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ON SOME FUNCTIONS CONCERNING FUZZY PG-CLOSED SETS

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Abstract

In this paper we consider new weak and strong forms of fuzzy preirresoluteness and fuzzy pre-closureness via the concept of fuzzy pg-closed sets which we call ap-Fp-irresolute functions, ap-Fp-closed functions and contra fuzzy pre-irresolute functions and we use it to obtain a characterization of fuzzy pre-T_1/2 spaces.

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1. Introduction and Preliminaries

In 1991 A. S. Binshahan [4] introduced and investigated the notions of fuzzy preopen and fuzzy preclosed sets. Since then the advent of these notions, several research papers with interesting results in di®erent respects came to existence ([1],[3],[5],[6],[7],[10,[12]).

In this paper we shall introduce new generalizations of fuzzy pre-closedness called ap-fuzzy pre-closedness by using fuzzy pg-closed sets and study some of their basic properties. This de⁻nition enables us to obtain conditions under which functions and inverse functions preserve fuzzy pg-closed sets, also in this paper we present a new generalization of fuzzy pre-irresoluteness called contra fuzzy pre-irresolute. We de⁻ne this last class of functions by the requirement that the inverse image of each fuzzy pre-open set in the co-domain is fuzzy preclosed in the domain. This notion is a stronger form of ap-fuzzy pre-irresolute and ap-fuzzy preclosed functions. Finally, we also characterize the class of fuzzy pre- $T_{1/2}$ spaces in terms of ap-fuzzy pre-irresolute and ap-fuzzy preclosed functions.

Throughout this paper (X,τ) , (Y,σ) and (Z,γ) (or simply X,Y and Z) represent non-empty fuzzy topological spaces on which no separation axioms are assumed, unless otherwise mentioned. Let A be a fuzzy subset of X then Cl(A) and Int(A) denote the closure of A and the interior of A. respectively.

1.1 **DEFINITION**: The fuzzy set A of X is called fuzzy preopen [4] if $A \cdot IntCl(A)$.

The fuzzy subset B of X is called fuzzy preclosed if, its complement B^c is fuzzy preopen in X .

By FPO(X) (rep. FPC(X)), we denote the family of all fuzzy preopen(resp. fuzzy preclosed) sets of X.

- 1.2 DEFINITION: The intersection of all fuzzy preclosed sets containing A is called the preclosure of A and is denoted by pCl(A) [4].
- 1.3 DEFINITION: The fuzzy interior of A [4] is denoted by pInt(A), and is de⁻ned by the union of all fuzzy preopen sets of X which are contained in A.

- 1.4 DEFINITION: Recall that a fuzzy set A is called fuzzy regular open (resp. regular closed) if A = IntCl(A) (resp. A = ClInt(A))[2].
 - 1.5 **DEFINITION**: A function $f:(X,\tau)!$ (Y,σ) is said to be:
- (i) fuzzy preirresolute [8] if, $f^{i-1}(V)$ is fuzzy preopen set in (X, τ) for each fuzzy open set V of (Y, σ) ,
- (ii) fuzzy M-preclosed [11](resp.fuzzy M-preopen[11]) if, for every fuzzy preclosed (resp.fuzzy preopen) set B of (X,τ) , f(B) is fuzzy preclosed(resp. fuzzy preopen) set in (Y,σ) ,
- (iii) fuzzy contra preclosed if f(A) is fuzzy preopen set in (Y, σ) for every fuzzy closed set A of $(X, \tau)[9]$.

ap-Fp-IRRESOLUTE AND ap-Fp-CLOSED FUNCTIONS

- 2.1 **DEFINITION**: A fuzzy set A of (X,τ) is said to be fuzzy pregeneralized closed (brie°y,Fpg-closed) if, $pCl(A) \cdot H$ whenever $A \cdot H$ and H is fuzzy preopen in X. A fuzzy set B is said to be fuzzy pre-generalized open (brie°y, Fpg-open) in X if its complement B^c is Fpg-closed set in X.
- 2.2 REMARK: (i) Every fuzzy preclosed set is Fpg-closed but not conversely.
- (ii) The concept of fuzzy g-closed sets and Fpg-closed sets are independent. For,
- **2.3 EXAMPLE**: Let X = fa, bg, fuzzy sets A, B, E, H and K are de-ned as:

$$A(a) = 0.3$$
, $A(b) = 0.4$;
 $B(a) = 0.6$, $B(b) = 0.5$;
 $E(a) = 0.2$, $E(b) = 0.2$;
 $H(a) = 0.3$, $H(b) = 0.7$;
 $K(a) = 0.3$, $K(b) = 0.8$.

