ASYMPTOTIC EQUILIBRIUM FOR CERTAIN TYPE OF DIFFERENTIAL EQUATIONS WITH MAXIMUM *

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Abstract

In this work we obtain asymptotic representations for the solutions of certain type of differential equations with maximum. We deduce the asymptotic equilibrium for this class of differential equations.

1991 Math. Subject Classification: 34A20

Keywords: Differential equations with maximum, asymptotic representation, asymptotic equilibrium, Banach fixed point theorem.

*This research was supported by FONDECYT Grant # 8990013
1. Introduction

Differential equations with maximum arise naturally when solving practical problems, in particular, in those which appear in the study of systems with automatic regulation. A classical example is that of an electric generator. In this case, the mechanism becomes activated when the maximum voltage variation that is permitted is reached in an interval of time $I_t = [t - h, t]$, with $h$ a positive constant. The equation which describes the actioning of this regulator has the form

$$V'(t) = -\delta V(t) + p \max_{s \in I_t} V(s) + F(t),$$

where $\delta$ and $p$ are constants that are determined by the characteristic of the system, $V(t)$ is the voltage and $F(t)$ is the effect of the perturbation that appears associated to the change of voltage [2].

Much work on these equations has been carried out in the last three decades. We mention the work in [1-2] and [8,11].

We study differential equations with maximum of the form

$$\begin{cases}
x'(t) = f(t, x(t), \max_{u \in I_t} x(u)), \quad \text{with } t \in I \\
x(t) = \varphi(t), \quad \text{with } t \in [-h, 0]
\end{cases}$$

(1.1)

where $I = [0, b)$ and the possibility that $b$ be infinity is not excluded. In addition, $f$ is a real-valued continuous function defined on $I \times R \times R$.

We denote by $\|\varphi\|$ the norm $\|\varphi\| = \max\{\|\varphi(t)\| / t \in [-h, 0]\}$.

**Definition 1.** A differential equation with maximum (1.1) as above has the property of asymptotic equilibrium if:

1. Every solution $x(t)$ of (1.1) with initial condition $x(t) = \varphi(t)$, for all $t \in [-h, 0]$, is defined for all $t \geq -h$ and there exists $\xi \in R$ which satisfies

$$\lim_{t \to \infty} x(t) = \xi$$

(1.2)
2. For all $\xi \in \mathbb{R}$, there exists a solution $x(t)$ of (1.1), which is defined on the interval $[-h, \infty)$ and verifies (2.1).

Our main result asserts, under certain hypotheses on the function $f$, equation (1.1) has asymptotic equilibrium. The techniques used in the proof are based on an inequality of Gronwall - Bellman type and successive approximations. The method used is analogous to those of [3-10].

Our principal results are applied to automatic control problems described by nonlinear equations of the type

$$V'(t) = -\delta V(t) + p \max_{s \in \mathcal{I}_t} V(s) + F(t, V(t)).$$

2. Main Results

In this paragraph we prove theorems about asymptotic behavior and boundedness of the solutions of equation (1.1). We first recall some basic inequality of Gronwall - Bellman type which we use in the study of the differential equation with maximum (1.1).

**Lemma 1.** Let $p, x$ be continuous and nonnegative functions on $[0, b)$ and $\varphi_{[-h, 0]}$ a nonnegative continuous function. If the inequality

$$x(t) \leq x(0) + \int_0^t p(s) \max_{u \in \mathcal{I}_t} x(u) \, ds$$

holds, for all $t \in [0, b)$, with $x(t) = \varphi(t)$ for all $t \in [-h, 0]$, then

$$x(t) \leq \|\varphi\| e^{\int_0^t p(s) \, ds}$$

for all $t \in [0, b).$
Theorem 1. Suppose that $f$ satisfies the following hypotheses:

[H$_{0}$] $f$ is continuous on $I \times R \times R$, where $I = [0, \infty)$.

[H$_{1}$] There exist $\lambda$ and $\beta$ integrable functions on $I$ such that, for all $(t, x, y) \in I \times R \times R$, we have

$$
|f(t, x, y)| \leq \lambda(t)(|x| + |y|) + \beta(t).
$$

Then, every solution $x(t)$ with $x(t) = \varphi(t)$, for all $t \in [-h, 0]$, is defined on $[-h, \infty)$ and satisfies (1.2) for some $\xi \in R$. In addition, we have $x(t) = \xi + O\left(\int_{t}^{\infty} [\lambda(s) + \beta(s)] \, ds\right)$

