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AN ABSTRACT ORLICS - PETTIS THEOREM AND APPLICATIONS

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Abstract

In this paper we establish two abstract versions of the classical Orlicz-Pettis Theorem for multiplier convergent series. We show that these abstract results yield known versions of the Orlicz-Pettis Theorem for locally convex spaces as well as versions for operator valued series. We also give applications to vector valued measures and spaces of continuous functions.

An Orlicz-Pettis theorem is a result which asserts that a series in a topological vector space which converges in some weak topology actually is convergent in some stronger topology. The original Orlicz-Pettis Theorem asserts that a series in a normed linear space which is subseries convergent in the weak topology is subseries convergent in the norm topology of the space ([O],[P]). In this note we present two Orlicz-Pettis results for multiplier convergent series in an abstract setting and then give applications to topics in vector valued measures, spaces of continuous functions and spaces of continuous linear operators. The abstract formulation allows the derivation of many known Orlicz-Pettis results from a single formulation. Let λ be a scalar sequence space which contains the space c_{00} of sequences which are eventually 0. The β-dual of λ is defined to be $\lambda^\beta = \{s = \{s_j\} : \sum_{j=1}^{\infty} t_j s_j = t \cdot s$ converges for every $t = \{t_j\} \in \lambda\}$ and the pair λ, λ^β form a dual pair under the bilinear pairing $(t, s) \to \sum_{j=1}^{\infty} t_j s_j = t \cdot s$. A series $\sum_j y_j$ in a topological vector space Y is λ multiplier convergent if the series $\sum_{j=1}^{\infty} t_j y_j$ converges in Y for every $t = \{t_j\} \in \lambda$; the elements of λ are called multipliers. For example, if m_0 is the space of all sequences with finite range, then a series $\sum_j y_j$ is m_0 multiplier convergent in Y iff the series is subseries convergent in Y. Thus, the Orlicz-Pettis Theorem can be viewed as result concerning multiplier convergent series, and we will establish a number of Orlicz-Pettis results for multiplier convergent series

In order to establish Orlicz-Pettis results for multiplier convergent series it is necessary to impose conditions on the multiplier space λ and we will now describe the condition which will be employed. An interval in $\mathbf{N}$ is a set of the form $[m, n] = \{j \in \mathbf{N} : m \leq j \leq n\}$, where $m \leq n$ and a sequence of intervals $\{I_j\}$ is increasing if $\max I_j < min I_{j+1}$. If $I \subset \mathbf{N}$,then χ_I will denote the characteristic function of I and if $t = \{t_j\}$ is any sequence, then $\chi_I t$ will denote the coordinatewise product of χ_I and t. The space λ has the signed weak gliding hump property (signed-WGHP) if whenever $t \in \lambda$ and $\{I_j\}$ is an increasing sequence of intervals, there exist a sequence of signs $\{s_j = \pm 1\}$ and a subsequence $\{n_j\}$ such that the coordinatewise sum of the series $\sum_{j=1}^{\infty} s_j \chi_{I_{n_j}} t \in \lambda$ ([St1],[St2],[Sw1]). If the signs above can all be chosen equal to 1 for every $t \in \lambda$, the space λ has the weak gliding hump property (WGHP) ([N]). For example, any monotone space such as c_{00}, c_0 or $l^p (1 < p \leq \infty)$ has WGHP while bs, the space of bounded series, has signed -WGHP but not WGHP ([St1],[St2],[Sw1]). See [BSS] for further examples.

We present several versions and applications of the Orlicz-Pettis Theorem in an abstract setting. We consider an abstract triple E, F, X where

E, F are vector spaces such that there is a bilinear mapping from $\cdot : E \times F \to X$, $(x, y) \to x \cdot y, x \in E, y \in F$, where X is a locally convex space. Of course, an example of this situation is when E, F are two vector spaces in duality and X is the scalar field; we give other examples in the applications which follow. Let $w(E, F)$ $[w(F, E)]$ be the weakest topology on E $[F]$ such that the linear maps $x \to x \cdot y$ $[y \to x \cdot y]$ from E into X $[F$ into $X]$ are continuous for all $y \in F$ $[x \in E]$. If E, F is a pair of vector spaces in duality, then $w(E, F)$ is just the weak topology $\sigma(E, F)$. A subset $K \subset F$ is said to be conditionally $w(F, E)$ sequentially compact if for every sequence $\{y_j\} \subset K$, there is a subsequence $\{y_{n_j}\}$ such that $\lim_j x \cdot y_{n_j}$ exists for every $x \in E$.

We give our first version of an Orlicz-Pettis theorem in this setting.

