take* $s=1$*, which means that* $\left|f^{'}\right|$ *is Godunova-Levin function, then (2.8) becomes the following inequality*

$$\left|\tfrac{(x-a)^{\frac{\alpha}{k}}+(b-x)^{\frac{\alpha}{k}}}{b-a}f(x)-\frac{\Gamma_k(\alpha+k)}{b-a}\left[J_{x^-}^{\alpha,k}f(a)+J_{x^+}^{\alpha,k}f(b)\right]\right|$$
$$\leq M\left[\tfrac{(x-a)^{\frac{\alpha}{k}+1}+(b-x)^{\frac{\alpha}{k}+1}}{b-a}\right]\left[\frac{1}{\frac{\alpha}{k}+1}\right]^{1-\frac{1}{q}}\left[\frac{k(\alpha+1)}{\alpha}\right]^{\frac{1}{q}},$$

where $\alpha,k>0$.

Now we give the following lemma for more results.

Lemma 2.13. *Let* $f:[a,b]\to\mathbf{R}$ *be a differentiable mapping on* (a,b) *with* $a<b$ *such that* $f^{'}\in L_1[a,b]$*. Then we have the following equality*

$$f(x)-\Gamma_k(\alpha+k)\left[\frac{1}{2(x-a)^{\frac{\alpha}{k}}}I_{x^-}^{\alpha,k}f(a)+\frac{1}{2(b-x)^{\frac{\alpha}{k}}}I_{x^+}^{\alpha,k}f(b)\right]$$
$$=\frac{x-a}{2}\int_0^1 t^{\frac{\alpha}{k}}f^{'}(tx+(1-t)a)dt-\frac{b-x}{2}\int_0^1 t^{\frac{\alpha}{k}}f^{'}(tx+(1-t)b)dt; x\in[a,b],$$

(2.12)

with $\alpha,k>0$.

Theorem 2.14. *Let* $f:I\to\mathbf{R}$ *be a differentiable mapping on* (a,b) *with* $a<b$ *such that* $f^{'}\in L_1[a,b]$*. If* $\left|f^{'}\right|$ *is s-Godunova-Levin function of second kind on* $[a,b]$ *and* $\left|f^{'}(x)\right|\leq M$*,* $x\in[a,b]$*, then the following inequality for Riemann-Liouville k-fractional integrals holds*

$$\left|f(x)-\Gamma_k(\alpha+k)\left[\frac{1}{2(x-a)^{\frac{\alpha}{k}}}I_{x^-}^{\alpha,k}f(a)+\frac{1}{2(b-x)^{\frac{\alpha}{k}}}I_{x^+}^{\alpha,k}f(b)\right]\right|$$
$$\leq\tfrac{M(b-a)}{2}\left[\frac{1}{\frac{\alpha}{k}+1-s}+\frac{\Gamma_k[\alpha+k]\,\Gamma_k(k-sk)}{\Gamma_k(\alpha+2k-sk)}\right],$$

(2.13)

where $\alpha,k>0$.

Proof. *Using Lemma 2.13 and s-Godunova-Levin function of second kind of $\left|f'\right|$ we proceed as follows*

