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PARABOLIC PERTURBATION IN THE FAMILY $z \mapsto 1 + 1/wz^d$

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Abstract

Consider the family of rational maps $\mathcal{F}_d = \{z \mapsto f_w(z) = 1 + \dfrac{1}{wz^d} : w \in \mathbf{C} \setminus \{\mathbf{0}\}\}$ $(\mathbf{d} \in \mathbf{N}, \mathbf{d} \geq \mathbf{2})$, *and the hyperbolic component* $A_1 = \{w : f_w$ *has an attracting fixed point* $\}$. *We prove that if* $w_0 \in \partial A_1$ *is a parabolic parameter with corresponding multiplier a primitive* $q-$*th root of unity,* $q \geq 2$, *then there exists a hyperbolic component* W_q, *attached to* A_1 *at the point* w_0, *which contains* $w-$*values for which* f_w *has an attracting periodic cycle of period* q.

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1. Introduction

For any $d \in \mathbf{N}, d \geq 2$, the family $\mathcal{F}_d = \{z \mapsto 1 + \dfrac{1}{wz^d} : w \in \mathbf{C} \setminus \{0\}\}$ is a normal form for the set of rational maps of degree d which have exactly two critical points, one of which maps onto the other under one iteration. These families have been considered in [3], [4] (for the special case $d = 2$), and in [1](for any d).

It is well known that a rational map f is hyperbolic if and only if all critical points of f tend to attracting cycles under iteration. Since the members of the family $\mathcal{F}_d$ have only one forward orbit of their critical points, f_w is hyperbolic if and only if f_w has an attracting periodic orbit. The connected components of the parametric set $\mathcal{H}_d = \{w : f_w(z) = 1 + 1/wz^d$ has an attracting periodic orbit$\}$ are called the hyperbolic components of the family $\mathcal{F}_d$.

Following the ideas used in [2], we can prove the following one.

Theorem *If f_{w_0} has a fixed point z_0 such that the multiplier $\lambda_0 = f'_{w_0}(z_0)$ is a primitive $q-$th root of unity, $q \geq 2$, then there exists a hyperbolic component W_q, which contains $w-$values for which f_w has an attracting periodic cycle of period q, with $w_0 \in \partial W_q$.*

In Section 2 we prove the Theorem.

2. Proof of Theorem

It is clear that for any $u \in \mathbf{C} \setminus \{0, -d\}$,

f_w has a fixed point of multiplier $u \iff w = -\dfrac{d}{u}\left(1 + \dfrac{u}{d}\right)^{d+1}$.

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In fact, $z(u) = \dfrac{d}{d+u}$ is the fixed point of multiplier u.

Let $g_u(z) := f_{-\frac{d}{u}\left(1+\frac{u}{d}\right)^{d+1}}(z)$, that is,

$$g_u(z) = 1 - \frac{d^d u}{(d+u)^{d+1} z^d}\ , \qquad u \in \mathbf{C}\setminus\{0, -d\}.$$

Therefore, g_u has at $z(u) = \dfrac{d}{d+u}$ a fixed point of multiplier u. Now, we will make an analytic conjugation :

Let $M_u(z) := z - z(u)$, and consider $h_u := M_u \circ g_u \circ M_u^{-1}$.

For any $u \in \mathbf{C}\backslash\{0, -d\}$, h_u is a rational map analytically conjugate to f_w (where $w = -\frac{d}{u}\left(1 + \frac{u}{d}\right)^{d+1}$), and which has at zero a fixed point of multiplier u.

Explicitly,

$$h_u(z) = \tfrac{u}{d+u} \cdot \tfrac{(z(d+u)+d)^d - d^d}{(z(d+u)+d)^d}, \qquad u \in \mathbf{C}\backslash\{\mathbf{0}, -\mathbf{d}\}.$$

Note that, for any $q \in \mathbf{N}$, $h_u^q(z) = u^q z \cdot \Phi_{u,q}(z)$, where $\Phi_{u,q}$ is a rational map with $\Phi_{u,q}(0) = 1$. Hence, in a neighbourhood of $z = 0$ we have :

$$h_u^q(z) = u^q z + a_2 z^2 + \ \ldots\ldots$$

In what follows, u_0 denotes a primitive $q-$th root of unity, $q \geq 2$ (that is, $u_0^q = 1$, and $u_0^k \neq 1$, for all $1 \leq k \leq q-1$). In order to prove the above theorem, we show the following results:

Lemma 1 : *$h_{u_0}^q$ has at 0 a fixed point of multiplicity $(q+1)$.*

Proof : Since h_{u_0} has at zero a fixed point of multiplier a primitive $q-$th root of unity, we have that in a neighbourhood of zero,

$$h_{u_0}^q(z) = z + az^{kq+1} + \ \ldots. \ \text{ where } a \neq 0, \text{ and } k \in \mathbf{N}.$$

From the fact that h_{u_0} has only one forward orbit of critical points, $k = 1$.