Theorem 1. *Let λ have signed-WGHP. If the series $\sum_j x_j$ is λ multiplier convergent in E with respect to $w(E, F)$, then for each $t \in \lambda$ and each conditionally $w(F, E)$ sequentially compact subset $K \subset F$, the series $\sum_{j=1}^{\infty} t_j x_j \cdot y$ converge uniformly for $y \in K$.*

Proof: If the conclusion fails to hold, there exists a neighborhood of 0, W, in X , $y_k \in K$ and an increasing sequence of intervals $\{I_k\}$ such that

$$(\#) \quad \sum_{l \in I_k} t_l x_l \cdot y_k \notin W$$

for every k. We may assume, by passing to a subsequence if necessary, that $\lim_k x \cdot y_k$ exists for every $x \in E$. Consider the matrix

$$M = [m_{ij}] = [\sum_{l \in I_j} t_l x_l \cdot y_i].$$

We claim that M is a signed $\mathcal{K}$-matrix ([Sw1]2.2.4). First, the columns of M converge. Next, given an increasing sequence of positive integers there exist a subsequence $\{n_j\}$ and a sequence of signs $\{s_j\}$ such that $u = \sum_{j=1}^{\infty} s_j \chi_{I_{n_j}} t \in \lambda$. Then

$$\{\sum_{j=1}^{\infty} s_j m_{i n_j}\}_i = \{\sum_{j=1}^{\infty} s_j \sum_{l \in I_{n_j}} t_l x_l \cdot y_i\}_i = \{\sum_{l=1}^{\infty} u_l x_l \cdot y_i\}_i$$

converges. Hence, M is a signed $\mathcal{K}$-matrix so the diagonal of M converges to 0 by the signed version of the Antosik-Mikusinski Matrix Theorem ([Sw1]2.2.4). But, this contradicts (#).

Using the result above we establish a further version of the Orlicz-Pettis Theorem in this setting.

Theorem 2. *Let λ have signed-WGHP. If the series $\sum_j x_j$ is λ multiplier convergent in E with respect to $w(E,F)$, then for each $w(F,E)$ compact (countably compact) subset $K \subset F$ and each $t \in \lambda$, the series $\sum_{j=1}^{\infty} t_j x_j \cdot y$ are convergent uniformly for $y \in K$.*

Proof: Let p be a continuous semi-norm on X. We need to show that the series $\sum_j t_j x_j \cdot y$ converge uniformly for $y \in K$ with respect to p. This will follow if we can show that this property holds in the quotient space X/p. Hence, we may assume that p is actually a norm. Define an equivalence relation $\sim$ on F by $y \sim z$ iff $x_j \cdot y = x_j \cdot z$ for all j. If $E_0 = \{\sum_{j=1}^{\infty} s_j x_j : s \in \lambda$, where $\sum_{j=1}^{\infty} s_j x_j$ is the $w(E,F)$ sum of the series$\}$, then $x \cdot y = x \cdot z$ for every $x \in E_0$ when $y \sim z$. Let y^- be the equivalence class of $y \in F$ and set $F^- = \{f^- : f \in F\}$. Define a metric d on F^- by

$$d(y^-, z^-) = \sum_{j=1}^{\infty} p(x_j \cdot (y - z))/2^j (1 + p(x_j \cdot (y - z)));$$

note that d is a metric since p is a norm. Define a bilinear mapping

$$\cdot : E_0 \times F^- \to (X, p)$$

by $x \cdot y^- = x \cdot y$ so we may consider the triple $E_0, F^-, (X,p)$ as above. The quotient map $F \to F^-$ is $w(F,E) - w(F^-, E_0)$ continuous and the inclusion $(F^-, w(F^-, E_0)) \subset (F^-, d)$ is continuous so K^- is compact (countably compact) with respect to $w(F^-, E_0)$ and d and ,therefore, $w(F^-, E_0) = d$ on K^- and K^- is $w(F^-, E_0)$ sequentially compact. Since the series $\sum_j x_j$ is λ multiplier convergent with respect to $w(E,F)$, the series $\sum_j x_j$ is λ multiplier convergent with respect to $w(E_0, F^-)$ in the abstract triple E_0,$F^-, (X,p)$. Since K^- is sequentially compact in $w(F^-, E_0)$,by Theorem 1 the series $\sum_{j=1}^{\infty} t_j x_j \cdot y^- = \sum_{j=1}^{\infty} t_j x_j \cdot y$ converge uniformly for $y^- \in K^-$ with respect to p.

We consider the special case of Theorems 1 and 2 when E, F are two vector spaces in duality. Let $\lambda(E,F)$ $[\gamma(E,F)]$ be the polar topology on E of uniform convergence on $\sigma(F,E)$ compact [conditionally $\sigma(F,E)$ sequentially compact] subsets of F and let $\tau(E,F)$ be the Mackey topology on E. The topology $\lambda(E,F)$ is obviously stronger than the Mackey topology and can be strictly stronger ([K]9.5.3); the topologies $\lambda(E,F)$ and $\gamma(E,F)$ are not comparable. From Theorems 1 and 2 we have

Corollary 3. *Let λ have signed-WGHP and let E, F be in duality. If the series $\sum_j x_j$ is λ multiplier convergent in E with respect to $\sigma(E,F)$, then*

$\sum_j x_j$ is λ multiplier convergent with respect to both $\lambda(E,F)$ and $\gamma(E,F)$. In particular, $\sum_j x_j$ is λ multiplier convergent with respect to $\tau(E,F)$.

