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REALIZABILITY BY SYMMETRIC NONNEGATIVE MATRICES *

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Abstract

Let $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ be a set of complex numbers. The nonnegative inverse eigenvalue problem (NIEP) is the problem of determining necessary and sufficient conditions in order that Λ may be the spectrum of an entrywise nonnegative $n \times n$ matrix. If there exists a nonnegative matrix A with spectrum Λ we say that Λ is realized by A. If the matrix A must be symmetric we have the symmetric nonnegative inverse eigenvalue problem (SNIEP). This paper presents a simple realizability criterion by symmetric nonnegative matrices. The proof is constructive in the sense that one can explicitly construct symmetric nonnegative matrices realizing Λ.

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1. Introduction

The *nonnegative inverse eigenvalue problem* (hereafter NIEP) is the problem of characterizing all possible spectra of entrywise nonnegative matrices (References [1-17]). This problem remains unsolved. In the general case, when the possible spectrum Λ is a set of complex numbers, the problem has only been solved for $n = 3$ by Loewy and London [8]. The cases $n = 4$ and $n = 5$ have been solved for matrices of trace zero by Reams [11] and Laffey and Meehan [7], respectively. When Λ is a set of real numbers (RNIEP), sufficient conditions have been obtained in [16], [9], [12], [6], [1], [13]. If Λ has to be the spectrum of a symmetric nonnegative matrix, we have the symmetric nonnegative inverse eigenvalue problem (SNIEP), which is the subject of this paper.

A set Λ of real numbers is said to be *realizable* if Λ is the spectrum of an entrywise nonnegative matrix. A set K of conditions is said to be a *realizability criterion* if any set of real numbers $\Lambda = \{\lambda_1, \lambda_2, ..., \lambda_n\}$ satisfying the conditions K is realizable.

In ([13], Theorem 11) the author gives a simple realizability criterion for the existence of an $n \times n$ nonnegative matrix with real prescribed spectrum. The goal of this work is to show that this criterion is also a realizability criterion for the symmetric nonnegative inverse eigenvalue problem.

Unlike several of the previous conditions which are sufficient for realizability of spectra, the proof of Theorem 11 in [13] is constructive in the sense that one can explicitly construct nonnegative matrices realizing the prescribed real spectrum. This is done by employing an extremely useful result, due to Brauer [3], which shows how to modify one single eigenvalue of a matrix via a rank-one perturbation, without changing any of the remaining eigenvalues.

In [4] Fiedler obtain some necessary and some sufficient conditions for a set of n real numbers $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ to be the spectrum of an $n \times n$ symmetric nonnegative. There, Fiedler also shows that Kellogg's realizability criterion [6] is sufficient for the existence of a symmetric nonnegative matrix with prescribed spectrum. In [10], Radwan shows that Borobia's realizability criterion [1] is also sufficient for the existence of a symmetric nonnegative matrix with prescribed spectrum. Soules [15] gives a realizability criterion for the existence of a symmetric doubly stochastic matrix and also shows how to construct a realizing matrix. Radwan, in [10], point out that the realizability criteria of Kellogg and Soules are not comparable. In [5], the authors show that the real nonnegative inverse eigenvalue problem and the symmetric nonnegative inverse eigenvalue problem are different, while Wuwen, in [17], shows that both problems are equivalent for $n \leq 4$.

This paper is organized as follows: In section 2, we introduce the notation and previous results, which will be necessary in order to prove Theorem 1 in section 3. In section 3 we prove that Soto's realizability criterion ([13], Theorem 11) established here as Theorem 1, is sufficient for the existence of an $n \times n$ symmetric nonnegative matrix with prescribed spectrum. In section 4 we consider the problem of constructing symmetric nonnegative matrices realizing spectra, which satisfy Theorem 1. Some examples are given in section 5.

