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REPRESENTATION THEOREMS OF LINEAR OPERATORS ON P-ADIC FUNCTION SPACES

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Abstract

Let X be a 0-dimensional Hausforff topological space, E, F non-archimedean Banach spaces and $C_b(X, E)$ the space of all continuous E-valued functions on X provided with two strict topologies. In this paper we show that every $F-$valued linear operator which is strictly continuous can be represented by a certain $\mathcal{L}(E, F)-$valued measure defined on the ring of all clopen subsets of X.

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1. Introduction and notations

By the classical Riesz Representation Theorem, a linear functional u on the space of continuous real functions on a compact Hausdorff space X, is continuous for the topology of uniform convergence if, and only if, there exists a bounded regular Borel measure m on X such that $u(f) = \int f dm$. The Riesz Representation Theorem has been extended to many other spaces (see [8]) and linear operators instead of linear functional (see [4]). The relation between vector measures, linear operators and strict topologies in the classical case have been studied by several authors (see [1] [4]). Analogous situation in the non-archimedean case is studied in [5].

This paper is devoted to extend the work given in [5] for another two strict topologies. Throughout this work, X will be a zero dimensional Hausdorff topological space, $\mathbf{K}$ a complete non-archimedean valued field with nontrivial valuation and E, F non-archimedean Banach spaces.

We will denote by $C_b(X,E)$ the space of all $E-$valued bounded and continuous functions on X and by $C_{rc}(X,E)$ the subspace of $C_b(X,E)$ of those functions whose image of X are relatively compact. If $E=\mathbf{K}$, we will write $C_b(X)$ and $C_{rc}(X)$ respectively.

We will denote by $\beta_\circ X$ the Banaschewski compactification of X [7] and understand by $\widehat{f}$ the unique continuous extension of f to $\beta_\circ X$. For $A \subset X$, we will denote by $\overline{A}^{\beta_\circ X}$ the closure of A in $\beta_\circ X$ and by $\mathcal{X}_A$ the $\mathbf{K}$-valued characteristic function of A. For an $E-$valued function f on X and $A \subset X$, we will denote

$$\|f\|_A = \sup_{x\in A} \|f(x)\|, \quad \|f\|_X = \|f\|$$

Let $\mathcal{S}(X)$ be the collection of all clopen subsets of X. An $\mathcal{L}(E,F)-$valued set function m on $\mathcal{S}(X)$ is said to be a measure if:

1. m is finitely additive
2. The set $m(\mathcal{S}(X))$ is bounded in $\mathcal{L}(E,F)$.

We will denote by $M(X,\mathcal{L}(E,F))$ the space of all these measures. For $m \in M(X,\mathcal{L}(E,F))$ and $A \in \mathcal{S}(X)$, we define

$$\|m\|(A) = \sup\left\{\|m(B)e\|_F : B \subset A, B \in \mathcal{S}(X), \|e\|_E \leq 1\right\}.$$

In order to introduce strict topologies, we will denote by Ω the collection of all compact subsets of $\beta_\circ X \setminus X$ and by Ω_u the collection of $Q \in \Omega$ such that Q admits a clopen partition $\{U_\alpha\}_{\alpha\in I}$ of X such that $\overline{U}_\alpha^{\beta_\circ X} \cap Q = \emptyset$ for all $\alpha \in I$.

The strict topology $\beta\,(\beta_u)$ on $C_b(X,E)$ is the inductive limit of the locally convex topologies β_Q, where β_Q is generated by the family of semi-norms $f \mapsto \|gf\|$, where $g \in C_Q = \left\{g \in C_{rc}(X) : \widehat{g}_{|Q} \equiv 0\right\}$ and $Q \in \Omega\,(\Omega_u)$ (see $[5]$, $[2,3]$).