This result was established in [SS]; see also [WL]. We consider a special case of Corollary 3. Let e^j be the sequence with 1 in the j^{th} coordinate and 0 in the other coordinates. If ν is locally convex topology on λ, then (λ, ν) is an AK-space if the series $\sum_{j=1}^{\infty} t_j e^j$ converges to $t = \{t_j\}$ with respect to ν for every $t \in \lambda$.

Corollary 4. *Let λ have signed-WGHP. The spaces $(\lambda, \lambda(\lambda, \lambda^\beta))$ and $(\lambda, \gamma(\lambda, \lambda^\beta))$ are AK-spaces.*

Proof: The series $\sum_j e^j$ is λ multiplier convergent in $\sigma(\lambda, \lambda^\beta)$ so by Corollary 3 the series is λ multiplier convergent in $\lambda(\lambda, \lambda^\beta)$ and $\gamma(\lambda, \lambda^\beta)$. This establishes the result.

This result was established in [SS] and then used to establish Corollary 3.

We now give several applications of the results above.

Example 5. *Let Σ be a σ-algebra of subsets of a set S and let $ca(\Sigma, X)$ be the space of all X valued countably additive set functions from Σ into X. Let $\mathcal{S}(\Sigma)$ =span$\{\chi_\sigma : \sigma \in \Sigma\}$ be the space of Σ simple scalar valued functions. If $E = \mathcal{S}(\Sigma)$ and $F = ca(\Sigma, X)$, then $f \cdot \mu = \int_S f d\mu$,$f \in E, \mu \in F$, defines a bilinear map from $E \times F$ into X (note that we are only integrating simple functions so no elaborate integration theory is involved). If $\{E_j\} \subset \Sigma$ is pairwise disjoint, then the series $\sum_j \chi_{E_j}$ is $w(\mathcal{S}(\Sigma), ca(\Sigma, X))$ subseries convergent. By Theorem 1 above, the series $\sum_{j=1}^{\infty} \mu(E_j)$ converge uniformly for μ belonging to any conditionally $w(ca(\Sigma, X), \mathcal{S}(\Sigma))$ sequentially compact subset of $ca(\Sigma, X)$. In particular, we have as a special case the Nikodym Convergence Theorem.*

Theorem 6. *Let $\{\mu_j\} \subset ca(\Sigma, X)$ be such that $\lim_j \mu_j(E) = \mu(E)$ exists for every $E \in \Sigma$. Then $\{\mu_j\}$ is uniformly countably additive and $\mu \in ca(\Sigma, X)$.*

Proof: By the observation above, since $\{\mu_j\}$ is conditionally $w(ca(\Sigma, X), \mathcal{S}(\Sigma))$ sequentially compact, $\{\mu_j\}$ is uniformly countably additive. That $\mu \in ca(\Sigma, X)$ then follows.

From Theorem 2 we can also derive a result of Graves and Ruess ([GR]) Lemma 6).

Theorem 7. *If $K \subset ca(\Sigma, X)$ is $w(ca(\Sigma, X), \mathcal{S}(\Sigma))$ compact, then K is uniformly countably additive.*

We can also obtain the Nikodym Boundedness Theorem for countably additive set functions from Theorem 2. Let $ca(\Sigma)$ be the space of scalar valued countably additive set functions on Σ.

Theorem 8. *(Nikodym) Let $A \subset ca(\Sigma)$ be pointwise bounded on Σ. Then A is uniformly bounded on Σ.*

Proof: If A is not uniformly bounded on Σ, there exist a pairwise disjoint sequence $\{E_j\} \subset \Sigma$ and $\{\nu_j\} \subset A$ such that $|\nu_j(E_j)| > j$ ([Sw1]4.7.1). Consider the abstract triple $E = \mathcal{S}(\Sigma)$, $F = ca(\Sigma)$ and the bilinear map $(f, \mu) \to f \cdot \mu = \int_S f d\mu$ from $E \times F \to \mathbf{R}$. Now $\sum_j \chi_{E_j}$ is $w(\mathcal{S}(\Sigma), ca(\Sigma))$ subseries convergent and $\{\frac{1}{j}\nu_j\}$ is $w(ca(\Sigma), \mathcal{S}(\Sigma))$ convergent to 0 so by Theorem 2 the series $\sum_{j=1}^{\infty} \frac{1}{k}\nu_k(E_j)$ converge uniformly for $k \in \mathbf{N}$. In particular, $\frac{1}{j}\nu_j(E_j) \to 0$ giving a contradiction.