$$\left|f(x)-\Gamma_k(\alpha+k)\left[\frac{1}{2(x-a)^{\frac{\alpha}{k}}}I_{x^-}^{\alpha,k}f(a)+\frac{1}{2(b-x)^{\frac{\alpha}{k}}}I_{x^+}^{\alpha,k}f(b)\right]\right|$$
$$\leq \tfrac{(x-a)}{2}\int_0^1 t^{\frac{\alpha}{k}}\left|f'(tx+(1-t)a)\right|dt$$
$$+\tfrac{(b-x)}{2}\int_0^1 t^{\frac{\alpha}{k}}\left|f'(tx+(1-t)b)\right|dt$$
$$\leq \tfrac{(x-a)}{2}\int_0^1\left[\tfrac{t^{\frac{\alpha}{k}}}{t^s}\left|f'(x)\right|+\tfrac{t^{\frac{\alpha}{k}}}{(1-t)^s}\left|f'(a)\right|\right]dt$$
$$+\tfrac{(b-x)}{2}\int_0^1\left[\tfrac{t^{\frac{\alpha}{k}}}{t^s}\left|f'(x)\right|+\tfrac{t^{\frac{\alpha}{k}}}{(1-t)^s}\left|f'(b)\right|\right]dt$$
$$\leq \tfrac{M(x-a)}{2}\int_0^1\left[\tfrac{t^{\frac{\alpha}{k}}}{t^s}+\tfrac{t^{\frac{\alpha}{k}}}{(1-t)^s}\right]dt$$
$$+\tfrac{M(b-x)}{2}\int_0^1\left[\tfrac{t^{\frac{\alpha}{k}}}{t^s}+\tfrac{t^{\frac{\alpha}{k}}}{(1-t)^s}\right]dt$$
$$=M\left[\tfrac{(x-a)+(b-x)}{2}\right]\int_0^1\left[t^{\frac{\alpha}{k}-s}+t^{\frac{\alpha}{k}}(1-t)^{-s}\right]dt$$
$$=\tfrac{M(b-a)}{2}\left[\frac{1}{\frac{\alpha}{k}+1-s}+\frac{\Gamma_k[\alpha+k]\,\Gamma_k(k-sk)}{\Gamma_k(\alpha+2k-sk)}\right].$$

The proof is completed. □

Remark 2.15. *If we put $k=1$ in (2.13), then we get the result involving Riemann-Liouville fractional integrals.*

Corollary 2.16. *In Theorem 2.14, if we take $s=0$, which means that $\left|f'\right|$ is p-function, then (2.13) becomes the following inequality*

$$\left|f(x)-\Gamma_k(\alpha+k)\left[\frac{1}{2(x-a)^{\frac{\alpha}{k}}}I_{x^-}^{\alpha,k}f(a)+\frac{1}{2(b-x)^{\frac{\alpha}{k}}}I_{x^+}^{\alpha,k}f(b)\right]\right|$$
$$\leq \frac{M(b-a)}{2}\left[\frac{1}{\frac{\alpha}{k}+1}+\frac{k\Gamma_k[k]}{\alpha+k}\right],$$

where $\alpha, k>0$.

Corollary 2.17. *In Theorem 2.14, if we take $s=1$, which means that $\left|f'\right|$ is Godunova-Levin function, then (2.13) becomes the following inequality*

$$\left|f(x)-\Gamma_k(\alpha+k)\left[\frac{1}{2(x-a)^{\frac{\alpha}{k}}}I_{x^-}^{\alpha,k}f(a)+\frac{1}{2(b-x)^{\frac{\alpha}{k}}}I_{x^+}^{\alpha,k}f(b)\right]\right|$$
$$\leq \frac{M(b-a)}{2}\left[\frac{k(\alpha+1)}{\alpha}\right],$$

where $\alpha, k>0$.

Theorem 2.18. *Let $f : I \to \mathbf{R}$ be a differentiable mapping on (a,b) with $a < b$ such that $f' \in L_1[a,b]$. If $\left|f'\right|^q, q > 1$, is s-Godunova-Levin function of second kind on $[a,b]$ and $\left|f'(x)\right| \leq M$, $x \in [a,b]$, then the following inequality for Riemann-Liouville k-fractional integrals holds*

$$\left|f(x) - \Gamma_k(\alpha+k)\left[\frac{1}{2(x-a)^{\frac{\alpha}{k}}} I_{x^-}^{\alpha,k} f(a) + \frac{1}{2(b-x)^{\frac{\alpha}{k}}} I_{x^+}^{\alpha,k} f(b)\right]\right|$$
$$\leq \frac{M(b-a)}{2(1+p\frac{\alpha}{k})^{\frac{1}{p}}} \left[\frac{2}{1-s}\right]^{\frac{1}{q}} \tag{2.14}$$

with $\alpha, k > 0$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. *Using Lemma 2.13 and then Holder's inequality, we have*

$$\left|f(x) - \Gamma_k(\alpha+k)\left[\frac{1}{2(x-a)^{\frac{\alpha}{k}}} I_{x^-}^{\alpha,k} f(a) + \frac{1}{2(b-x)^{\frac{\alpha}{k}}} I_{x^+}^{\alpha,k} f(b)\right]\right|$$
$$\leq \frac{(x-a)}{2} \int_0^1 t^{\frac{\alpha}{k}} \left|f'(tx+(1-t)a)\right| dt$$
$$+\frac{(b-x)}{2} \int_0^1 t^{\frac{\alpha}{k}} \left|f'(tx+(1-t)b)\right| dt$$
$$\leq \frac{(x-a)}{2} \left(\int_0^1 t^{p\frac{\alpha}{k}} dt\right)^{\frac{1}{p}} \left(\int_0^1 \left|f'(tx+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}$$
$$+\frac{(b-x)}{2} \left(\int_0^1 t^{p\frac{\alpha}{k}} dt\right)^{\frac{1}{p}} \left(\int_0^1 \left|f'(tx+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}. \tag{2.15}$$