Next, we will define the integrability of an $E-$valued function f on X with respect to a $m \in M(X,\mathcal{L}(E,F))$. For $A \in \mathcal{S}(X)$, $A \neq \emptyset$, let $\mathcal{D}_A$ denote the family of all $\alpha = \{A_1,\cdots,A_n; x_1,\cdots,x_n\}$, where $\{A_1,\cdots,A_n\}$ is a clopen partition of A and $x_i \in A_i$. We will introduce the following relation: $\alpha_1 \geq \alpha_2$ iff the partition of A in α_1 is a refinement of the partition of A in α_2. We will denote by Ω_A the collection of all these α. Ω_A will become to be a directed set. For f, m and $\alpha \in \Omega_A$, $\alpha = \{A_1,\cdots,A_n; x_1,\cdots,x_n\}$, we will define

$$\varpi\,(f,m) = \sum_{n=1}^{n} m(A_i)\,(f(x_i))\,.$$

Note that $\varpi\,(f,m) \in F$.

We will say that f is $m-$integrable over A if $\lim_\alpha \varpi\,(f,m)$ exists; in such a case, we will denote this limit by

$$\int_A f dm = \lim_\alpha \varpi\,(f,m)\,.$$

If $A = \emptyset$, then we will define $\int_\emptyset f dm = 0$. For $A = X$, we will simply write $\int f dm$. It is easy to see that if f is $m-$integrable over X, then f is $m-$integrable over every $A \in \mathcal{S}(X)$.

We will present the following very well-known technical result.

Lemma 1. *Let $\varepsilon > 0$ and $f \in C_{rc}(X,E)$. Then, there exist disjoint clopen sets $A_1, A_2, ..., A_n$ covering X and elements $e_1, e_2, ..., e_n$ of E such that*

$$\left\| f - \sum_{i=1}^{n} \mathcal{X}_{A_i} e_i \right\| \leq \varepsilon,$$

where $\mathcal{X}_{A_i} e_i\,(x) = e_i$, if $x \in A_i$ and the null element θ of E otherwise.

2. The space $\mathcal{L}(C_{rc}(X,E),F)$.

In this section we will study the relation between measure theory and $F-$valued continuous linear operators on $C_{rc}(X,E)$. We will denote by $\mathcal{L}(C_{rc}(X,E),F)$ the space of all these operators.

Theorem 2 : If $f \in C_{rc}(X,E)$ and $m \in M(X,\mathcal{L}(E,F))$, then f is m-integrable over A, for each $A \in \mathcal{S}(X)$.

Proof. Without loss of generality, we can assume that $A = X$ and $\|m\|(X) \leq 1$. Let $\mu \in \mathbf{K}$, with $0 < |\mu| \leq 1$, and $\varepsilon > 0$. We take $\nu \in \mathbf{K}$ such that $0 < |\nu| < |\mu|\,\varepsilon$, Since $f \in C_{rc}(X,E)$, there exists a finite clopen partition $\{A_1, A_2, ..., A_n\}$ of X and finite subset $\{e_1, e_2, ..., e_n\}$ of E such that

$$\|f(x) - e_i\|_E \leq |\nu| \ , \forall x \in A_i$$

Choose $x_i \in A_i$ and consider $\alpha_\circ = \{A_1, A_2, ..., A_n; x_1, x_2, ..., x_n\}$. Take $\alpha = \{B_1, B_2, ..., B_k; y_1, y_2, ..., y_k\}$ such that $\alpha \geq \alpha_\circ$. Then, for $B_j \subset A_i$, we have

$$\begin{aligned} \|f(y_j) - f(x_i)\|_E &= \|f(y_j) - e_i + e_i - f(x_i)\|_E \\ &\leq \max\{\|f(y_j) - e_i\|_E, \|e_i - f(x_i)\|_E\} \\ &\leq |\nu| \end{aligned}$$

Thus, if $\lambda = \nu^{-1}\mu$, then

$$\|\lambda\,[f(y_j) - f(x_i)]\|_E \leq |\lambda|\,|\nu| \leq |\mu| \leq 1$$

and then

$$\|m(A)(\lambda\,[f(y_j) - f(x_i)])\| \leq 1$$

which implies

$$\|m(A)(f(y_j) - f(x_i))\|_F \leq |\nu|\,|\mu|^{-1} \leq \varepsilon.$$

Therefore,

$$\begin{aligned} \|\omega_\alpha(f,m) - \omega_{\alpha_\circ}(f,m)\|_F &= \left\| \sum_{j=1}^{N} m(A_j)\, f(y_i) - \sum_{i=1}^{k} m(B_i)\, f(x_i) \right\|_F \\ &\leq \max_{1\leq j\leq N} \|m(A_j)(f(y_i) - f(x_i))\|_F \\ &\leq \varepsilon \end{aligned}$$