The version of the Nikodym Boundedness Theorem for vector valued measures with values in a locally convex space follows directly from the scalar version above and the Uniform Boundedness Principle.

Next we derive a version of a result of Thomas ([Th]).

Example 9. *Let S be a sequentially compact Hausdorff space. Let $E = SC(S, X)$ be the space of sequentially continuous functions from S into X and let $F = span\{\delta_t : t \in S\}$, where δ_t is the Dirac measure concentrated at t. Then $f \cdot t = f(t)$ defines a mapping from $E \times S$ into X which can be extended to a bilinear map from $E \times F$ into X. Note that S is conditionally $w(span\{\delta_t : t \in S\}, SC(S, X))$ sequentially compact since S is sequentially compact [here we are identifying t with δ_t]. Thus, from Theorem 1 above if λ has signed-WGHP and $\sum_j f_j$ is λ multiplier convergent in $SC(S, X)$with respect the topology of pointwise convergence on S, then for each $t \in \lambda$ the series $\sum_j t_j f_j$ converges uniformly on S. Similarly, if S is compact (countably compact), then S is $w(span\{\delta_t : t \in S\}, C(S, X))$ compact (countably compact) so from Theorem 2 if λ has signed-WGHP and the series $\sum_j f_j$ is λ multiplier convergent in $C(S, X)$ with respect to the topology of pointwise convergence on S, then for each $t \in \lambda$ the series $\sum_j t_j f_j$ converges uniformly on S. The subseries version of this result is due to Thomas ([Th]).*

We can also use Theorems 1 and 2 above to derive a version of the Orlicz-Pettis Theorem for continuous linear operators. Let Z be a locally convex space and $L(Z,X)$ the space of all continuous linear operators from Z into X. If $A \subset Z$ is a family of bounded subsets of Z, we denote the topology of uniform convergence on the elements of A by $L_A(Z,X)$. If A consists of the singleton subsets of Z, the topology $L_A(Z,X)$ is just the topology of pointwise convergence on Z and will be denoted by $L_s(Z,X)$; if A consists of the family of all bounded subsets of Z, then the topology $L_A(Z,X)$ will be denoted by $L_b(Z,X)$. Similar notation will be employed for any subspace of $L(Z,X)$.

Example 10. *Let Z be a locally convex space. Set $E = L(Z,X)$ and $F = Z$ and define a bilinear mapping $\cdot : E \times F \to X$ defined by $\cdot : (T,z) \to T \cdot z = Tz$. Then $w(E,F)$ is just the topology of pointwise convergence on Z or $L_s(Z,X)$. If $K \subset Z$ is sequentially compact (compact), then K is conditionally $w(Z, L(Z,X))$ sequentially compact ($w(Z, L(Z,X))$ compact) so if $\mathcal{K}$ (C)denotes the set of all sequentially compact (compact) subsets of Z, from Theorem 1(Theorem 2) above , we have*

Theorem 11. *Let λ have signed-WGHP. If $\sum_j T_j$ is λ multiplier convergent in $L_s(Z,X)$, then $\sum_j T_j$is λ multiplier convergent in $L_{\mathcal{K}}(Z,X)$ ($L_C(Z,X)$).*

An operator $T \in L(Z,X)$ is completely continuous if T carries weakly convergent sequences into convergent sequences; denote all such operators by $CC(Z,X)$. Note that if T is completely continuous, then T carries weak Cauchy sequences into Cauchy sequences. Now consider the abstract triple $E = CC(Z,X)$, $F = Z$ and the bilinear map $\cdot : E \times F \to X$ defined by $\cdot : (T,z) \to T \cdot z = Tz$. If a subset $K \subset Z$ is conditionally weakly sequentially compact, then K is conditionally $w(CC(Z,X),Z)$ sequentially compact. If CW denotes the set of all conditionally weakly sequentially compact subsets of Z, then from Theorem 1 we have

Theorem 12. *Let λ have signed-WGHP. If the series $\sum_j T_j$ is λ multiplier convergent in $CC_s(Z,X)$, then $\sum_j T_j$ is λ multiplier convergent in $CC_{CW}(Z,X)$.*

An operator $T \in L(Z,X)$ is weakly compact if T carries bounded sets to relatively weakly compact sets; denote all such operators by $W(Z,X)$. The space Z has the Dunford-Pettis property if every weakly compact operator from Z into any locally convex space X carries weak Cauchy sequences into convergent sequences ([E]). Consider the abstract triple $E = W(Z,X)$, $F = Z$ and the bilinear map $\cdot : E \times F \to X$ defined by $\cdot : (T,z) \to T \cdot z = Tz$. If $K \subset Z$ is conditionally weakly sequentially compact and Z has the Dunford-Pettis property, then K is conditionally $w(W(Z,X),Z)$ sequentially compact. If CW denotes the set of all conditionally weakly compact subsets of Z, then from Theorem 1 we have