Since $\left|f'\right|^q$ is s-Godunova-Levin function of second kind and $\left|f'(x)\right| \leq M$, therefore we have

$$\begin{aligned}\int_0^1 \left|f'(tx+(1-t)a)\right|^q dt &\leq \int_0^1 \left[\frac{1}{t^s}\left|f'(x)\right|^q + \frac{1}{(1-t)^s}\left|f'(a)\right|^q\right] dt \\ &\leq M^q \int_0^1 \left[\frac{1}{t^s} + \frac{1}{(1-t)^s}\right] dt = \frac{2M^q}{1-s}\end{aligned} \tag{2.16}$$

similarly

$$\int_0^1 \left|f'(tx+(1-t)b)\right|^q dt \leq \frac{2M^q}{1-s}. \tag{2.17}$$

We also have

$$\int_0^1 t^{p\frac{\alpha}{k}} dt = \frac{1}{1+p\frac{\alpha}{k}}. \tag{2.18}$$

Using (2.16), (2.17) and (2.18) in (2.15) we can get (2.14). □

Remark 2.19. *If we put $k = 1$ in (2.14) then we get the result involving Riemann-Liouville fractional integrals.*

Corollary 2.20. *In Theorem 2.18, if we take $s = 0$, which means that $\left|f'\right|$ is p-function, then (2.14) becomes the following inequality*

$$\left|f(x) - \Gamma_k(\alpha+k)\left[\frac{1}{2(x-a)^{\frac{\alpha}{k}}} I_{x^-}^{\alpha,k} f(a) + \frac{1}{2(b-x)^{\frac{\alpha}{k}}} I_{x^+}^{\alpha,k} f(b)\right]\right| \leq \frac{M(b-a)}{(2)^{1-\frac{1}{q}}(p\frac{\alpha}{k}+1)^{\frac{1}{p}}},$$

where $\alpha, k > 0$.

Theorem 2.21. *Let $f : I \to \mathbf{R}$ be a differentiable mapping on I° such that $f' \in L_1[a,b]$, where $a, b \in I$, with $a < b$. If $\left|f'\right|^q, q > 1$ is s-Godunova-Levin function of second kind on $[a,b]$ and $\left|f'(x)\right| \leq M$, $x \in [a,b]$, then the following inequality for Riemann-Liouville k-fractional integrals holds*

$$\left|f(x) - \Gamma_k(\alpha+k)\left[\frac{1}{2(x-a)^{\frac{\alpha}{k}}} I_{x^-}^{\alpha,k} f(a) + \frac{1}{2(b-x)^{\frac{\alpha}{k}}} I_{x^+}^{\alpha,k} f(b)\right]\right| \leq \frac{M(b-a)}{2\left[1+\frac{\alpha}{k}\right]^{1-\frac{1}{q}}}\left[\frac{1}{\frac{\alpha}{k}-s+1} + \frac{\Gamma_k(\alpha+k)\,\Gamma_k(k-sk)}{\Gamma_k(\alpha+2k-sk)}\right]^{\frac{1}{q}},$$

(2.19)

with $\alpha, k > 0$.