Let $\tau = f0, A, 1g$ and $\sigma = f0, H, 1g$. Then,

- (i) B is fuzzy pg-closed but not fuzzy preclosed set in (X, τ) .
- (ii) E is fuzzy pg-closed but not fuzzy g-closed in (X, τ) and K is fuzzy g-closed but not fuzzy pg-closed set in (X, σ) .

- 2.4 **DEFINITION**: A function $f:(X,\tau)$! (Y,σ) is said to be approximately fuzzy preirresolute (brie°y, ap-Fp- irresolute) if, $pCl(A) \cdot f^{i-1}(H)$ whenever H is a fuzzy preopen subset of (Y,σ) , A is a fuzzy pg-closed subset of (X,τ) and $A \cdot f^{i-1}(H)$.
- 2.5 **DEFINITION**: A function $f:(X,\tau)$! (Y,σ) is said to be approximately fuzzy preclosed (brie°y, ap-Fp-closed)if $_{,}f(B) \cdot _{pInt}(A)$ whenever A is a fuzzy pg-closed subset of $(Y,\sigma)_{,}B$ is a fuzzy preclosed subset of (X,τ) and $f(B) \cdot _{A}$.
- 2.6 THEOREM : (i)A function $f:(X,\tau)$! (Y,σ) is an ap-Fp-irresolute if, $f^{i-1}(A)$ is fuzzy preclosed in (X,τ) for every $A \supseteq FPO(Y)$.
- (ii) $f:(X,\tau)$! (Y,σ) is ap-Fp-closed if, f(B) 2 FPO(Y) for every fuzzy preclosed subset B of (X,τ) .
- PROOF : (i) Let $H \cdot f^{i}(A)$ where $A \ 2 \ FPO(Y)$ and H is a fuzzy pg-closed subset of (X, τ) . Therefore $pCl(H) \cdot pCl(f^{i}(A)) = f^{i}(A)$. Thus, f is ap-Fp-irresolute.
- (ii) Let $f(B) \cdot A$, where B is a fuzzy preclosed subset of (X, τ) and A a fuzzy pg-open subset of (Y, σ) . Therfore, pInt(f(B)) · pInt(A).Then, f(B) · pInt(A).Thus, fisap ¡ Fp ¡ closed.

Clearly, fuzzy preirresolute functions are ap-Fp-irresolute. Also, fuzzy M-preclosed functions are ap-Fp- closed. The converse implication do not hold as it is shown in the following example.

2.7 **EXAMPLE**: Let X = fa, bg and Y = fx, yg. Fuzzy sets E,A,B and H be de⁻ned as:

$$E(a) = 0$$
, $E(b) = 0.8$;
 $A(a) = 0.3$, $A(b) = 0.4$;
 $H(x) = 0.3$, $H(y) = 0.4$;
 $B(x) = 0$, $B(y) = 0.8$.

Let $\tau=\mathsf{f0},E.1\mathsf{g}$ and $\sigma=\mathsf{f0},H.1\mathsf{g}$, $_{\mathsf{i}}=\mathsf{f0},A.1\mathsf{g}$ and $\nu=\mathsf{f0},B.1\mathsf{g}$. Then, the function $f:(X,\tau)$! (Y,σ) de ned by $\mathsf{f(a)}=\mathsf{x}$, $\mathsf{f(b)}=\mathsf{y}$ is ap-Fp-irresolute but not fuzzy preirresolute since $\mathsf{Z}(\mathsf{x})=\mathsf{0.1}$ and $\mathsf{Z}(\mathsf{y})=\mathsf{0.1}$ are fuzzy preopen in Y but $f^{\mathsf{i}}(Z)$ is not fuzzy preopen in (X,τ) , and the function $h:(X,\mathsf{j})$! (Y,ν) de ned by $\mathsf{h(a)}=\mathsf{x}$, $\mathsf{h(b)}=\mathsf{y}$, is ap-Fp-closed but not fuzzy M-preclosed.