Theorem 13. *Let λ have signed-WGHP and assume that Z has the Dunford-Pettis property. If the series $\sum_j T_j$ is λ multiplier convergent in $W_s(Z,X)$, then $\sum_j T_j$ is λ multiplier convergent in $W_{CW}(Z,X)$.*

A space Z is almost reflexive if every bounded sequence contains a weak Cauchy subsequence ([LW]). For example, Banach spaces with separable duals, quasi-reflexive Banach spaces and $c_0(S)$ are almost reflexive ([LW]). If Z is almost reflexive and has the Dunford-Pettis property, then every bounded set is conditionally $w(W(Z,X),Z)$ sequentially compact so from Theorem 1, we have

Theorem 14. *Let λ have signed-WGHP and assume that Z is almost reflexive with the Dunford-Pettis property. If the series $\sum_j T_j$ is λ multiplier convergent in $W_s(Z,X)$, then $\sum_j T_j$ is λ multiplier convergent in $W_b(Z,X)$.*

As another application of Theorem 2, we derive an Orlicz-Pettis result of Stiles for a locally convex TVS with a Schauder basis ([Sti]). Stiles' version of the Orlicz-Pettis Theorem is for subseries convergent series with values in an F-space with a Schauder basis and his proof uses the metric properties of the space. We will establish a version of Stiles' result for multiplier convergent series which requires no metrizability assumptions.

Let X be a LCTVS with a Schauder basis $\{b_j\}$ and associated coordinate functionals $\{f_j\}$. That is, every $x \in X$ has a unique series representation $x = \sum_{j=1}^{\infty} t_j b_j$ and $f_j : X \to \mathbf{R}$ is defined by $\langle f_j, x\rangle = t_j$. We do not assume that the coordinate functionals are continuous although this is the case when X is an F-space ([Sw2] 10.1.13). Define $P_i : X \to X$ by $P_i x = \sum_{j=1}^{i} \langle f_j, x\rangle\, b_j$. Let $E = X$, $F = span\{P_i : i \in \mathbf{N}\}$ and let the bilinear mapping from $E \times F$ into X be the extension to F of the mapping $x \cdot P_i = P_i x$. Let $G = span\{f_i : i \in \mathbf{N}\}$.

Theorem 15. *Let λ have signed-WGHP. If $\sum_j x_j$ is λ multiplier convergent with respect to $\sigma(X,G)$, then $\sum_j x_j$ is λ multiplier convergent with respect to the original topology of X.*

Proof: Since a sequence in X is $\sigma(X,G)$ convergent iff the sequence is $w(E,F)$ convergent, the series $\sum_j x_j$ is λ multiplier convergent with respect to $w(E,F)$. Now $\{P_i : i \in \mathbf{N}\}$ is relatively $w(F,E)$ sequentially compact since $P_i x \to x$ for every $x \in X$. By Theorem 2, for every $t \in \lambda$ the series $\sum_{j=1}^{\infty} t_j P_i x_j$ converge uniformly for $i \in \mathbf{N}$. Let U be a closed neighborhood of 0 in X. There exists N such that $\sum_{j=m}^{\infty} t_j P_i x_j = P_i(\sum_{j=m}^{\infty} t_j x_j) \in U$ for $m \geq N, i \in \mathbf{N}$. Let $i \to \infty$ giving $\sum_{j=m}^{\infty} t_j x_j \in U$ for $m \geq N$.

Note that we did not use the continuity of the coordinate functionals in the proof so the topology of X may not even be comparable to $\sigma(X,G)$.

We next consider a more general situation than that encountered in Stiles' result. Assume that there exists a sequence of linear operators $P_i : X \to X$ such that for each $x \in X$, $x = \sum_{i=1}^{\infty} P_i x$ [convergence in X]. When each P_i is continuous, $\{P_i\}$ is called a Schauder decomposition . If X has a Schauder basis $\{b_i\}$ with coordinate functionals $\{f_i\}$, then $P_i x = \langle f_i, x \rangle b_i$ is an example of this situation. Let $E = X, F = \mathit{span}\{P_i : i \in \mathbf{N}\}$ and let the bilinear mapping from $E \times F$ into X be the extension of the map $(x, P_i) \to x \cdot P_i = P_i x$.