Now, if $\alpha_1, \alpha_2 \geq \alpha_\circ$, then

$\|\omega_{\alpha_1}(f,m) - \omega_{\alpha_2}(f,m)\|_F = \|\omega_{\alpha_1}(f,m) - \omega_{\alpha_\circ}(f,m) + \omega_{\alpha_\circ}(f,m) - \omega_{\alpha_2}(f,m)\|_F$
$\leq \max\{\|\omega_{\alpha_1}(f,m) - \omega_{\alpha_0}(f,m)\|_F \ , \|\omega_{\alpha_0}(f,m) - \omega_{\alpha_2}(f,m)\|_F\} \leq \varepsilon.$

This proves that the net $\{\omega_\alpha(f,m)\}$ is Cauchy in F and hence convergent, since F is complete.

Lemma 3 : Let $f : X \to E$ be a m-integrable function over $A \in \mathcal{S}(X)$.

1. If the valuation of $\mathbf{K}$ is dense or the valuation is discrete and $\|E\| \subset |\mathbf{K}|$, then

$$\left\| \int_A f dm \right\| \leq \|f\|_A \|m\| (A).$$

2. If the valuation of $\mathbf{K}$ is discrete and if $\rho > 1$ is the generator of $|\mathbf{K}| \setminus \{0\}$, then

$$\left\| \int_A f dm \right\| \leq \rho \|f\|_A \|m\| (A).$$

Proof. 1.) We can assume that $0 < \|f\|_A < \infty$, since otherwise, the statement is trivial. Under the denseness conditions, for each $\varepsilon > 0$, there exists $\lambda \in \mathbf{K}$ such that

$$\|f\|_A \leq \lambda < \|f\|_A + \varepsilon.$$

Since $\|\lambda^{-1} f(x)\| \leq 1$, $\forall\, x \in A$, we have

$$\begin{aligned} |\lambda|^{-1} \|\omega_\alpha(f, m)\| &= \|\omega_\alpha(\lambda^{-1} f, m)\| \\ &= \left\| \sum_{i=1}^{n} m(A_i) \lambda^{-1} f(x_i) \right\| \\ &\leq \|m\| (A) \end{aligned}$$

for each $\alpha \in \Omega_A$. It follows that

$$\left\| \int_A f dm \right\| \leq \|f\|_A \|m\| (A).$$

2.) If we consider the norm $\|e\|^* = \inf \{|\lambda| : \lambda \in \mathbf{K},\ |\lambda| \geq \|e\|\}$ on E, then $\|E\|^* \subset |\mathbf{K}|$ and $\|\cdot\|^* \leq \rho \|\cdot\|$. Therefore, from 1.), we have

$$\left\| \int_A f dm \right\| \leq \|f\|_A^* \|m\|^* (A) \leq \rho \|f\|_A \|m\| (A).$$

Remark 4 : The previous lemma proves that if $m \in M(X, \mathcal{L}(E, F))$, then the linear operator $T_m : C_{rc}(X, E) \to F$ defined by $T_m(f) = \int f dm$ is a $\mathcal{T}_u-$continuous linear operator, where $\mathcal{T}_u$ denotes the uniform convergence topology on $C_{rc}(X, E)$.

Theorem 5 : If $T : C_{rc}(X, E) \to F$ is a $\mathcal{T}_u-$continuous linear operator, then there exists $m \in M(X, \mathcal{L}(E, F))$ such that $T = T_m$.