- 2.8 REMARK: Converse of the Theorem 2.6 do not hold for .
- 2.9 EXAMPLE: The function f de ned in Example 2.3 is ap-Fp-irresolute. Fuzzy set K(x) = 0.9, K(y) = 0.9 are fuzzy preopenin Y but find (X) is not fuzzy preclosed in (X,τ) . The function function h de ned in Example 2.7 is ap-Fp-closed and the fuzzy set W(a) = 0.1, W(b) = 0.1 is fuzzy preclosed set in (X,τ) , but h(W) is not fuzzy preopen set in (Y,ν) .

In the following theorem, we get under certain conditions the converse of the Theorem 2.6 is true.

- 2.10 THEOREM : Let $f:(X,\tau)$! (Y,σ) be a function. Then,
- (i) Let all fuzzy subsetsof (X,τ) be fuzzy clopen, then f is ap-Fp-irresolute if and only if, $f^{\text{i-1}}(H)$ is fuzzy preclosed in (X,τ) for every H 2 FPO(Y).
- (ii)Let all fuzzy subsets of (Y, σ) be fuzzy clopen, then f is ap-Fp-closed if and only if, f(B) 2 FPO(Y) for every fuzzy preclosed subset B of (X, τ) .
- PROOF: (i) Assume f is ap-Fp-irresolute. Let A be an arbitrary fuzzy subset of (X,τ) such that $A\cdot Q$ where Q 2 FPO(X). Then by hypothesis $pCl(A)\cdot pCl(Q)=Q$. Therefore all fuzzy subsets of (X,τ) are fuzzy pg-closed (and hence all are fuzzy pg-open). So for any H 2 FPO(Y), $f^{i-1}(H)$ is fuzy pg-closed in X . Since f is ap-Fp-irresolute, $pCl(f^{i-1}(H))\cdot f^{i-1}(H)$. Therefore, $pCl(f^{i-1}(H))=f^{i-1}(H)$, i.e., $f^{i-1}(H)$ is fuzzy preclosed in (X,τ) .

The converse is clear by Theorem 2.6.

(ii) Assume f is ap-Fp-closed. Reasoning as in (i), we obtain that all fuzzy subsets of (Y, σ) are fuzzy pg-open. Therefore, for any fuzzy preclosed subset B of (X, τ) , f(B) is fuzzy pg-open in Y. Since f is ap-Fp-closed, $f(B) \cdot pInt(f(B))$. Therefore, f(B) = pInt(f(B)), i.e., f(B) is fuzzy preopen.

The converse is clear by Theorem 2.6.

As an immediate consequence of Theorem 2.10, we have the following. (ii)

2.11 COROLLARY : Let $f:(X,\tau)$! (Y,σ) be a function. Then,

- (i) Let all the fuzzy subsets of (X, τ) be fuzzy clopen, then f is ap-Fp-irresolute if and only if f is fuzzy preirresolute.
- (ii) Let all the fuzzy subsets of (Y, σ) be fuzzy clopen, the f is ap-Fp-closed if and only if f is fuzzy M-preclosed.

3. CONTRA FUZZY PRE-IRRESOLUTE FUNCTIONS

- 3.1 DEFINITION : A function $f:(X,\tau)$! (Y,σ) is called :
- (i) contra fuzzy preirresolute if f^{i} ¹(A) is fuzzy preclosed in (X, τ) for each $A \supseteq FPO(Y)$,
- (ii) contra fuzzy M-preclosed if f(B) 2 FPO(Y) for each fuzzy preclosed set B of (X, τ) .
- 3.2 REMARK: The concepts of contra fuzzy preirresolutenes and fuzzy preirresoluteness are independent notions. For,
- 3.3 EXAMPLE : Let X = fa, bg, Y = fx, yg and $Z = p, q.De^-ne$ fuzzy sets A,E and B as :

$$A(a) = 0.7$$
, $A(b) = 0.8$;

$$E(x) = 0.4$$
, $E(y) = 0.3$;

$$B(p) = 0.3 , B(q) = 0.3 .$$

Let $\tau=\mathsf{f0},A,\mathsf{1g},\ \mathsf{i}=\mathsf{f0},E,\mathsf{1g}$ and $\sigma=\mathsf{f0},B,\mathsf{1g}.\mathsf{Then}$ the function $f:(X,\tau)!$ (Y,i) de ned by $\mathsf{f(a)}=\mathsf{x}$, $\mathsf{f(b)}=\mathsf{y}$ is contra fuzzy preirresolute but not fuzzy preirresolute and the function $h:(Y,\mathsf{i})!$ (Z,σ) de ned by $\mathsf{h(x)}=\mathsf{p}$ and $\mathsf{h(y)}=\mathsf{q}$ is fuzzy preirresolute but not contra fuzzy preirresolute.