Therefore, $h_{u_0}^q(z) = z + az^{q+1} + \ \ldots\ldots$ □

Next, we will show that for u near to u_0, the $(q+1)-$fold fixed point zero of $h_{u_0}^q$ will split up into $(q+1)$ simple fixed points of h_u^q, which are : 0, and $\{z_1(u), z_2(u), \ \ldots\ldots, \ z_q(u)\}$; the latter forms a periodic orbit of period q of h_u.

Lemma 2 : *There exist $\varepsilon > 0$ and $r > 0$ such that for each $u \in \mathbf{C}$ with $0 < |u - u_0| < r$, the rational map h_u^q has precisely q fixed points in the punctured disc $0 < |z| < \varepsilon$. Furthermore, these q points form a cycle of length q for h_u.*

Proof : Since the zeros of an analytic function (not identically zero) are isolated, there exist ε, $\quad 0 < \varepsilon < \dfrac{d-1}{d+2}$, such that :

$$h^k_{u_0}(z) - z \neq 0 \qquad \text{for} \qquad 0 < |z| < \varepsilon', \qquad \text{and} \qquad 1 \leq k \leq q,$$

where $\varepsilon' := \dfrac{2}{d-1}[(2d-1)^d - d^d] \cdot \varepsilon$. (Note that $\varepsilon < \varepsilon'$).

Let $\gamma_\varepsilon = \{z \in \mathbf{C} : |z| = \varepsilon\}$, $\quad \gamma_{\varepsilon'} = \{z \in \mathbf{C} : |z| = \varepsilon'\}$, and

$$\alpha := \min_{1 \leq k \leq q} \{|h^k_{u_0}(z) - z| \ : \ z \in \gamma_\varepsilon \cup \gamma_{\varepsilon'}\} \ (> 0).$$

It is clear that there exists r, $\quad 0 < r < 1$, such that :
$|h^k_u(z) - z| \geq \dfrac{\alpha}{2}$, for all $|u - u_0| < r$, $z \in \gamma_\varepsilon \cup \gamma_{\varepsilon\prime}$, and $1 \leq k \leq q$.

From the Argument Principle, the number $N_{k,\varepsilon}(u)$ (resp. $N_{k,\varepsilon'}(u)$) of fixed points of h^k_u in the disk $|z| < \varepsilon$(Resp. $|z| < \varepsilon'$) for $|u - u_0| < r$ and $1 \leq k \leq q$, is given by :

$$N_{k,\varepsilon}(u) = \frac{1}{2\pi i} \oint_{|z|=\varepsilon} \frac{(h^k_u)'(z) - 1}{h^k_u(z) - z} \ dz$$

$$(\text{resp.}\ N_{k,\varepsilon'}(u) = \frac{1}{2\pi i} \oint_{|z|=\varepsilon'} \frac{(h^k_u)'(z) - 1}{h^k_u(z) - z} \ dz)$$

From above we conclude that $u \mapsto N_{k,\varepsilon}(u)$, and $u \mapsto N_{k,\varepsilon'}(u)$, are conti- nuous, and hence are constant since they are integer-valued.

Therefore,

$$N_{k,\varepsilon}(u) = N_{k,\varepsilon}(u_0) \quad \text{and} \quad N_{k,\varepsilon'}(u) = N_{k,\varepsilon'}(u_0)$$
$$\text{for } |u - u_0| < r \quad \text{and} \quad \text{for} \quad 1 \leq k \leq q.$$

Hence, $|u - u_0| < r \quad \Rightarrow$

$$\begin{cases} N_{k,\varepsilon}(u) = N_{k,\varepsilon'}(u) = 1 & , \quad \text{if} \qquad 1 \leq k \leq (q-1) \\ \\ N_{k,\varepsilon}(u) = N_{k,\varepsilon'}(u) = (q+1) & , \quad \text{if} \qquad k = q. \end{cases}$$

We conclude that 0 is the unique fixed point of $h^k_u(1 \leq k \leq (q-1))$ in the disk $|z| < \varepsilon'$. On the other hand, for $0 < |u - u_0| < r$, $\quad h^q_u$ has at zero a simple fixed point, and has other q fixed points in the punctured disk $0 < |z| < \varepsilon$. Note that for $|u - u_0| < r$, $\quad h^q_u$ has no fixed points in $\varepsilon < |z| < \varepsilon'$, because $N_{q,\varepsilon}(u) = N_{q,\varepsilon'}(u)$.

Furthermore, using the facts that $\varepsilon < \frac{d-1}{d+2}$, $\ r < 1$, a simple calculation shows that :

$$\forall \ u \ \in \ \{u \ : \ |u - u_0| < r\} \ , \qquad |z| < \varepsilon \quad \Rightarrow \quad |h_u(z)| < \varepsilon' \ .$$

Hence, if $z_1(u)$ denotes one of the fixed points of h_u^q with $0 < |z_1(u)| < \varepsilon$, then $z_j(u) = h_u^j(z_1(u))$, for $0 \le j \le (q-1)$, are the q fixed points of h_u^q in the punctured disk $0 < |z| < \varepsilon$ (they are clearly different pairwise).