Proof. For each $A \in \mathcal{S}(X)$, we define

$$\begin{array}{rll} m(A): & E & \to F \\ & e & \mapsto m(A)e = T(\mathcal{X}_A e) \end{array}.$$

Since T is bounded, with bound $M > 0$, we have

$$\begin{array}{ll} \|m(A)e\| & = \|T(\mathcal{X}_A e)\| \\ & \leq M \|\mathcal{X}_A e\| \\ & \leq M \|e\| \end{array},$$

that is, $m(A) \in \mathcal{L}(E, F)$. We claim that the set-function

$$\begin{array}{rll} m: & \mathcal{S}(X) & \to \mathcal{L}(E, F) \\ & A & \mapsto m(A) \end{array}$$

is a measure. In fact, trivially m is well-define and finitely additive. To prove that $\{m(A) : A \in \mathcal{S}(X)\}$ is equicontinuous, take $\varepsilon > 0$ and choose $\delta = \varepsilon / M$; hence,

$$(\forall A \in \mathcal{S}(X))\,(\|e\| \leq \delta \Rightarrow \|m(A)e\| \leq M \|e\| \leq M\delta = \varepsilon)\,.$$

Finally, we claim that $T = T_m$. In fact, if $f = \sum\limits_{i=1}^{n} \mathcal{X}_{A_i} e_i$, then it is immediately to prove $T_m(f) = T(f)$. On the other hand, by the facts that $\langle\{\mathcal{X}_A e : e \in E, A \in \mathcal{S}(X)\}\rangle$ is $\mathcal{T}_u-$dense in $C_{rc}(X, E)$ (see Lemma 1) and both T_m and T are $\mathcal{T}_u-$continuous, we get $T_m(f) = T(f), \forall\, f \in C_{rc}(X, E)$.

Corollary 6 : The mapping

$$\begin{array}{rll} \Psi: & M(X, \mathcal{L}(E;F)) & \to \mathcal{L}(C_{rc}(X, E), F) \\ & m & \longmapsto \Psi(m) = T_m \end{array}$$

is an algebraic isomorphism.

3. τ and u-additive measures.

This section will devote to study certain class of members of $M(X, \mathcal{L}(E, F))$ and study the behavior of the associated $F-$valued continuous linear operators given in the previous section.

Definition 7 : Let $m \in M(X, \mathcal{L}(E, F))$. We will say that

1. m is τ-additive if for each decreasing net $\{A_\alpha\}_{\alpha\in I}$ in $\mathcal{S}(X)$ such that $A_\alpha \downarrow \emptyset$, we have
$$\|m\|(A_\alpha) \to 0.$$

2. m is u-additive if for each clopen partition $\{U_\alpha\}_{\alpha\in I}$ of X, we have
$$\|m\|\left(X \setminus \bigcup_{j\in J} U_j\right) \to 0$$
where the limit has to be taken over the directed set of all finite subsets $J \subset I$.

Proposition 8 : $M_\tau(X,\mathcal{L}(E,F)) \subseteq M_u(X,\mathcal{L}(E,F))$

Proof. Let $m \in M_\tau(X, L(E,F))$ and $(U_\alpha)_{\alpha\in I}$ be a clopen partition of X. For any finite subset J of I, we define the decreasing net $\{A_J\}_J$, where $A_J = X \setminus \bigcup_{j\in J} U_j$. Now, since $(U_\alpha)_{\alpha\in I}$ is a clopen partition of X, we have that $A_j \downarrow \emptyset$; therefore, $\|m\|(A_J) = \|m\|(X \setminus \bigcup_{j\in J} \mathcal{U}_j) \to 0$. Therefore, $m \in M_u(X,\mathcal{L}(E,F))$.

In the previous section we proved that if $f \in C_{rc}(X,E)$ and $m \in M(X,\mathcal{L}(E,F))$, then f is $m-$integrable over any $A \in \mathcal{S}(X)$. The next theorem will extend this result.

Theorem 9 : If $f \in C_b(X,E)$ and $m \in M_u(X,\mathcal{L}(E,F))$, then f is m-integrable over A, for each $A \in \mathcal{S}(X)$.