Proof. If $x(t)$ is a solution of equation (1.1) such that $x(t) = \varphi(t)$, for all $t$ in $[-h, 0]$, and defined on a subinterval $J = [-h, T)$ of $[-h, \infty)$, then

$$
(2.2) \quad x(t) = x(0) + \int_{0}^{t} f(s, x(s), \max_{u \in I_{s}} x(u)) \, ds,
$$

for all $t$ in $[0, T)$. Therefore, for all $t$ in $[0, T)$, we have

$$
|x(t)| \leq |x(0)| + \int_{0}^{t} |\lambda(s)| \left(|x(s)| + \max_{u \in I_{s}} x(u)\right) \, ds + \int_{0}^{t} \beta(s) \, ds
$$

$$
\leq |x(0)| + \int_{0}^{\infty} \beta(s) \, ds + \int_{0}^{t} 2 \lambda(s) \max_{u \in I_{s}} |x(u)| \, ds.
$$

By lemma 1,

$$
|x(t)| \leq \left[ \|\varphi\| + \int_{0}^{\infty} \beta(s) \, ds \right] e^{\int_{0}^{t} 2 \lambda(s) \, ds} \leq \left[ \|\varphi\| + \int_{0}^{\infty} \beta(s) \, ds \right] e^{\int_{0}^{\infty} 2 \lambda(s) \, ds}
$$

for all $t$ in the interval $J$, which shows that $x(t)$ is bounded in the interval $J$. Therefore, the left limit $x(T - 0)$ exists, when $t$ tends to $T$. Since the initial value problem

$$
y'(t) = f(t, y(t), \max_{u \in I_{t}} y(u)),
$$

with initial condition $y(t) = x(t)$ in $[T - h, T)$ and where $y(T) = x(T - 0)$ has a local solution, we conclude that it is possible to extend
x beyond $T$. This shows that solution $x(t)$ of equation (1.1) is defined on $[-h, \infty)$ and there exists a constant $M$ such that $|x(t)| \leq M$, for all $t \in [-h, \infty)$.

Since $f(s, x(s), \max_{u \in I_s} x(u))$ is integrable on $I$, from equation (2.2) it follows that

$$x(t) = x(0) + \int_0^\infty f(s, x(s), \max_{u \in I_s} x(u)) \, ds - \int_0^\infty f(s, x(s), \max_{u \in I_s} x(u)) \, ds.$$

Let $\xi$ denote the real number given by

$$x(0) + \int_0^\infty f(s, x(s), \max_{u \in I_s} x(u)) \, ds,$$

then, we have

$$|x(t) - \xi| \leq \int_t^\infty \lambda(s) \left( |x(s)| + \max_{u \in I_s} |x(u)| \right) \, ds + \int_t^\infty \beta(s) \, ds \leq 2M \int_t^\infty \lambda(s) + \beta(s) \, ds$$

thus,

$$x(t) = \xi + O \left( \int_t^\infty \lambda(s) + \beta(s) \, ds \right).$$

**Theorem 2.** If $f$ satisfies conditions

[H$_0$] $f$ is continuous on $I \times \mathbb{R} \times \mathbb{R}$ and $f(t, 0, 0)$ is integrable on $I$, where $I = [0, \infty)$.

[H$_2$] There exists a positive and integrable function $\mu$ on $I$ such that, for all $(t, x_1, y_1), (t, x_2, y_2)$ in $I \times \mathbb{R} \times \mathbb{R}$, we have

$$|f(t, x_1, y_1) - f(t, x_2, y_2)| \leq \mu(t) (|x_1 - x_2| + |y_1 - y_2|)$$

and there exist a nonnegative real constant $K$ such that, for all $s \in [0, \infty)$

$$\mu(s) \leq K \mu(s - h).$$

Then for all $\xi \in \mathbb{R}$ there exists a solution $x(t)$ of (1.1) which is defined on $[-h, \infty)$ and verifies (1.2). In addition, we have
\[ x(t) = \xi + \int_{t}^{\infty} f(s, 0, 0) \, ds + O \left( \int_{t}^{\infty} \mu(s) \, ds \right). \]

**Proof.** Let \( B \) denote the Banach space of the real bounded functions \( g \) defined on \([-h, \infty)\) with the norm defined by
\[
\| g \| = \left\{ \sup \{ |g(t)| / t \in [-h, \infty) \} \right\}.
\]
Consider the operator \( T : B \rightarrow B \) defined by
\[
T(x)(t) = \xi - \int_{t}^{\infty} f(s, x(s), \max_{u \in I_s} x(u)) \, ds \quad \text{for all} \quad t \geq 0,
\]
and by \( T(x)(t) = \xi \), for all \( t \in [-h, 0] \).

In order to prove that \( T : B \rightarrow B \) has a fixed point, we must show that there exists an integer \( n \) such that \( T^n \) is a contractive operator. For this, it suffices to show that, for all \( x, y \) in \( B \) and for all positive integer \( n \), we have
\[
| (T^n y)(t) - (T^n x)(t) | \leq \frac{1}{K n!} \left( \int_{t-h}^{\infty} 2K \mu(s) \, ds \right)^n \| y - x \|.
\]
(2.4)

We proceed by induction.