2. Preliminaries and notation

Following the notation in [2], the set

$$\mathrm{A} \equiv \{\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \subset \mathbf{R} : \lambda_1 \geq |\lambda_i|, \ i = 2, \ldots, n\}$$

includes all possible real spectra of nonnegative matrices. We denote

$$\mathrm{AR} = \{\Lambda \in \mathcal{A} \, : \, \Lambda \textit{ is realizable}\}.$$

We denote by $\mathcal{N}_n$ the set of all $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \in \mathcal{AR}$, where $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. Similarly, we denote by $\mathcal{S}_n$ $(\widehat{\mathcal{S}_n})$ the set of all $\Lambda \in \mathcal{N}_n$ for which there exists an $n \times n$ symmetric nonnegative (*positive*) matrix with spectrum Λ. We shall only consider real sets $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ satisfying

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_p \geq 0 > \lambda_{p+1} \geq \ldots \geq \lambda_n,$$

since if $\lambda_n \geq 0$, then $A = diag\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ is a symmetric nonnegative matrix.

The following result, due to Fiedler, shows that if A and B are symmetric matrices of order n and m, respectively, then we may construct a new symmetric matrix of order $n+m$ as follows:

Lemma 1. (Fiedler [4]) *Let A be a symmetric $n\times n$ matrix with eigenvalues $\alpha_1, \alpha_2, \ldots, \alpha_n$. Let $\mathbf{u}, \|\mathbf{u}\| = 1$, be a unit eigenvector of A corresponding to α_1. Let B be a symmetric $m\times m$ matrix with eigenvalues $\beta_1, \beta_2, \ldots, \beta_m$. Let $\mathbf{v}, \|\mathbf{v}\| = 1$, be a unit eigenvector of B corresponding to β_1. Then for any ρ the matrix*

$$C = \begin{pmatrix} A & \rho\mathbf{u}\mathbf{v}^T \\ \rho\mathbf{v}\mathbf{u}^T & B \end{pmatrix}$$

has eigenvalues $\alpha_2, \ldots, \alpha_n, \beta_2, \ldots, \beta_m, \gamma_1, \gamma_2$, where γ_1 and γ_2 are eigenvalues of the matrix

$$\widehat{C} = \begin{pmatrix} \alpha_1 & \rho \\ \rho & \beta_1 \end{pmatrix}.$$

The next relevant result, due also to Fiedler [4], is necessary for the proof of the main result in section 3. Here we present the Wuwen version of it [17]:

Lemma 2. (Fiedler [4]) *If $\{\alpha_1, \alpha_2, \ldots, \alpha_n\} \in \mathcal{S}_n$, $\{\beta_1, \beta_2, \ldots, \beta_m\} \in \mathcal{S}_m$ and $\varepsilon \geq \max\{0, \beta_1 - \alpha_1\}$, then $\{\alpha_1 + \varepsilon, \beta_1 - \varepsilon, \alpha_2, \ldots, \alpha_n, \beta_2, \ldots, \beta_m\} \in \mathcal{S}_{n+m}$.*

We shall also need the following lemma:

Lemma 3. (Fiedler [4]) *If $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \in \mathcal{S}_n$ and if $\varepsilon > 0$ then*

$$\Lambda_\varepsilon = \{\lambda_1 + \varepsilon, \lambda_2, \ldots, \lambda_n\} \in \widehat{\mathcal{S}}_n.$$

In ([13], Theorem 11) we give the following simple realizability criterion, which also shows how to construct a realizing matrix.

Theorem 1. *(**Soto** [13]) Let $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ be a set of real numbers, such that*

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_p \geq 0 > \lambda_{p+1} \geq \ldots \geq \lambda_n.$$

If

$$\lambda_1 \geq -\lambda_n - \sum_{S_k<0} S_k \tag{2.1}$$

where $S_k = \lambda_k + \lambda_{n-k+1}$, $k = 2, 3, \ldots, \left[\frac{n}{2}\right]$ and $S_{\frac{n+1}{2}} = \min\{\lambda_{\frac{n+1}{2}}, 0\}$ for n odd, then Λ is realized by a nonnegative matrix A (with constant row sums).

Observe that if $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ satisfies the sufficient condition (2.1), then

$$\Lambda' = \{-\lambda_n - \sum_{S_k<0} S_k, \lambda_2, \ldots, \lambda_n\}$$

is a realizable set.

3. Realizability by a symmetric nonnegative matrix

In this section we show that the realizability criterion given by Theorem 1 is sufficient for the existence of a symmetric nonnegative matrix with prescribed spectrum Λ.