Proof. Without loss of generality, we can assume that $\|f\| \le 1$, and $\|m\|(X) \le 1$. For a given $\varepsilon > 0$, we define the following equivalence relation
$$x \sim y \Leftrightarrow \|f(x) - f(y)\| \le \varepsilon$$

Note that the corresponding equivalent classes $\{A_i\}_{i\in I}$ form a clopen partition of X. Let us choose $x_i \in A_i$ and define $g = \sum_{i\in I} \mathcal{X}_{A_i} f(x_i)$. Clearly, $g \in C_b(X,E)$. For a finite subsets J of I, we denote by $B_J = X \setminus \bigcup_{j\in J} \cup A_j$.

Each B_J is clopen and $B_J \downarrow \emptyset$. Since m is $u-$ additive, we have that there exists a finite subset $J_\circ$ of I such that if J is another finite subset of I with $J_\circ \subseteq J$, then $\|m\|(B_J) \leq \varepsilon$. For such a J, we define the following functions

$$g_J = \sum_{j \in J} \mathcal{X}_{A_j} f(x_j)$$

$$h_J = g - g_J = \sum_{i \notin J} \mathcal{X}_{A_i} f(x_i)$$

Let us consider the finite clopen partition $\{A_j : j \in J\} \cup \{B_J\}$ of X and take $\{D_1, , ...D_n\}$ a refinement of $\{A_j : j \in J\} \cup \{B_J\}_j$. If we choose $y_i \in D_i$, then $g_J(y_i) = f(x_j)$ and $h_J(y_i) = 0$, if $D_i \subseteq A_j$ for some $j \in J$, or $g_J(y_i) = 0$ and $h_J(y_i) = f(x_j)$, for some $j \notin J$, if $D_i \subset B_J$. Therefore, if we denote by $\alpha = \{D_1, D_2, ...D_n; y_1, \cdots, y_n\}$ and $\alpha_\circ = \{A_1, ...A_m, B_J; x_1, \cdots x_m, x_J\}$, then

$$\|\varpi_\alpha(h, m)\| = \left\| \sum_{D_K \subseteq A_j} m(D_k)(h_J(y_k)) + \sum_{D_k \subset B_J} m(D_k)(f(x_i)) \right\|$$
$$= \left\| \sum_{D_k \subset B_J} m(D_k)(f(x_i)) \right\|$$
$$\leq \max \{\|m(D_k)(f(x_i))\| : D_k \subseteq B_J\}$$
$$\leq \|m\|(B_J) \leq \varepsilon.$$

On the other hand,

$$\begin{aligned} \varpi_\alpha(g_J, m) &= \sum_{D_k \subset A_j} m(D_k)(f(x_j)) \\ &= \sum_{j \in J} m(A_j)(f(x_j)) \end{aligned}$$

Therefore,

$$\begin{aligned} &\left\| \varpi_\alpha(g, m) - \sum_{j \in J} m(A_j)(f(x_j)) \right\| \\ &= \left\| \sum_{k=1}^{n} m(D_k)(g(y_k)) - \sum_{k=1}^{n} m(D_k)(g_J(y_k)) \right\| \\ &= \|\varpi_\alpha(g - g_J, m)\| \\ &= \|\varpi_\alpha(h_J, m)\| \leq \varepsilon. \end{aligned}$$

Also, since $\|f(y_k) - f(x_i)\| \leq \varepsilon$,

$$\begin{aligned} \|\varpi_\alpha(f-g,m)\| &= \left\| \sum_{k=1}^{n} m(D_k)(f(y_k) - f(x_i)) \right\| \\ &\leq \max\{\|m(D_k)(f(y_k) - f(x_i))\|\,; k = 1, ..., n\} \\ &\leq \max\{\|m(D_k)\|\,\|f(y_k) - f(x_i)\|\,; k = 1, ..., n\} \\ &\leq \varepsilon \max\{\|m(D_k)\|\,; k = 1, ..., n\} \end{aligned}$$

