Proyecciones
Vol. 26, N^o 2, pp .219-236, August 2007.
Universidad Católica del Norte
Antofagasta - Chile
DOI: 10.4067/S0716-09172007000200005

A CONVERGENCE RESULT FOR ASYNCHRONOUS ALGORITHMS AND APPLICATIONS

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Received : August 2006. Accepted : March 2007

Abstract

We give in this paper a convergence result concerning parallel asynchronous algorithm with bounded delays to solve a nonlinear fixed point problems. This result is applied to calculate the solution of a strongly monotone operator. Special cases of these operators are used to solve some problems related to convex analysis like minimization of functionals, calculus of saddle point and variational inequality problem.

Keywords : *Asynchronous algorithm, nonlinear problems, monotone operators, fixed point, optimization problem, variational inequality problem, convex analysis.*

2000 Mathematics Subject Classification. *Primary 68W10, 47H10;Secondary 47Hxx.*

1. Introduction

In this paper, we give a convergence result for parallel asynchronous iterations with bounded delays. The convergence result of these algorithms was shown by many authors. Chazan and Miranker in [5], treated the chaotic iterations using a linear and contractive mapping. In 1975, Miellou [8] extended the works of Chazan and Miranker to the nonlinear case, using a contraction mapping and proposes a model with bounded delays. In 1978, Baudet in [3] generalizes the chaotic iterations of Chazan-Miranker and Miellou and proposes a model where the delays considered can be infinite. In a different context, El Tarazi [6] also established this result by a contraction technique according to a suitable scalar norm. Recently, Bahi [2] gave a convergence result concerning parallel asynchronous algorithm, to solve a linear fixed point problems, using nonexpansive linear mappings with respect to a weighted maximum norm. Our goal is to establish a convergence result concerning parallel asynchronous algorithm to solve a nonlinear fixed point problems, using a nonlinear and nonexpansive mapping. We regard this study as a generalization to the asynchronous case, of all results stated by Benahmed and Addou in [1] and so, we repeat the proofs given in [1] by including the modifications which requires the asynchronous case. Section 2 is devoted to a brief description of asynchronous parallel algorithm. In section 3 we prove the main result concerning the convergence of the general algorithm to a fixed point of a nonlinear operator from $\mathbf{R}^n$ to $\mathbf{R}^n$. This result is applied in section 4 to the operator $F = (I + cT)^{-1}$ $(c > 0)$ which is called the proximal mapping associated with the maximal monotone operator cT (see Rockafellar [13]) to calculate a solution of the operator T. Special cases of these operators are also studied to solve optimization problems and variational inequality problem.

2. Preliminaries

$\mathbf{R}^n$ is considered as the product space $\prod_{i=1}^{\alpha} \mathbf{R}^{n_i}$, where $\alpha \in \mathbf{N} - \{0\}$ and $n = \sum_{i=1}^{\alpha} n_i$. All vectors $x \in \mathbf{R}^n$ considered in this study are splitted in the form $x = (x_1, ..., x_\alpha)$ where $x_i \in \mathbf{R}^{n_i}$. Let $\mathbf{R}^{n_i}$ be equipped with the inner product $\langle ., . \rangle_i$ and the associated norm $\|..\|_i = \langle ., . \rangle_i^{\frac{1}{2}} \cdot \mathbf{R}^n$ will be equipped with the inner product $\langle x, y \rangle = \sum_{i=1}^{\alpha} \langle x_i, y_i \rangle_i$ where $x, y \in \mathbf{R}^n$ and the associated norm $\|x\| = \langle x,x\rangle^{\frac{1}{2}} = \left(\sum\limits_{i=1}^{\alpha}\|x_i\|_i^2\right)^{\frac{1}{2}}$. It will be equipped also with the uniform norm $\|x\|_\infty = \max\limits_{1\le i\le \alpha}\|x_i\|_i$

Definition 2.1. *Define* $J = \{J(p)\}_{p\in\mathbf{N}}$ *a sequence of non empty sub sets of* $\{1,...,\alpha\}$ *and* $S = \{(s_1(p),...,s_\alpha(p))\}_{p\in\mathbf{N}}$ *a sequence of* $\mathbf{N}^\alpha$ *and consider an operator* $F = (F_1,...,F_\alpha) : \mathbf{R}^n \longrightarrow \mathbf{R}^n$. *The asynchronous algorithm associated with* F *is defined by,*

$$\tag{2.1} \begin{cases} x^0 = (x_1^0,...,x_\alpha^0) \in \mathbf{R}^n \\ x_i^{p+1} = \begin{cases} x_i^p & if \quad i \notin J(p) \\ F_i(x_1^{s_1(p)},...,x_\alpha^{s_\alpha(p)}) & if \quad i \in J(p) \end{cases} \\ i = 1,...,\alpha \\ p = 0,1,.. \end{cases}$$

It will be denoted by (F, x^0, J, S). This algorithm describes the behavior of iterative process executed asynchronously on a parallel computer with α processors. At each iteration $p+1$, the i^{th} processor computes x_i^{p+1} by using (2.1).