In this same manner one can prove that, contra fuzzy M-preclosed functions and fuzzy M-preclosed functions are indepedent notions.

- 3.4 DEFINITION : A function $f:(X,\tau)$! (Y,σ) is called contra fuzzy precontinuous if f_i 1(A) is fuzzy preclosed set in (X,τ) for each fuzzy open set A of (Y,σ) .
- 3.5 REMARK: Every contra fuzzy preirresolute function is contra fuzzy precontinuous, but not conversely. For,

3.6 EXAMPLE : Let $X=\mathsf{f} a,b\mathsf{g}$ and $Y=\mathsf{f} x,y\mathsf{g}$. Fuzzy sets A and B are de⁻ned as :

$$A(a) = 0.7$$
 , $A(b) = 0.8$; $B(x) = 0$, $B(y) = 0.6$.

Let $\tau = \text{f0}, A, \text{1g}$ and $\sigma = \text{f0}, B, \text{1g}$. Then the function $f:(X, \tau)$! (Y, σ) de⁻ned by f(a) = x, f(b) = y is contra fuzzy precontinuous but not contra fuzzy preirresolute.

The following result can be easily veri⁻ed. Its proof is straightforward.

- 3.7 THEOREM : Let $f:(X,\tau)$! (Y,σ) be a function. Then the following conditions are equivalent :
 - (i) f is contra fuzzy preirresolute,
- (ii) The inverse image of each fuzzy preclosed set in Y is fuzzy preopen in X.
- 3.8 REMARK: By Theorem 2.6, we have that every contra fuzzy preirresolute function is ap-Fp-irresolute and every contra fuzzy preclosed function is ap-Fp-closed, the converse implication do not hold (see Remark 2.8)

Now, we de ne the following.

3.9 DEFINITION : A function $f:(X,\tau)$! (Y,σ) is called perfectly contra fuzzy preirresolute if the inverse image of every fuzzy preopen set in Y is fuzzy pre-clopen in X

Every perfectly contra fuzzy preirresolute function is contra fuzzy preirresolute and fuzzy preirresolute. But the converse may not be true. For,

3.10 EXAMPLE: The functions f and h de ned in Example 3.3 are contra fuzzy preirresolute and fuzzy preirresolute respectively but not perfectly contra fuzzy preirresolute.

Clearly, the following diagram holds and none of its implications is reversible.

Where contra f.p.i = contra fuzzy preirresolute, ap-Fp-i.= ap-Fp-irresolute, perfectly contra f.p.i.= perfectly contra fuzzy preirresolute.

The following theorem is a decomposition of perfectly contra fuzzy preirresoluteness.

- 3.11 THEOREM : For a function $f:(X,\tau)$! (Y,σ) the following conditions are equivalent:
 - (i) f is perfectly contra fuzzy preirresolute.
 - (ii) f is both contra fuzzy preirresolute and fuzzy preirresolute.
- 3.12 THEOREM : If a function $f:(X,\tau)$! (Y,σ) is ap-Fp-irresolute and fuzzy M-preclosed, then for every Fpg-closed set A of (X,τ) , f(A) is Fpg-closed in (Y,σ) .

PROOF: Let A be a Fpg-closed subset of (X,τ) . Let $f(A) \cdot B$ where $B \ 2 \ FPO(Y)$. Then $A \cdot f^{\text{i-1}}(B)$ holds. Since f is ap-Fp-irresolute $pCl(A) \cdot f^{\text{i-1}}(B)$ and hence $f(pCl(A)) \cdot B$. Therefore, we have, $pCl(f(A)) \cdot pCl(f(pCl(A))) = f(pCl(A)) \cdot B$. Hence f(A) is fuzzy pg-closed in (Y,σ) .

However the following theorem holds. The easy proof of it is omitted.

- 3.13 THEOREM : A function $f:(X,\tau)$! (Y,σ) and $g:(Y,\sigma)$! (Z,γ)) be two functions such that $g\circ f:(X,\tau)$! (Z,γ) . Then,
- (i) gof is contra fuzzy preirresolute, if g is fuzzy preirresolute and f is contra fuzzy preirresolute.
- (ii) gof is contra fuzzy preirresolute, if g is contra fuzzy preirresolute and f is fuzzy preirresolute.

In an analogous way, we have the following.