Thus, $\left\| \varpi_\alpha(f,m) - \sum_{j\in J} m(A_j)(f(x_j)) \right\|$

$$= \left\| \varpi_\alpha(f,m) - \varpi_\alpha(g,m) \ + \varpi_\alpha(g,m) - \sum_{j\in J} m(A_j)(f(x_j)) \right\|$$

$$= \max\left\{ \|\varpi_\alpha(f,m) - \varpi_\alpha(g,m)\|, \left\| \varpi_\alpha(g,m) - \sum_{j\in J} m(A_j)(f(x_j)) \right\| \right\}$$

$$\leq \varepsilon$$

Now, if $\{G_1, G_2, ..., G_s\}$ is another refinement of $\{A_j : j \in J\} \cup B_J$ and $\beta = \{G_1, G_2, ..., G_s; z_1, \cdots z_s\}$, then

$$\|\varpi_\alpha(f,m) - \varpi_\beta(f,m)\|$$

$$= \|\varpi_\alpha(f,m) - \sum m(A_j)(f(x_j)) + \sum m(A_j)(f(x_j)) - \varpi_\beta(f,m)\|$$

$$= \max\{\|\varpi_\alpha(f,m) - \sum m(A_j)(f(x_j))\|, \|\sum m(A_j)(f(x_j)) - \varpi_\beta(f,m)\|\}$$

$$\leq \varepsilon$$

Therefore, $\{\omega_\alpha(f,m)\}$ is a Cauchy net in F, which is convergent since F is Banach. This proves that f is m -integrable over A.

Remark 10 : From Prop. 8, if m is $\tau-$additive measure and $f \in C_b(X,E)$, then f is m-integrable over A. In $[6]$, it has been proved that $M_t(X, \mathcal{L}(E,F))$ and the space of all $F-$valued and β_o-continuous linear operators on $C_b(X,E)$ are algebraically isomorphic. The next theorems will show similar results for $M_\tau(X, \mathcal{L}(E,F))$ and $M_u(X, \mathcal{L}(E,F))$.

Theorem 11 : If $T \in \mathcal{L}(C_b(X,E), F)$, then the following statements are equivalent:

1. T is β-continuous.

2. The associated measure m is $\tau-$additive.

Proof. 1.) $\Rightarrow$ 2.) Let $\{A_\alpha\}_{\alpha\in I}$ be a net of clopen subsets of X such that $A_\alpha \downarrow \emptyset$. By the continuity of T,

$$\mathcal{W} = \{f \in C_b(X,E) : \|T(f)\| \le 1\}$$

is a β-neighborhood of 0, and by the definition of β, $\mathcal{W}$ is a β_K-neighborhood of 0, for all $K \in \Omega$. Now, since $\overline{A_\alpha}^{\beta_\circ X} \downarrow Q \in \Omega$, there exists $h \in C_Q(X)$ such that

$$\mathcal{U} = \{f \in C_b(X,E) : \|f\|_h \le 1\} \subset \mathcal{W}.$$

For a given $\varepsilon > 0$, we choose $\mu \in \mathbf{K}$, with $0 < |\mu| < \varepsilon$ and define

$$G = \left\{x \in \beta_o X : \left|\widehat{h}(x)\right| \le |\mu|\right\}.$$

Note that $Q \subset G$; hence, there exists $\alpha_\circ \in I$ such that $\overline{A_{\alpha_\circ}}^{\beta_\circ X} \subset G$. Now, if $\alpha \ge \alpha_\circ$ and $\|e\| \le 1$, then for any $A \subset A_\alpha,\ A \in \mathcal{S}(X)$,

$\left\|\mu^{-1}\mathcal{X}_A e\right\|_h = \sup_{x\in A} |h(x)|\,|\mu|^{-1}\,\|e\|$

$\le |\mu|^{-1} \sup_{x\in A} |h(x)|$

$\le |\mu|^{-1}\,|\mu| = 1,$

that is, $\mu^{-1}\mathcal{X}_A e \in \mathcal{U}$, and then $\left\|T(\mu^{-1}\mathcal{X}_A e)\right\| \le 1$ or equivalently

$$\|m(A)e\| \le |\mu| \le \varepsilon.$$

Therefore, since e and A are arbitrary, we have $\|m\|(A) \le \varepsilon$.

therefore $m \in M_\tau(X, \mathcal{L}(E,F))$.