Theorem 1. *Let $\Lambda = \{\lambda_1; \lambda_2, \ldots, \lambda_n\}$ be a set of real numbers such that*

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_p \geq 0 > \lambda_{p+1} \geq ... \geq \lambda_n.$$

If Λ satisfies the realizability criterion given by Theorem 1, then Λ is realized by an $n \times n$ symmetric nonnegative matrix.

Proof. Suppose that Λ satisfies the condition (2.1) of Theorem 1. That is,

$$\lambda_1 \geq -\lambda_n - \textstyle\sum_{S_i<0} S_i,$$

where $S_k = \lambda_k + \lambda_{n-k+1}$, $k = 2, 3, \ldots, \left[\frac{n}{2}\right]$ and $S_{\frac{n+1}{2}} = \min\{\lambda_{\frac{n+1}{2}}, 0\}$ for n odd.

It suffices to prove the statement for $\lambda_1 = -\lambda_n - \sum_{S_k<0} S_k$. In fact, if $\lambda_1 > -\lambda_n - \sum_{S_k<0} S_k$ then we take $\widetilde{\Lambda} = \{\mu_1, \lambda_2, \ldots, \lambda_n\}$ with $\mu_1 = -\lambda_n - \sum_{S_k<0} S_k$. Thus, if $\widetilde{\Lambda} \in \mathcal{S}_n$ then we apply Lemma 3 with $\varepsilon = \lambda_1 - \mu_1 > 0$ to show that $\Lambda \in \mathcal{S}_n$ (actually $\Lambda \in \widehat{\mathcal{S}_n}$).

Let

$$\begin{aligned} \Lambda_k &= \{\lambda_k, \lambda_{n-k+1}\};\ k = 1, 2, \ldots, \left[\frac{n}{2}\right] \text{ and} \\ \Lambda_{\frac{n+1}{2}} &= \{\lambda_{\frac{n+1}{2}}\} \text{ for } n \text{ odd.} \end{aligned}$$

Consider the partition

$$\Lambda = \cup_{k=1}^{\left[\frac{n}{2}\right]}\Lambda_k \quad \text{with} \quad \Lambda = \cup_{k=1}^{\left[\frac{n}{2}\right]}\Lambda_k \cup \Lambda_{\frac{n+1}{2}} \quad \text{for } n \text{ odd.}$$

Observe that some subsets Λ_k can be realizable thenselves, in particular by the symmetric nonnegative matrix

$$B_k = \frac{1}{2}\begin{pmatrix} \lambda_k + \lambda_{n-k+1} & \lambda_k - \lambda_{n-k+1} \\ \lambda_k - \lambda_{n-k+1} & \lambda_k + \lambda_{n-k+1} \end{pmatrix}. \tag{3.1}$$

Without loss of generality we may reorder the subsets Λ_k, in such a way that $\Lambda_2, \Lambda_3, \ldots, \Lambda_{t,}$ $t \leq \left[\frac{n}{2}\right]$, are nonrealizable ($S_k < 0$), while $\Lambda_{t+1}, \ldots, \Lambda_{\left[\frac{n}{2}\right]}$ are realizable ($S_k \geq 0$). Consider, if there is someone, the realizable sets Λ_k : If B_k in (3.1) realizes Λ_k, then the direct sum $B = \oplus B_k$, $k = t+1, \ldots, \left[\frac{n}{2}\right]$, with $B_{\frac{n+1}{2}} = \left(\lambda_{\frac{n+1}{2}}\right)$ if $\lambda_{\frac{n+1}{2}} \geq 0$ for n odd, is a symmetric nonnegative matrix realizing $\cup_{k=t+1}^{\left[\frac{n}{2}\right]}\Lambda_k$ $\left(\cup_{k=t+1}^{\left[\frac{n}{2}\right]}\Lambda_k \cup \Lambda_{\frac{n+1}{2}}\right.$ for n odd).