- 3.14 THEOREM : Let function $f:(X,\tau)$! (Y,σ) and $g:(X,\tau)$! (Z,γ) be two functions such that $gof:(X,\tau)$! (Z,γ) . Then,
 - (i) gof is ap-Fp-closed, if f is fuzzy M-preclosed and g is ap-Fp-closed.
- (ii) gof is ap-Fp-closed, if f is ap-Fp-closed and g is fuzzy M-preopen and g^{i-1} preserves Fpg-closed sets.
- (iii) gof is ap-Fp-irresolute, if f is ap-Fp-irresolute and g is fuzzy preirresolute.

PROOF : (i) Suppose B is an arbitrary fuzzy preclosed subset in (X, τ) and A is a Fpg-open subset of (Z, γ) for which $(g \circ f)(B) \cdot A$. Then f(B)

is fuzzy preclosed in (Y, σ) because f is fuzzy M-preclosed. Since g is ap-Fp-closed, $g(f(B)) \cdot pInt(A)$. This implies that $g \circ f$ is ap-Fp-closed.

- (ii) Suppose B is an arbitrary fuzzy preclosed subset of (X,τ) and A is a fuzzy preopen subset of (Z,γ) for which $(gof)(B) \cdot A$. Hence $f(B) \cdot g^{i-1}(A)$. Then $f(B) \cdot pInt(g^{i-1}(A))$ since $g^{i-1}(A)$ is Fpg-open and f is ap-Fp-closed. Thus, $(gof)(B) = g(f(B)) \cdot g(pInt(g^{i-1}(A))) \cdot pInt(gg^{i-1}(A)) \cdot pInt(A)$. This shows that gof is ap-Fp-closed.
- (iii) Suppose H is an arbitrary Fpg-closed subset of (X, τ) and E 2 FPO(Z) for which $H \cdot (gof)^{\frac{1}{2}}(E)$. Then $g^{\frac{1}{2}}(E)$ 2 FPO(Y) since g is fuzzy preirresolute. Since f is ap-Fp-irresolute, $pCl(A) \cdot f^{\frac{1}{2}}(g^{\frac{1}{2}}(E)) = (gof)^{\frac{1}{2}}(E)$. This proved that gof is ap-Fp-irresolute.

In the following result we o®er a new characterization of the class of fuzzy pre- $T_{1/2}$ spaces by using the concepts of ap-Fp-irresolute functions and ap-Fp- closed functions.

- 3.15 DEFINITION : A fuzzy topological space (X, τ) is said to be fuzzy pre- T_i 1/2 space if every Fpg-closed set is fuzzy preclosed in it.
- 3.16 THEOREM : Let (X, τ) be a fuzzy topological space. Then the following statements are equivalent:
 - (i) (X, τ) is fuzzy pre-T; 1/2 space,
- (ii) For every space (Y, σ) and every function $f: (X, \tau) ! (Y, \sigma)$, f is ap-Fp-irresolute.
- PROOF: (i)) (ii) . Let E be a Fpg-closed subset of (X,τ) and suppose that $E \cdot f^{i-1}(H)$ where H 2 FPO(Y). Since (X,τ) is fuzzy pre- $T_{1/2}$ space, E is fuzzy preclosed (i.e., E = pCl(E)). Therefore, $pCl(E) \cdot f^{i-1}(H)$. Then f is ap-Fp-irresolute.
- (ii)) (i). Let B be a Fpg-closed subset of (X,τ) and let Y be the fuzzy set of X with the topology $\sigma=0,B,1$. Finally, let $f:(X,\tau)$! (Y,σ) be the identity function. By assumption f is ap-Fp-irresolute. Since B is Fpg-closed in (X,τ) and fuzzy preopen in (Y,σ) and $B \cdot f^{\text{i-1}}(B)$, it follows that $pCl(B) \cdot f^{\text{i-1}}(B) = B$. Hence B is fuzzy preclosed in (X,τ) and therefore, (X,τ) is a fuzzy pre- $T_{1/2}$ space.
- 3.17 THEOREM : Let (Y, σ) be a fuzzy topological space. Then the following statements are equivalent :

- (i) (Y, σ) is a fuzzy pre- $T_{1/2}$ space,
- (ii) For every space (X, τ) and every function $f: (X, \tau)$! (Y, σ) , f is ap-Fp-closed.

Proof: Analogous to Theorem 3.16 making the obvious changes.

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