Theorem 16. *Let λ have signed-WGHP and assume that each P_i is $w(E,F)-X$ continuous. If the series $\sum_j x_j$ is λ multiplier convergent with respect to $w(E,F)$, then the series $\sum_j x_j$ is λ multiplier convergent in X with respect to the original topology.*

Proof: Define $S_n : X \to X$ by $S_n = \sum_{i=1}^{n} P_i$. Then $\{S_n : n \in \mathbf{N}\}$ is $w(F,E)$ sequentially compact so by Theorem 2 for each $t \in \lambda$ the series $\sum_{j=1}^{\infty} t_j S_n x_j$ converge uniformly for $n \in \mathbf{N}$. Let U be a closed neighborhood of 0 in X. There exists N such that $\sum_{j=m}^{\infty} t_j S_n x_j = S_n(\sum_{j=m}^{\infty} t_j x_j) \in U$ for $m \geq N, n \in \mathbf{N}$. Letting $n \to \infty$ gives $\sum_{j=m}^{\infty} t_j x_j \in U$ for $m \geq N$.

We give an example where the theorem above is applicable.

Example 17. *Let Y be a LCTVS and let X be a vector space of Y valued sequences containing the space of sequences which are eventually 0. Then X is an AK-space if the coordinate functionals $f_j : X \to Y$, $f_j(\{x_j\}) = x_j$ are continuous for every j and each $x = \{x_j\}$ has a representation $x = \sum_{j=1}^{\infty} e^j \otimes x_j$ [here $e^j \otimes x$ denotes the sequence with x in the j^{th} coordinate and 0 in the other coordinates]. The space X has the property (I) if the injections $x \to e^j \otimes x$ are continuous from Y into X. If $P_j : X \to X$ is defined by $P_j(\{x_j\}) = e^j \otimes x_j$, then $\{P_j\}$ is a Schauder decomposition for X. If X has property (I), then the topology of coordinatewise convergence is equal to $w(E,F)$ so the result above applies and* *if λ has signed-WGHP, then any series which is λ multiplier convergent in the topology of coordinatewise convergence converges in the topology of X.*

For examples where the result above applies let Y be a normed space. If $1\le p<\infty$, then $l^p(Y)$ and $c_0(Y)$ are AK-spaces satisfying the conditions in the example above.

We use Theorem 1 to derive Stuart's Theorem on the completeness of β-duals. First, we have the following uniform convergence result.

Theorem 18. *Let λ have signed-WGHP. Assume that $\sum_j x_{ij}$ is λ multiplier convergent for every $i\in\mathbf{N}$ and that $\lim_i\sum_{j=1}^\infty t_jx_{ij}$ exists for every $t\in\lambda$ with $x_j=\lim_i x_{ij}$ for every j. Then for every $t\in\lambda$ the series $\sum_{j=1}^\infty t_jx_{ij}$ converge uniformly for $i\in\mathbf{N}$, the series $\sum_j x_j$ is λ multiplier convergent and $\lim_i\sum_{j=1}^\infty t_jx_{ij}=\sum_{j=1}^\infty t_jx_j$.*

Proof: For every $i\in\mathbf{N}$ define a linear map $f_i:\lambda\to X$ by $f_i(t)=\sum_{j=1}^\infty t_jx_{ij}$ and set $F=span\{f_i:i\in\mathbf{N}\}$. Consider the abstract triple $E=\lambda,F$ and X and let the bilinear mapping from $E\times F$ into X be the extension of the map $(t,f_i)\to t\cdot f_i=f_i(t)$. We first claim that the series $\sum_j e^j$ is λ multiplier convergent with respect to $w(E,F)$. For if $t\in\lambda$,

$$\sum_{j=1}^\infty t_je^j\cdot f_i=\sum_{j=1}^\infty t_jf_i(e^j)=\sum_{j=1}^\infty t_jx_{ij}$$

converges for every i. Now $\{f_i\}$ is conditionally $w(F,E)$ sequentially compact since $\{t\cdot f_i\}=\{\sum_{j=1}^\infty t_jx_{ij}\}$ converges for every $t\in\lambda$. Theorem 1 implies that the series $\sum_{j=1}^\infty t_jf_i(e^j)=\sum_{j=1}^\infty t_jx_{ij}$ converge uniformly for $i\in\mathbf{N}$.

Let $x=\lim_i\sum_{j=1}^\infty t_jx_{ij}$. We claim that $x=\sum_{j=1}^\infty t_jx_j$. Let U be a neighborhood of 0 in X and pick a neighborhood ,V, of 0 such that $V+V+V\subset U$. There exists N such that $\sum_{j=m}^\infty t_jx_{ij}\in V$ for $m\ge N$ and all $i\in\mathbf{N}$. Fix $m\ge N$ and pick $i=i(m)$ such that $x-\sum_{j=1}^\infty t_jx_{ij}\in V$ and $\sum_{j=1}^m t_j(x_{ij}-x_j)\in V$. Then

$$x-\sum_{j=1}^m t_jx_j=x-\sum_{j=1}^\infty t_jx_{ij}+\sum_{j=1}^m t_j(x_{ij}-x_j)+\sum_{j=m+1}^\infty t_jx_{ij}\in V+V+V\subset U$$

and the result follows.