Therefore, $\{z_1(u), h_u(z_1(u)), \ldots\ldots, h_u^{q-1}(z_1(u))\}$ are the q fixed points of h_u^q in the punctured disk $0 < |z| < \varepsilon$, and they form a cycle of length q of h_u, for $u \in \{u : 0 < |u - u_0| < r\}$. $\square$

For $u \in \{u : 0 < |u - u_0| < r\}$, $\lambda(u)$ denotes the multiplier of the periodic cycle of period q of h_u, contained in the punctured disk $0 < |z| < \varepsilon$.

Lemma 3 : *$u \mapsto \lambda(u)$ is an analytic function in the disk $\{u : |u - u_0| < r\}$. Furthermore, $\lambda(u_0) = 1$.*

Proof : For $0 < |u - u_0| < r$, let $\{z_1(u), z_2(u), \ldots\ldots, z_q(u)\}$ be the periodic cycle of period q of h_u, contained in the punctured disk $0 < |z| < \varepsilon$. Furthermore, for $u = u_0$, let $z_1(u_0) = z_2(u_0) = \ldots. = z_q(u_0) = 0$.

Consider the polynomial $P_u(z) = \prod_{j=1}^q (z - z_j(u))$.

We know that $P_u(z) = z^q + a_{q-1}(u) z^{q-1} + \ldots. + a_1(u) z + a_0(u)$, where,

$$a_{q-k}(u) = (-1)^k \cdot \sum_{1 \le j_1 < j_2 < \ldots. < j_k \le q} z_{j_1}(u) z_{j_2}(u) \ldots z_{j_k}(u)$$

are the elementary symmetric functions in $z_1(u), z_2(u), \ldots., z_q(u)$.

Consider the polynomials :

$$\sigma_k(u) = \sum_{j=1}^q (z_j(u))^k, \qquad k = 1, 2 \ldots, q$$

A calculation shows that each elementary symmetric function can be written as a polynomial in $\sigma_1(u), \sigma_2(u), \ldots\ldots, \sigma_q(u)$. Indeed, we have that :

$a_{q-1}(u) = -\sigma_1(u)$ $a_{q-2}(u) = \dfrac{1}{2}[(\sigma_1(u))^2 - \sigma_2(u)]$

$a_{q-3}(u) = \dfrac{1}{6}[3\sigma_1(u)\sigma_2(u) - 2\sigma_3(u) - (\sigma_1(u))^3]$ and, so on.

On the other hand, by the Residue Theorem we have that for $0 < |u - u_0| < r,$ and $1 \leq k \leq q,$

$$\sigma_k(u) = \frac{1}{2\pi i} \oint_{|z|=\varepsilon} z^k \frac{(h_u^q)'(z) - 1}{h_u^q(z) - z}\ dz$$

Note that the above formula holds also for $u = u_0$.

Hence, by the Leibniz's rule we conclude that :

$$\forall\ k\ \in\ \{1, 2,\ .\ .\ .,\ q\}, \quad u \mapsto \sigma_k(u)$$

is holomorphic in the disk $|u - u_0| < r$.

Therefore, $a_0(u), a_1(u),\ .\ .\ .\ ., a_{q-1}(u)$ are holomorphic functions in the disk $|u - u_0| < r$.

For the multiplier, we have :

$$\lambda(u) = (h_u^q)'(z_1(u)) = \prod_{j=1}^{q} h_u'(z_j(u)) = \prod_{j=1}^{q} \frac{ud^{d+1}}{[z_j(u)(d+u) + d]^{d+1}}$$

Hence,

$$\lambda(u) = \frac{d^{(d+1)q} u^q}{[\prod_{j=1}^q (z_j(u)(d+u) + d)]^{d+1}} \quad , \quad \forall \quad |u - u_0| < r$$

Since, $\forall\ u\ \in\ \{u\ :\ |u - u_0| < r\}$, $\prod_{j=1}^q (z_j(u)(d+u) + d) = (-(d+u))^q \prod_{j=1}^q (\frac{-d}{d+u} - z_j(u)) = (-(d+u))^q P_u\left(\frac{-d}{d+u}\right)$,

we conclude that $u \mapsto \lambda(u)$ is analytic in $|u - u_0| < r$. Finally, is clear that $\lambda(u_0) = 1$.

Proof of Theorem: If $f_{w_0}(z) = 1 + 1/w_0 z^d$ has a fixed point z_0 with corresponding multiplier $u_0 = f_{w_0}'(z_0)$ a primitive $q-$th root of unity, $q \geq 2$, then h_{u_0} has at zero a fixed point of multiplier u_0.

By lemma 2, there exists $r > 0$ such that for each $u \in \mathbf{C}$ with $0 < |u - u_0| < r$, the rational map h_u has a periodic orbit $\{z_1(u), z_2(u),\ .\ .\ .\ .\ .\ ,\ z_q(u)\}$ of period q. Furthermore, by lemma 3 the multiplier $\lambda(u)$ of that periodic orbit, is an analytic function in the disk $B(u_0, r) := \{u \ : \ |u - u_0| < r\}$, where $\lambda(u_0) = 1$. λ is clearly non- constant, and therefore is open. Then we conclude that there exists a hyperbolic component of period q, W_q, such that $w_0 \in \partial W_q$.

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