Now we consider, if there is someone, the nonrealizable sets Λ_k, $k = 2, 3, \ldots, t$ together with the realizable set $\Lambda_1 = \{\lambda_1, \lambda_n\}$ and we renumber the $2t$ elements in $\cup\Lambda_k$ as

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_t \geq \lambda_{t+1} \geq \ldots \geq \lambda_{2t-1} \geq \lambda_{2t}.$$

For each one of these sets Λ_k, $k = 1, 2, \ldots, t$, we define the associated set

$$\Gamma_k = \{-\lambda_{2t-k+1}, \lambda_{2t-k+1}\}, \tag{3.2}$$

which is realizable by the symmetric nonnegative matrix

$$A_k = \begin{pmatrix} 0 & -\lambda_{2t-k+1} \\ -\lambda_{2t-k+1} & 0 \end{pmatrix} \tag{3.3}$$

with $\Gamma_{\frac{2t+1}{2}} = \{0\}$ if $\lambda_{\frac{2t+1}{2}} < 0$ for n odd, which is realized by the symmetric nonnegative matrix $A_{\frac{2t+1}{2}} = (0)$.
Now, we procede as follows: First, we merge the sets

$$\begin{aligned} \Gamma_1 &= \{-\lambda_{2t}, \lambda_{2t}\} \in \mathcal{S}_2 \quad \text{and} \\ \Gamma_2 &= \{-\lambda_{2t-1}, \lambda_{2t-1}\} \in \mathcal{S}_2 \end{aligned}$$

to obtain, from Lemma 2, a new set $\Delta_2 \in \mathcal{S}_4$. In fact, we take $\varepsilon_2 = -S_2 = -(\lambda_2 + \lambda_{2t-1}) > 0$. Then

$$\begin{aligned} -\lambda_{2t} + \varepsilon_2 &= -\lambda_{2t} - S_2 = -\lambda_{2t} - (\lambda_2 + \lambda_{2t-1}) \\ -\lambda_{2t-1} - \varepsilon_2 &= -\lambda_{2t-1} + S_2 = -\lambda_{2t-1} + (\lambda_2 + \lambda_{2t-1}) = \lambda_2 \end{aligned}$$

and

$$\Delta_2 = \{-\lambda_{2t} - S_2, \lambda_2, \lambda_{2t-1}, \lambda_{2t}\} \in \mathcal{S}_4.$$

Next we merge Δ_2 with $\Gamma_3 = \{-\lambda_{2t-2}, \lambda_{2t-2}\}$. Let $\varepsilon_3 = -S_3 = -(\lambda_3 + \lambda_{2t-2}) > 0$. Then

$$\begin{aligned} -\lambda_{2t} - S_2 + \varepsilon_3 &= -\lambda_{2t} - S_2 - S_3 \\ -\lambda_{2t-2} - \varepsilon_3 &= -\lambda_{2t-2} + S_3 = -\lambda_{2t-2} + (\lambda_3 + \lambda_{2t-2}) = \lambda_3 \end{aligned}$$

and from Lemma 2

$$\Delta_3 = \{-\lambda_{2t} - S_2 - S_3, \lambda_3, *, \dots, *\} \in \mathcal{S}_6.$$

Observe that in each step we recover the first element $\lambda_k \in \Lambda_k$ from $-\lambda_{2t-k+1} - \varepsilon_k = \lambda_k$.
In the $j-th$ step of the procedure ($j \geq 2$), we merge the sets

$$\begin{aligned} \Delta_j &= \{-\lambda_{2t} - S_2 - S_3 - \cdots - S_j, \lambda_j, *, \dots, *\} \text{ and} \\ \Gamma_{j+1} &= \{-\lambda_{2t-j}, \lambda_{2t-j}\}. \end{aligned}$$

Then for $\varepsilon_{j+1} = -S_{j+1} = -(\lambda_{j+1} + \lambda_{2t-j}) > 0$ we have

$$\begin{aligned} -\lambda_{2t} - \sum_{k=2}^{j} S_k + \varepsilon_{j+1} &= -\lambda_{2t} - \sum_{k=2}^{j+1} S_k \\ -\lambda_{2t-j} - \varepsilon_{j+1} &= \lambda_{j+1} \end{aligned}$$

and from Lemma 2
$\Delta_{j+1} = \{-\lambda_{2t} - \sum_{k=2}^{j+1} S_k, \lambda_{j+1}, *, \dots, *\} \in \mathcal{S}_{2j+2}$.
In the last step ($(t-1)-$step) we merge the sets

$$\begin{aligned} \Delta_{t-1} &= \{-\lambda_{2t} - \sum_{k=2}^{t-1} S_k, \lambda_{t-1}, *, \dots, *\} \in \mathcal{S}_{2t-2} \text{ and} \\ \Gamma_t &= \{-\lambda_{t+1}, \lambda_{t+1}\}. \end{aligned}$$