First, we verify that the inequality is true for \( n = 1 \).
\[
| (Ty)(t) - (Tx)(t) | = \left| \int_{t}^{\infty} \left( f(s, y(s), \max_{u \in I_s} y(u)) - f(s, x(s), \max_{u \in I_s} x(u)) \right) \, ds \right|
\leq \int_{t}^{\infty} \mu(s) \left\{ |y(s) - x(s)| + \max_{u \in I_s} |y(u) - x(u)| \right\} \, ds
\leq \int_{t}^{\infty} 2\mu(s) \max_{u \in I_s} |y(u) - x(u)| \, ds
\leq \left( \int_{t}^{\infty} 2\mu(s) \, ds \right) \| y - x \|
\leq \frac{1}{K 1!} \left( \int_{t-h}^{\infty} 2K \mu(s) \, ds \right) \| y - x \|
Next, if the formula were true for \( n - 1 \), then
\[
| (T^n y)(t) - (T^n x)(t) |
= \left| \int_t^\infty \left( f \left( s, (T^{n-1} y)(s), \max_{u \in I_s} (T^{n-1} y)(u) \right) - 
\quad \quad f \left( s, (T^{n-1} x)(s), \max_{u \in I_s} (T^{n-1} x)(u) \right) \right) ds \right|
\leq \int_t^\infty \mu(s) \left\{ \left| (T^{n-1} y)(s) - (T^{n-1} x)(s) \right| + \max_{u \in I_s} (T^{n-1} y)(u) - (T^{n-1} x)(u) \right\} ds
\leq \int_t^\infty 2\mu(s) \max_{u \in I_s} (T^{n-1} y)(u) - (T^{n-1} x)(u) ds
\leq \int_t^\infty 2\mu(s) \frac{1}{K (n-1)!} \left( \int_t^\infty 2K \mu(\tau) d\tau \right)^{n-1} \| y - x \| ds
\leq \int_t^\infty 2K \mu(s-h) \frac{1}{K (n-1)!} \left( \int_t^\infty 2K \mu(\tau) d\tau \right)^{n-1} \| y - x \| ds
\leq \frac{1}{K n!} \left( \int_t^\infty 2K \mu(\tau) d\tau \right)^n \| y - x \|.
\]

Since \( \mu(t) \) is integrable on \( I \), we can find a positive integer \( n \) such that the operator \( T^n \) is a contraction. Therefore, there exists a fixed point \( x(t) \) of \( T^n \), and then \( x(t) = \xi - \int_t^\infty f(s, x(s), \max_{u \in I_s} x(u)) ds \) for all \( t \geq 0 \), and \( x(t) = \xi \), for all \( t \in [-h, 0] \), i.e. \( x(t) \) is a solution of equation (1.1) such that \( x(t) = \varphi(t) \), for all \( t \) in \([-h, 0]\).

Since \( \left| x(t) - \xi - \int_t^\infty f(s, 0, 0) ds \right| \leq K \int_t^\infty \mu(s) ds \) we can see that (2.3) is verified.

**Example 1**: Consider the differential equation with maximum
\[
\begin{cases}
    x'(t) = \frac{2}{(1+t)^2} \max_{u \in I_t} x(u) \\
    x(t) = e^{-2}, \text{ for } t \in [-h, 0]
\end{cases}
\]

We have that \( f(t, x, y) = \frac{2}{(1+t)^2} y \) is a real-valued, continuous, integrable function on \( I \times R \times R \) and
\[
| f(t, x, y) | = \left| \frac{2}{(1+t)^2} \cdot y \right| \leq \frac{2}{(1+t)^2} (|x| + |y|).
\]
The function \( \lambda(t) = \frac{2}{(1 + t)^2} \) is continuous and integrable on \( I \). Thus \( f \) satisfies conditions \( H_0 \) and \( H_1 \) and by Theorems 1 and 2, every solution \( x(t) \) is defined on \([-h, \infty)\) and verifies

\[
(2.5) \quad x(t) = \xi + O\left(\int_{t}^{\infty} \lambda(s)ds\right) = \xi + O\left(\frac{2}{1+t}\right)
\]

and reciprocally for any \( \xi \in \mathbb{R} \) there exists a solution \( x \) defined on \([-h, \infty]\) and satisfying asymptotic formula (2.5).

In this example, the function \( x(t) = \begin{cases} \exp\left(\frac{-2}{1+t}\right), & \text{for } t \in I \\ \exp(-2), & \text{for } t \in [-h, 0] \end{cases} \)

is an explicit solution of the differential equation which in effect satisfies formula (2.5).