Proof. Using Lemma 2.13 and power mean inequality, we have

$$\begin{aligned}&\left|f(x) - \Gamma_k(\alpha+k)\left[\frac{1}{2(x-a)^{\frac{\alpha}{k}}} I_{x^-}^{\alpha,k} f(a) + \frac{1}{2(b-x)^{\frac{\alpha}{k}}} I_{x^+}^{\alpha,k} f(b)\right]\right| \\ &\leq \tfrac{(x-a)}{2} \int_0^1 t^{\frac{\alpha}{k}} \left|f'(tx+(1-t)a)\right| dt \\ &+ \tfrac{(b-x)}{2} \int_0^1 t^{\frac{\alpha}{k}} \left|f'(tx+(1-t)b)\right| dt \\ &\leq \tfrac{(x-a)}{2}\left(\int_0^1 t^{\frac{\alpha}{k}} dt\right)^{1-\frac{1}{q}} \left(\int_0^1 t^{\frac{\alpha}{k}} \left|f'(tx+(1-t)a)\right|^q dt\right)^{\frac{1}{q}} \\ &+ \tfrac{(b-x)}{2}\left(\int_0^1 t^{\frac{\alpha}{k}} dt\right)^{1-\frac{1}{q}} \left(\int_0^1 t^{\frac{\alpha}{k}} \left|f'(tx+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}.\end{aligned}$$

(2.20)

Since $\left|f'\right|^q$ is s-Godunova-Levin function of second kind on $[a,b]$ and $\left|f'(x)\right| \leq M$, therefore we have

$$\int_0^1 t^{\frac{\alpha}{k}}\left|f'(tx+(1-t)a)\right|^q dt$$
$$\leq \int_0^1\left[\frac{t^{\frac{\alpha}{k}}}{t^s}\left|f'(x)\right|^q+\frac{t^{\frac{\alpha}{k}}}{(1-t)^s}\left|f'(a)\right|^q\right]dt$$
$$\leq M^q\int_0^1\left[\frac{t^{\frac{\alpha}{k}}}{t^s}+\frac{t^{\frac{\alpha}{k}}}{(1-t)^s}\right]dt=M^q\left[\frac{1}{(\frac{\alpha}{k}-s+1)}+\frac{\Gamma_k[\alpha+k]\ \Gamma_k(k-sk)}{\Gamma_k(\alpha+2k-sk)}\right]$$

(2.21)

similarly

$$\int_0^1 t^{\frac{\alpha}{k}}\left|f'(tx+(1-t)b)\right|^q dt\leq M^q\left[\frac{1}{(\frac{\alpha}{k}-s+1)}+\frac{\Gamma_k[\alpha+k]\ \Gamma_k(k-sk)}{\Gamma_k(\alpha+2k-sk)}\right].$$

(2.22)

Using (2.21) and (2.22) in (2.20) we can attain (2.19). □

Remark 2.22. *If we put $k=1$ in (2.19), then we get the result involving Riemann-Liouville fractional integrals.*

Corollary 2.23. *In Theorem 2.21, if we take $s=0$, which means that $\left|f'\right|$ is p-function, then (2.19) becomes the following inequality*

$$\left|\frac{(x-a)^{\frac{\alpha}{k}}+(b-x)^{\frac{\alpha}{k}}}{b-a}f(x)-\frac{\Gamma_k(\alpha+k)}{b-a}\left[J_{x^-}^{\alpha,k}f(a)+J_{x^+}^{\alpha,k}f(b)\right]\right|$$
$$\leq\frac{M(b-a)}{2\left[1+\frac{\alpha}{k}\right]^{1-\frac{1}{q}}}\left[\frac{1}{\frac{\alpha}{k}+1}+\frac{k\Gamma_k[k]}{\alpha+k}\right]^{\frac{1}{q}},$$

where $\alpha,k>0$.

Corollary 2.24. *In Theorem 2.21, if we take $s=1$, which means that $\left|f'\right|$ is Godunova-Levin function, then (2.19) becomes the following inequality*

$$\left|\frac{(x-a)^{\frac{\alpha}{k}}+(b-x)^{\frac{\alpha}{k}}}{b-a}f(x)-\frac{\Gamma_k(\alpha+k)}{b-a}\left[J_{x^-}^{\alpha,k}f(a)+J_{x^+}^{\alpha,k}f(b)\right]\right|$$
$$\leq\frac{M(b-a)}{2}\left[\frac{1}{\frac{\alpha}{k}+1}\right]^{1-\frac{1}{q}}\left[\frac{k(\alpha+1)}{\alpha}\right]^{\frac{1}{q}},$$

where $\alpha,k>0$.

Acknowledgment: The research work of Ghulam Farid is supported by the Higher Education Commission of Pakistan under NRPU 2016, Project No. 5421.

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