Let $\varepsilon_t = -S_t = -(\lambda_t + \lambda_{t+1})$. Then from Lemma 2 we obtain

$$\begin{aligned}\Delta_t &= \{-\lambda_{2t} - \sum_{k=2}^{t} S_k, \lambda_t, *, \ldots, *\} \\ &= \{\lambda_1, \lambda_2, \ldots, \lambda_t, \lambda_{t+1}, \ldots, \lambda_{2t-1}, \lambda_{2t}\} \in \mathcal{S}_{2t}.\end{aligned}$$

Now, if n is odd with $\lambda_{\frac{2t+1}{2}} < 0$ then we also merge Δ_t with $\Gamma_{\frac{2t+1}{2}} = \{0\}$ to obtain

$$\begin{aligned}\Delta_t' &= \{-\lambda_{2t} - \sum_{k=2}^{t} S_k - S_{\frac{2t+1}{2}}, \lambda_{\frac{2t+1}{2}}, \lambda_t, *, \ldots, *\} \\ &= \{\lambda_1, \ldots, \lambda_t, \lambda_{\frac{2t+1}{2}}, \lambda_{t+1}, \ldots, \lambda_{2t}\} \in \mathcal{S}_{2t+1}.\end{aligned}$$

Thus, if A is a symmetrix nonnegative matrix realizing $\Delta_t = \cup_{k=1}^{t}\Lambda_k$ $(\Delta_t' = \cup_{k=1}^{t}\Lambda_k \cup \Lambda_{\frac{n+1}{2}})$, then $A \oplus B$ realizes $\Lambda = \{\lambda_1; \lambda_2, \ldots, \lambda_n\}$. That is $\Lambda \in \mathcal{S}_n$. $\square$

4. Constructing the realizing matrix

Let $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ be as in Theorem 1 with $\lambda_1 = -\lambda_n - \sum_{S_k<0} S_k$. Consider the partition

$$\Lambda = \cup_{k=1}^{[\frac{n}{2}]}\Lambda_k \quad \text{with} \quad \Lambda = \cup_{k=1}^{[\frac{n}{2}]}\Lambda_k \cup \Lambda_{\frac{n+1}{2}} \quad \text{for } n \text{ odd},$$

where $\Lambda_k = \{\lambda_k, \lambda_{n-k+1}\}$; $k = 1, 2, \ldots, [\frac{n}{2}]$ and $\Lambda_{\frac{n+1}{2}} = \left\{\lambda_{\frac{n+1}{2}}\right\}$ for n odd. For $k = 2, 3, \ldots, [\frac{n}{2}]$ let

$$\mathbf{A} = \{\Lambda_k : S_k = \lambda_k + \lambda_{n-k+1} < 0\}$$
$$\mathbf{B} = \{\Lambda_k : S_k = \lambda_k + \lambda_{n-k+1} \geq 0\}.$$

Note that $\mathbf{A}$ or $\mathbf{B}$ can be empty, $n \geq 3$, and $\Lambda_{\frac{n+1}{2}}$ can be in $\mathbf{A}$.or $\mathbf{B}$. Each set $\Lambda_k \in \mathbf{B}$ is realizable in particular by the symmetric nonnegative matrix B_k in (3.1). Then the direct sum $B = \oplus B_k$, with $B_{\frac{n+1}{2}} = \left(\lambda_{\frac{n+1}{2}}\right)$ if $\lambda_{\frac{n+1}{2}} \geq 0$ for n odd, is a symmetric nonnegative matrix realizing $\cup\Lambda_k$ with $\Lambda_k \in \mathbf{B}$. Now we consider the nonrealizable sets $\Lambda_k \in \mathbf{A}$, which can be numbered as $\Lambda_2, \Lambda_3, \ldots, \Lambda_t$, $t \leq [\frac{n}{2}]$ with $\Lambda_{\frac{2t+1}{2}} = \{\lambda_{\frac{2t+1}{2}}\}$ if $\lambda_{\frac{2t+1}{2}} < 0$ for n odd. For each $\Lambda_k \in \mathbf{A}$ we define the associated set Γ_k, $k = 2, 3, \ldots, t$, as in (3.2) and $\Gamma_1 = \{-\lambda_{2t}, \lambda_{2t}\}$, which are realizable in particular by the symmetric nonnegative matrix A_k, $k = 1, 2, \ldots, t$, as in (3.3).