**Example 2.** In the differential equation with maximum

\[
\begin{aligned}
  x'(t) &= \frac{2+\sin t}{1+t^2} x(t) + \frac{2}{(1+t)^2} \max_{u \in I} x(u) \\
  x(t) &= \sigma
\end{aligned}
\]

the function \( f(t, x, y) = \frac{2 + \sin t}{1 + t^2} - x + \frac{2}{(1 + t)^2} y \) is continuous for \( t > -1, x \in \mathbb{R}, y \in \mathbb{R} \).

Moreover,

\[
|f(t, x_1, y_1) - f(t, x_2, y_2)| = \mu_1(t) |x_1 - x_2| + \mu_2(t) |y_1 - y_2|
\]

\[
\leq \mu(t) (|x_1 - x_2| + |y_1 - y_2|)
\]

with \( \mu(t) = \max\{\mu_1(t), \mu_2(t)\} \), where \( \mu_1(t) = \frac{|2 + \sin t|}{1 + t^2} \) and \( \mu_2(t) = \frac{2}{(1 + t)^2} \).

Since \( f \) satisfies conditions \( H_0, H_1, \) and \( H_2 \) we have that for every solution \( x(t) \) of the given equation, there exists \( \xi \in \mathbb{R} \) such that \( x(t) = \xi + O(\frac{\pi}{2} - \arctan t) \).

Conversely, for each \( \xi \in \mathbb{R} \), there exist a solution \( x(t) \) defined on
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the interval $[-h, \infty)$ such that

$$x(t) = \xi + O \left( \int_{t}^{\infty} \lambda(s) ds \right) = \xi + O\left( \frac{\pi}{2} - \arctan t \right).$$

**Example 3 :** (Application to semilinear electrogenerators) : Consider the differential equation with maximum

$$\begin{cases} 
  x'(t) = a(t)x(t) + g(t, x(t), \max_{s \in I_t} x(s)), & \text{for } t \in [0, \infty) \\
  x(t) = e^{\int_{0}^{t} a(s) ds} \xi, & \text{for } t \in [-h, 0]
\end{cases}$$

where $a$ is a continuous function on $I$ and $g$ is a continuous function on $I \times R \times R$, such that $f(t,x,y) = e^{-\int_{0}^{t} a(s) ds} g(t, e^{\int_{0}^{t} a(s) ds} x, e^{\int_{0}^{t} a(s) ds} y)$ verifies conditions $H_0$ and $H_2$.

Substitution $x(t) = e^{\int_{0}^{t} a(s) ds} v(t)$ in this equation gives

$$v'(t) = e^{-\int_{0}^{t} a(s) ds} g(t, e^{\int_{0}^{t} a(s) ds} v(t), \max_{s \in I_t} e^{\int_{s}^{t} a(\tau) d\tau} v(s))$$

By Theorem 2, there exists a solution $v$ of the differential equation

$$\begin{cases} 
  v'(t) = f(t, v(t), \max_{s \in I_t} v(s)) \\
  v(t) = \xi, & t \in [-h, 0]
\end{cases}$$

defined in $[-h, \infty)$ such that

$$v(t) = \xi + \int_{t}^{\infty} f(s, 0, 0) ds + O \left( \int_{t}^{\infty} \mu(s) ds \right) \quad \text{then}$$

$$x(t) = e^{\int_{0}^{t} a(s) ds} \left\{ \xi + \int_{t}^{\infty} e^{-\int_{s}^{t} a(\tau) d\tau} g(s, 0, 0) ds + O \left( \int_{t}^{\infty} \mu(s) ds \right) \right\}$$

If function $a$ is integrable in the interval $I$ then the solutions of the equation are stables. If $\int_{0}^{t} a(s) ds$ tends to $-\infty$ as $t$ approaches infinity then, the solutions of the system are asymptotically stables.

We can apply this result to an automatic control problem described by a nonlinear equation of the type

$$\begin{cases} 
  V'(t) = -\delta V(t) + p(t) \max_{s \in I_t} V(s) + F(t), & \text{if } t \in I \\
  V(t) = e^{-\delta t} \xi, & \text{if } t \in [-h, 0]
\end{cases}$$

where $\delta$ is a positive constant, $p, F$ are continuous and integrable functions on $I$ and there exists a nonnegative real constant $K$, such that, for all $s \in [0, \infty), p(s) \leq K p(s-h)$. We can see there exists a solution
of this equation such that the following asymptotic representation is verified

\[ V(t) = e^{-\delta t} \left\{ \xi + \int_t^\infty e^{-\delta s} F(s) \, ds + O(\int_t^\infty p(s) \, ds) \right\} \]

3. References


Received : December 2001.

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