2.) $\Rightarrow$ 1.) We will prove that

$$\mathcal{W} = \{f \in C_b(X,E) : \|Tf\| \le 1\}$$

is a β-neighborhood of 0. Let us take $d > 0$ and choose $\lambda, \gamma \in \mathbf{K}$ such that $|\lambda| \ge d$ and $|\gamma|\,\|m\|(X) \le 1$. Let $Q \in \Omega$; hence, there exists a net $\{B_\alpha\}_{\alpha\in I}$

in $\beta_\circ X$ with $B_\alpha \downarrow Q$. Now, if $A_\alpha = B_\alpha \cap X$, then $\{A_\alpha\}_{\alpha \in I}$ is net of clopen subsets of X with $A_\alpha \downarrow \emptyset$. Thus, there exists $\alpha_\circ$ such that

$$\alpha \geq \alpha_\circ \Rightarrow \|m\|(A_\alpha) \leq |\lambda|^{-1}.$$

The clopen subset $D = X \setminus A_{\alpha_\circ}$ of X satisfies $\overline{D}^{\beta_o X} \cap Q = \emptyset$. We claim that the β_Q-neighborhood of 0

$$\mathcal{U} = \{f \in C_b(X,E) : \|f\| \leq d \wedge \|f\|_D \leq |\gamma|\}$$

is contained in $\mathcal{W}$. In fact, if $f \in \mathcal{U}$, then

$$\begin{aligned} \|T(f)\| &= \|\textstyle\int f dm\| \\ &= \left\|\textstyle\int_D f dm + \int_{A_{\alpha_\circ}} f dm\right\|. \end{aligned}$$

Now,

$$\begin{aligned} \left\|\textstyle\int_{A_{\alpha_\circ}} f dm\right\| &\leq \|f\| \, \|m\| (A_{\alpha_\circ}) \\ &\leq |\lambda| \, |\lambda|^{-1} = 1 \end{aligned}$$

and

$$\begin{aligned} \|\textstyle\int_D f dm\| &\leq \|f\|_D \, \|m\| (D) \\ &\leq |\gamma| \, \|m\| (D) \\ &\leq |\gamma| \, \|m\| (X) \\ &\leq |\gamma| \, |\gamma|^{-1} = 1 \end{aligned},$$

therefore $f \in \mathcal{W}$.

Theorem 12 : If $m \in M_\tau(X, L(E,F))$, then the linear operator

$$\begin{array}{lll} T_m : & C_b(X,E) & \to F \\ & f & \mapsto T_m(f) = \int f dm \end{array}$$

is β-continuous.

Proof. The same arguments used in the previous theorem proves this statement and then we omit the proof.

Theorem 13 :$M_\tau(X, \mathcal{L}(E,F))$ and the space of all $F-$valued and β-continuous linear operators on $C_b(X,E)$ are algebraically isomorphic.

Proof. In order to prove this theorem, we need to prove that the linear map

$$\begin{aligned} \Psi : \quad M_\tau(X, \mathcal{L}(E,F)) \quad &\to \mathcal{L}_\beta(C_b(X,E),F)) \\ m \quad &\mapsto \Psi(m) = T_m \end{aligned}$$

is an algebraic isomorphism, where $\mathcal{L}_\beta(C_b(X,E),F))$ denotes the space of all β-continuous linear operators.

It is easy to see that the map Ψ is linear and one to one. To prove that Ψ is onto, take $T : C_b(X,E) \to F$ and prove that $T = T_m$ for the associated measures m.

By Th. 11, $m \in M_\tau(X, \mathcal{L}(E,F))$ and $T = T_m$ on $C_b(X,E)$ follows from the $\beta-$denseness of $C_{rc}(X,E)$ in $C_b(X,E)$ and the $\beta-$continuity of both T and T_m.

Theorem 14 : If $m \in M_u\left(X, \mathcal{L}\left(E,F\right)\right)$, then T_m is β_u-continuous.

Proof. Let $\mathcal{W} = \{f \in C_b(X,E) : \|T_m(f)\| \leq 1\}$ and $Q \in \Omega_u$. There exists a clopen partition $\{A_i\}_{i\in I}$ of X such that $Q \cap \overline{A_i}^{\beta_\circ X} = \emptyset$, $\forall i \in I$.