As in the proof of Theorem 1 we merge the sets Γ_1 and Γ_2 to obtain $\Delta_2 = \{-\lambda_{2t} - S_2, \lambda_2, \lambda_{2t-1}, \lambda_{2t}\} \in \mathcal{S}_4$. Then a symmetric nonnegative matrix which realizes Δ_2 is

$$M_4 = \begin{pmatrix} A_1 & \rho_2 v_2 u_2^T \\ \rho_2 u_2 v_2^T & A_2 \end{pmatrix}$$

where $v_2^T = u_2^T = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ and $\rho_2 = \sqrt{(\lambda_2 + \lambda_{2t})(\lambda_2 + \lambda_{2t-1})}$. Next we merge Δ_2 with Γ_3 to obtain

$$\Delta_3 = \{-\lambda_{2t} - S_2 - S_3, \lambda_2, \lambda_3, \lambda_{2t-2}, \lambda_{2t-1}, \lambda_{2t}\} \in \mathcal{S}_6,$$

which, according to Lemma 1, is realized by the symmetric nonnegative matrix

$$M_6 = \begin{pmatrix} M_4 & \rho_3 v_3 u_3^T \\ \rho_3 u_3 v_3^T & A_3 \end{pmatrix},$$

where $M_4 v_3 = (-\lambda_{2t} - S_2) v_3$, $\|v_3\| = 1$ and $A_3 u_3 = (-\lambda_{2t-2}) u_3$, $\|u_3\| = 1$ and ρ_3 must be such that

$$C_3 = \begin{pmatrix} -\lambda_{2t} - S_2 & \rho_3 \\ \rho_3 & -\lambda_{2t-2} \end{pmatrix}$$

has eigenvalues $-\lambda_{2t} - S_2 - S_3$ and λ_3. The process shows that, in the $(k-1)-$step, we may compute the matrix

$$M_{2k} = \begin{pmatrix} M_{2k-2} & \rho_k v_k u_k^T \\ \rho_k u_k v_k^T & A_k \end{pmatrix}, \quad k = 2, 3, \ldots, t,$$

where M_{2k-2} is the symmetric nonnegative matrix with spectrum Δ_{k-1}, v_k and u_k are unit eigenvectors of M_{2k-2} and A_k, respectively, corresponding to the eigenvalues $-\lambda_{2t} - \sum_{j=2}^{k-1} S_j$ and λ_{2t-k+1}, respectively,.and ρ_k must be such that the matrix

$$C_k = \begin{pmatrix} -\lambda_{2t} - \sum_{j=2}^{k-1} S_j & \rho_k \\ \rho_k & -\lambda_{2t-k+1} \end{pmatrix}$$

has eigenvalues $-\lambda_{2t} - \sum_{j=2}^{k} S_j$ and λ_k.

Now we compute symmetric nonnegative matrices with spectrum Λ for $n = 4$ and $n = 5$.

Let $\Lambda = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ satisfying the realizability criterion of Theorem 1. We have two cases:

i) $\lambda_1 \geq -\lambda_4$ with $\lambda_2 + \lambda_3 \geq 0$. Then

$$A = \frac{1}{2}\begin{pmatrix} \lambda_1+\lambda_4 & \lambda_1-\lambda_4 & 0 & 0 \\ \lambda_1-\lambda_4 & \lambda_1+\lambda_4 & 0 & 0 \\ 0 & 0 & \lambda_2+\lambda_3 & \lambda_2-\lambda_3 \\ 0 & 0 & \lambda_2-\lambda_3 & \lambda_2+\lambda_3 \end{pmatrix}.$$

ii) $\lambda_1 \geq -\lambda_4 - (\lambda_2 + \lambda_3)$. Then

$$A = \frac{1}{2}\begin{pmatrix} 0 & -2\lambda_4 & \rho & \rho \\ -2\lambda_4 & 0 & \rho & \rho \\ \rho & \rho & 0 & -2\lambda_3 \\ \rho & \rho & -2\lambda_3 & 0 \end{pmatrix},$$

where $\rho = \sqrt{\lambda_3\lambda_4 - \lambda_1\lambda_2}$.