Stuart's result follows immediately from Theorem 18. Recall the β-dual of λ with respect to X is $\lambda^{\beta X}=\{x=\{x_j\}:\sum_{j=1}^\infty t_jx_j=t\cdot x$ converges for every $t = \{t_j\} \in \lambda\}$. Then $E = \lambda, F = \lambda^{\beta X}$ and X form an abstract triple with the bilinear map $(t, x) \to t \cdot x$. Stuart's result asserts that $w(\lambda^{\beta X}, \lambda)$ is sequentially complete if X is sequentially complete ([St1],[St2],[Sw1]). If $\{x^k = \{x_j^k\}\}$ is $w(\lambda^{\beta X}, \lambda)$ Cauchy, then for each k the sequence $\{x_j^k\}_j$ is Cauchy in X and,therefore, convergent in X. Thus, the conditions of Theorem 18 are satisfied and Stuart's result follows.

Theorem 18 can be viewed as a weak form of the Hahn-Schur Theorem ([Sw1]). If stronger gliding hump assumptions are placed on the multiplier space λ, stronger versions of the Hahn-Schur Theorem can be obtained (see [Sw2]).

We can also derive a version of Kalton's Theorem on subseries convergence in the space of compact operators ([Sw1] 10.5.6). Let X and Y be normed spaces and let $K(X, Y)$ be the space of all compact operators from X into Y (an operator $T \in L(X, Y)$ is compact if T carries bounded sets into relatively compact sets). The space X has the DF property if every weak* subseries convergent series in X' is $\|\cdot\|$ subseries convergent ([DF]; Diestel and Faires have shown that for B-spaces this is equivalent to X' containing no subspace isomorphic to l^∞).

Theorem 19. *Let X and Y be normed spaces and let X have the DF property. If the series $\sum_j T_j$ is subseries convergent in the weak operator topology of $K(X, Y)$, then the series is subseries convergent in the norm topology of $K(X, Y)$.*

Proof: Each T_j has separable range so we may assume that Y is separable by replacing Y with $\cup_{j=1}^{\infty} T_j X$. By Lemma 10.1.8 of [Sw1] it suffices to show that $\|T_j\| \to 0$ or, equivalently, $\left\|T_j'\right\| \to 0$. Pick $y_j' \in Y'$, $\left\|y_j'\right\| = 1$, such that $\left\|T_j'\right\| \leq \left\|T_j' y_j'\right\| + 1/j$. By the separability of Y there exists a subsequence $\{y_{n_j}'\}$ which is weak* convergent to some $y' \in Y'$; for convenience assume that the sequence $\{y_j'\}$ is weak* convergent to y'. Consider the abstract triple $E = \{T' : T \in K(X, Y)\}, F = Y'$ and $(X', \|\cdot\|)$ with the bilinear map $E \times F \to (X', \|\cdot\|)$ defined by $(T', y') \to T' \cdot y' = T'y'$. For each $z' \in Y'$, the series $\sum_j T_j' z'$ is weak* subseries convergent in X' and is, therefore, subseries convergent in $(X', \|\cdot\|)$ by the DF property. Hence, the series $\sum_j T_j'$ is $w(E, F)$ subseries convergent. The sequence $\{y_j'\}$ is relatively $w(F, E)$ sequentially compact since $\|\cdot\| - \lim T' y_j' = T' y'$ for every $T \in K(X, Y)$ ([DS]VI.5.6). By Theorem 2 the series $\sum_{j=1}^{\infty} T_j' y_i'$ converge uniformly for $i \in \mathbf{N}$. In particular, $\left\|T_j' y_j'\right\| \to 0$ so $\left\|T_j'\right\| = \|T_j\| \to 0$ as desired.

Finally, we note that the conclusions of Theorems 1 and 2 can be strengthened if the multiplier space λ satisfies a stronger gliding hump condition. A subset $\Lambda \subset \lambda$, where λ is a K-space, has the signed strong gliding hump property (signed-SGHP) if whenever $\{t^k\}$ is a bounded sequence in Λ and $\{I_k\}$ is an increasing sequence of intervals, there exist a sequence of signs $\{s_k\}$ and a subsequence $\{n_k\}$ such that the coordinate sum $\sum_{k=1}^{\infty} s_k \chi_{I_{n_k}} t^{n_k} \in \Lambda$; if all of the signs can be chosen equal to 1, then Λ has the strong gliding hump property (SGHP). For example, l^∞ has SGHP while bs has signed-SGHP but not SGHP ([Sw2]); the subset $\{\chi_\sigma : \sigma \subset \mathbf{N}\} \subset m_0$ has SGHP while the space m_0 does not.