For any finite subset J of I, we define $B_J = X \setminus \bigcup\limits_{i\in J} A_i$. Since $m \in M_u\left(X, \mathcal{L}\left(E,F\right)\right)$, there exists a finite subset $J_\circ$ of I such that for a given $\delta > 0$, we have

$$\|m\|\left(B_J\right) \leq \delta^{-1},$$

for any finite subset J of I with $J_\circ \subset J$. If $B = i \in J_\circ \cup A_i$, then B is clopen in X and $\overline{B}^{\beta_\circ X} \cap Q = \emptyset$.

We claim that $\mathcal{U} = \{f \in C_b(X,E) : \|f\| \leq \delta, \|f\|_B \leq 1\} \subset \mathcal{W}$. In fact, if $f \in C_b(X,E)$, then

$$\left\| \int_B f \ dm \right\| \leq \|f\|_B \, \|m\| \, (B) \leq \|f\|_B \, \|m\| \, (X) \leq 1.$$

On the other hand,

$$\left\| \int_{X\setminus B} f \ dm \right\| \leq \|f\| \, \|m\| \, (B_J) \leq \delta\delta^{-1} = 1$$

Thus,

$$\|T_m\,(f)\| \leq \max\left\{ \left| \int_B f \ dm \right|, \left| \int_{X\setminus B} f \ dm \right| \right\} \leq 1$$

Therefore, since $Q \in \Omega_u$ is arbitrary, the continuity of T_m follows.

Theorem 15 : $M_u(X, L(E,F))$ and the space of all $F-$valued and β_u-continuous linear operators on $C_b(X,E)$ are algebraically isomorphic.

Proof. As before, we only have to prove that if T is a $F-$valued and β_u-continuous linear operators on $C_b(X,E)$, then $T=T_m$.
Let T be given and m be the associated measure, that is, $T(f)=\int f\,dm$, $f\in C_{rc}(X,E)$. We need to prove first that m is u-additive.
Let $\{A_i\}_{i\in I}$ be a clopen partition of X and $Q=\beta_\circ X\setminus\bigcup_{i\in I}\overline{A_i}^{\beta_\circ X}$. Clearly, $Q\in\Omega_u$ and then T is β_Q-continuous. Thus, there exists $h\in C_Q$ such that

$$\mathcal{V}_1=\{f\in C_b(X,E):\|f\|_h\le 1\}\subset\mathcal{W}=\{f\in C_b(X,E):\|T(f)\|\le 1\}.$$

Take $\varepsilon>0$ and choose $\lambda\in\mathbf{K}$ with $0<|\lambda|\le\varepsilon$. The set

$$A=\left\{x\in\beta_\circ X:\left|\widehat{h}(x)\right|\le|\lambda|\right\}$$

is clopen in $\beta_\circ X$, $Q\subset\beta_\circ$ and $X\backslash A\subset\bigcup_{i\in I}\overline{A_i}^{\beta_\circ X}$. Now, by the compactness of $\beta_o X\backslash A$, there exists a finite subset $J_\circ$ of I such that $\beta_\circ X\backslash A\subset\bigcup_{i\in J_\circ}\cup\overline{A_i}^{\beta_\circ X}$.

Takes a finite subset $J\subset I$ with $J_\circ\subset J$, and take any clopen B of X contained in $X\setminus\bigcup_{i\in J}A_i$. If $e\in E$ with $\|e\|\le 1$, then it is easy to see that $\lambda^{-1}\mathcal{X}_B e\in\mathcal{V}_1$. Therefore, $\|m(B)e\|\le\varepsilon$ and then $\|m\|\left(X\setminus\bigcup_{i\in J}A_i\le\varepsilon\right.$. This proves $m\in M_u(X,\mathcal{L}(E,F))$.

Finally, $T=T_m$ on $C_b(X,E)$ follows from the β_u-denseness of $C_{rc}(X,E)$ in $C_b(X,E)$ and the β_u-continuity of both T and T_m. $\square$

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