Let $\Lambda = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\}$ satisfying the realizability criterion of Theorem 1. We have four cases:

i) $\lambda_1 \geq -\lambda_5$ with $\lambda_2 + \lambda_4 \geq 0$ and $\lambda_3 \geq 0$. Then

$$A = \frac{1}{2}\begin{pmatrix} \lambda_1+\lambda_5 & \lambda_1-\lambda_5 & 0 & 0 & 0 \\ \lambda_1-\lambda_5 & \lambda_1+\lambda_5 & 0 & 0 & 0 \\ 0 & 0 & \lambda_2+\lambda_4 & \lambda_2-\lambda_4 & 0 \\ 0 & 0 & \lambda_2-\lambda_4 & \lambda_2+\lambda_4 & 0 \\ 0 & 0 & 0 & 0 & 2\lambda_3 \end{pmatrix}.$$

ii) $\lambda_1 \geq -\lambda_5 - (\lambda_2 + \lambda_4)$ with $\lambda_3 \geq 0$. Then

$$A = \frac{1}{2}\begin{pmatrix} 0 & -2\lambda_5 & \rho & \rho & 0 \\ -2\lambda_5 & 0 & \rho & \rho & 0 \\ \rho & \rho & 0 & -2\lambda_4 & 0 \\ \rho & \rho & -2\lambda_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\lambda_3 \end{pmatrix},$$

where $\rho = \sqrt{\lambda_4\lambda_5 - \lambda_1\lambda_2}$.

iii) $\lambda_1 \geq -\lambda_5 - \lambda_3$ with $\lambda_2 + \lambda_4 \geq 0$. Then

$$A = \frac{1}{2}\begin{pmatrix} 0 & -2\lambda_5 & \sqrt{2}\rho & 0 & 0 \\ -2\lambda_5 & 0 & \sqrt{2}\rho & 0 & 0 \\ \sqrt{2}\rho & \sqrt{2}\rho & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2+\lambda_4 & \lambda_2-\lambda_4 \\ 0 & 0 & 0 & \lambda_2-\lambda_4 & \lambda_2+\lambda_4 \end{pmatrix},$$

where $\rho = \sqrt{-\lambda_1\lambda_3}$.
iv) $\lambda_1 \geq -\lambda_5 - (\lambda_2 + \lambda_4) - \lambda_3$. Then

$$A = \frac{1}{2}\left(\begin{array}{cccc|c} 0 & -2\lambda_5 & \rho & \rho & \\ -2\lambda_5 & 0 & \rho & \rho & 2\eta v \\ \rho & \rho & 0 & -2\lambda_4 & \\ \rho & \rho & -2\lambda_4 & 0 & \\ & & 2\eta v^T & & 0 \end{array}\right),$$

where $v^T = (v_1, v_2, v_3, v_4)$ with $v_1 = v_2 = \frac{p}{\sqrt{2p^2+2q^2}}$, $v_3 = v_4 = \frac{q}{\sqrt{2p^2+2q^2}}$, $p = \mu_1 + \lambda_4 + \rho$, $q = \mu_1 + \lambda_5 + \rho$, $\mu_1 = -\lambda_5 - (\lambda_2 + \lambda_4)$, $\rho = \sqrt{\lambda_4\lambda_5 - \lambda_1\lambda_2}$ and $\eta = \sqrt{-\lambda_1\lambda_3}$.