Theorem 20. *Let $\Lambda \subset \lambda$ have signed-SGHP. If $\sum_j x_j$ is λ multiplier convergent with respect to $w(E, F)$, then for each conditionally $w(F, E)$ sequentially compact ($w(F, E)$ compact, $w(F, E)$ countably compact) subset $K \subset F$ and each bounded subset $B \subset \Lambda$, the series $\sum_{j=1}^{\infty} t_j x_j \cdot y$ converge uniformly for $y \in K, t \in B$.*

Proof: If the conclusion fails to hold, there exist a neighborhood, W, in X ,$y_k \in K, t^k \in B$ and an increasing sequence of intervals $\{I_k\}$ such that

$$(\#) \quad \sum_{l \in I_k} t_l^k x_l \cdot y_k \notin W$$

for every k. We may assume, by passing to a subsequence if necessary, that $\lim_k x \cdot y_k$ exists for every $x \in E$. Consider the matrix

$$M = [m_{ij}] = [\sum_{l \in I_j} t_l^j x_l \cdot y_i].$$

We claim that M is a signed $\mathcal{K}$ matrix as in Theorem 1 ([Sw1]2.2.4). First, the columns of M converge. Next given an increasing sequence of positive integers, there exist a sequence of signs $\{s_j\}$ and a subsequence $\{n_j\}$ such that $u = \sum_{k=1}^{\infty} s_k \chi_{I_{n_k}} t^{n_k} \in \Lambda$. Then

$$\{\sum_{j=1}^{\infty} s_j m_{in_j}\}_i = \{\sum_{j=1}^{\infty} s_j \sum_{l \in I_{n_j}} t_l^{n_j} x_l \cdot y_i\}_i = \{\sum_{l=1}^{\infty} u_l x_l \cdot y_i\}_i$$

converges. Hence, M is a signed $\mathcal{K}$ matrix so the diagonal of M converges to 0 by the signed version of the Antosik-Mikusinski Matrix Theorem ([Sw1]2.2.4). But, this contradicts (#).

The proof of the statements in parentheses follow as in the proof of Theorem 2.

If the multiplier space $\Lambda \subset \lambda$ has signed-SGHP, then the conclusion of Corollary 3 can be improved.

Corollary 21. *Let $\Lambda \subset \lambda$ have signed-SGHP and let E, F be in duality. If the series $\sum_j x_j$ is λ multiplier convergent with respect to $\sigma(E,F)$, then the series $\sum_{j=1}^{\infty} t_j x_j$ converge uniformly for t belonging to bounded subsets of Λ with respect to both $\lambda(E,F)$ and $\gamma(E,F)$.*

Corollary 21 covers the case of subseries convergent series ($\Lambda = \{\chi_\sigma : \sigma \subset \mathbf{N}\} \subset m_0 = \lambda$) and bounded multiplier convergent series (Λ the unit ball of l^∞).

Using Theorem 20 we can also obtain an improved conclusion in Theorem 6. In particular, if $\{E_j\}$ is a pairwise disjoint sequence from Σ, then the series $\sum_{j=1}^{\infty} \chi_E(j)\mu_i(E_j)$ converge uniformly for $i \in \mathbf{N}, E \in \Sigma$. That is, the series $\sum_{j=1}^{\infty} \mu_i(E_j)$ are uniformly unordered convergent for $i \in \mathbf{N}$.

Similarly, we can obtain an improvement to the statements in Example 9 if the multiplier space λ has signed-SGHP. If λ has signed-SGHP and the series $\sum_j f_j$ is λ multiplier convergent in $SC(S,X)$ ($C(S,X)$) with respect to the topology of pointwise convergence on S, then the series $\sum_{j=1}^{\infty} t_j f_j(s)$ converge uniformly for $s \in S$ and t belonging to bounded subsets of λ (Theorem 20).

We can also obtain a strengthened version of Stuart's completeness result given in Theorem 18.

Theorem 22. *Let λ have signed-SGHP. Assume that $\sum_j x_{ij}$ is λ multiplier convergent for each $i \in \mathbf{N}$ and that $\lim_i \sum_{j=1}^{\infty} t_j x_{ij}$ exists for each $t \in \lambda$ with $x_j = \lim_i x_{ij}$ for every j. Then the series $\sum_{j=1}^{\infty} t_j x_{ij}$ converge uniformly for t belonging to bounded subsets of Λ, the series $\sum_j x_j$ is λ multiplier convergent and $\lim_i \sum_{j=1}^{\infty} t_j x_{ij} = \sum_{j=1}^{\infty} t_j x_j$ uniformly for t belonging to bounded subsets of λ.*

The proof of Theorem 18 carries forward using Theorem 20 in place of Theorem 1. Theorem 22 can be viewed as a version of the Hahn-Schur Theorem for multiplier convergent series (see [Sw2]).

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