5. Examples

Example 1. *Let* $\Lambda = \{9, 5, 3, 3, -5, -5, -5, -5\}$. *We have the partition* $\Lambda = \Lambda_1 \cup \Lambda_2 \cup \Lambda_3 \cup \Lambda_4$, *where* $\Lambda_1 = \{9, -5\}$, $\Lambda_2 = \{3, -5\}$, $\Lambda_3 = \{3, -5\}$ *and* $\Lambda_4 = \{5, -5\}$. *We define the associated sets* $\Gamma_1 = \{5, -5\}$, $\Gamma_2 = \{5, -5\}$ *and* $\Gamma_3 = \{5, -5\}$. *Then we merge* Γ_1 *with* Γ_2 *to obtain*

$$A_4 = \begin{pmatrix} 0 & 5 & 1 & 1 \\ 5 & 0 & 1 & 1 \\ 1 & 1 & 0 & 5 \\ 1 & 1 & 5 & 0 \end{pmatrix}$$

havig spectrum $\Delta_2 = \{7, 3, -5, -5\}$. *Next we merge* Δ_2 *with* Γ_3 *and obtain*

$$A_6 = \begin{pmatrix} 0 & 5 & 1 & 1 & 1 & 1 \\ 5 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 5 & 1 & 1 \\ 1 & 1 & 5 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 5 \\ 1 & 1 & 1 & 1 & 5 & 0 \end{pmatrix}$$

with spectrum $\Delta_3 = \{9, 3, 3, -5, -5, -5\}$. *Finally we have*

$$A_8 = \begin{pmatrix} 0 & 5 & 1 & 1 & 1 & 1 & 0 & 0 \\ 5 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 5 & 1 & 1 & 0 & 0 \\ 1 & 1 & 5 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 5 & 0 & 0 \\ 1 & 1 & 1 & 1 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 & 0 \end{pmatrix}.$$

with spectrum $\Lambda \in \mathcal{S}_8$.

Example 2. *Let* $\Lambda = \{7, 5, 1, -3, -4, -6\}$. *Observe that* Λ *does not satisfy Theorem 1. However we still may obtain a symmetrix nonnegative matrix realizing* Λ : *Consider the partition* $\Lambda = \Lambda_1 \cup \Lambda_2$, *where* $\Lambda_1 = \{7, -6\}$ *and* $\Lambda_2 = \{5, 1, -3, -4\}$. *Define* $\Gamma_1 = \{6, -6\}$ *and* $\Gamma_2 = \{6, 1, -3, -4\}$. *Then*

$$A_2 = \begin{pmatrix} 0 & 4 & \frac{\sqrt{6}}{2} & \frac{\sqrt{6}}{2} \\ 4 & 0 & \frac{\sqrt{6}}{2} & \frac{\sqrt{6}}{2} \\ \frac{\sqrt{6}}{2} & \frac{\sqrt{6}}{2} & 0 & 3 \\ \frac{\sqrt{6}}{2} & \frac{\sqrt{6}}{2} & 3 & 0 \end{pmatrix}$$

realizes Γ_2 *while*

$$A_1 \begin{pmatrix} 0 & 6 \\ 6 & 0 \end{pmatrix} \quad \textit{realizes } \Gamma_1.$$

By applying Lemma 2 to Γ_2 *and* Γ_1 *we obtain* Λ *and from Lemma 1 we may compute the realizing matrix*

$$A = \begin{pmatrix} 0 & 4 & \frac{\sqrt{6}}{2} & \frac{\sqrt{6}}{2} & \sqrt{\frac{3}{20}} & \sqrt{\frac{3}{20}} \\ 4 & 0 & \frac{\sqrt{6}}{2} & \frac{\sqrt{6}}{2} & \sqrt{\frac{3}{20}} & \sqrt{\frac{3}{20}} \\ \frac{\sqrt{6}}{2} & \frac{\sqrt{6}}{2} & 0 & 3 & \frac{1}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ \frac{\sqrt{6}}{2} & \frac{\sqrt{6}}{2} & 3 & 0 & \frac{1}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ \sqrt{\frac{3}{20}} & \sqrt{\frac{3}{20}} & \frac{1}{\sqrt{10}} & \frac{1}{\sqrt{10}} & 0 & 6 \\ \sqrt{\frac{3}{20}} & \sqrt{\frac{3}{20}} & \frac{1}{\sqrt{10}} & \frac{1}{\sqrt{10}} & 6 & 0 \end{pmatrix}.$$

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