$J(p)$ is the subset of the indexes of the components updated at the p^{th} step.
$p - s_i(p)$ is the delay due to the i^{th} processor when it computes the i^{th} block at the p^{th} iteration.

If we take $s_i(p) = p$ $\forall i \in \{1,...,\alpha\}$, then (2.1) describes synchronous algorithm (without delay). During each iteration, every processor executes a number of computations that depend on the results of the computations of other processors in the previous iteration. Within an iteration, each processor does not interact with other processors, all interactions takes place at the end of iterations.

If we take
$$\begin{cases} s_i(p) = p & \forall p \in \mathbf{N}, \forall i \in \{1,...,\alpha\} \\ J(p) = \{1,...,\alpha\} & \forall p \in \mathbf{N} \end{cases}$$
then (2.1) describes the algorithm of Jacobi.

If we take
$$\begin{cases} s_i(p) = p & \forall p \in \mathbf{N}, \forall i \in \{1,...,\alpha\} \\ J(p) = p + 1\,(mod\ \alpha) & \forall p \in \mathbf{N} \end{cases}$$

then (2.1) describes the algorithm of Gauss-Seidel.

For more details about asynchronous algorithms see [5], [8], [3], [6] and [4].

Definition 2.2. *An operator F from $\mathbf{R}^n$ to $\mathbf{R}^n$ is said to be nonexpansive with respect to the norm $\|..\|$ if, $\|F(x) - F(x')\| \leq \|x - x'\|$ for all $x, x' \in \mathbf{R}^n$.*

3. The main result

We establish in this section the convergence of the general parallel asynchronous algorithm with bounded delays to a fixed point of a nonlinear operator $F : \mathbf{R}^n \to \mathbf{R}^n$.

Theorem 3.1. *Suppose*
(h_0) $\exists$ a subsequence $\{p_k\}_{k\in\mathbf{N}}$ such that, $\forall i \in \{1, ..., \alpha\}$, $i \in J(p_k)$ and $s_i(p_k) = p_k$
(h_1) $\exists s \in \mathbf{N}$, such that, $\forall i \in \{1, ..., \alpha\}$, $\forall p \in \mathbf{N}$, $p - s \leq s_i(p) \leq p$
(h_2) $\exists u \in \mathbf{R}^n$, $F(u) = u$
(h_3) $\forall x, x' \in \mathbf{R}^n$, $\|F(x) - F(x')\|_\infty \leq \|x - x'\|_\infty$
(h_4) $\forall x, x' \in \mathbf{R}^n$, $\|F(x) - F(x')\|^2 \leq \langle F(x) - F(x'), x - x' \rangle$
Then, for all $x^0 \in \mathbf{R}^n$ the sequence (2.1) is convergent in $\mathbf{R}^n$ to a fixed point x^* of F.

Proof. We follow the steps given in Addou-Benahmed [1] Theorem 4, with important modifications in the step *(i)*. The steps *(ii)* and *(iii)* are similar. We proceed then in three steps:

(i) First, we show that the sequence $\{\|x^p - u\|\}_{p\in\mathbf{N}}$ is convergent. For $p \in \mathbf{N}$, we consider the $(s+1)$ iterates $x^p, x^{p-1}, ..., x^{p-s}$ in the process and put
$$z^p = \max_{0\leq l\leq s} \left\|x^{p-l} - u\right\|_\infty = \max_{p-s\leq l\leq p} \left\|x^l - u\right\|_\infty$$
Then $\forall\, i \in \{1, ..., \alpha\}$ we have, either $i \notin J(p)$ so,
$$\begin{aligned} \left\|x_i^{p+1} - u_i\right\|_i &= \|x_i^p - u_i\|_i \\ &\leq \|x^p - u\|_\infty \\ &\leq \max_{0\leq l\leq s} \left\|x^{p-l} - u\right\|_\infty \\ &= z^p \end{aligned}$$

or $i \in J(p)$ so,

$$\begin{array}{rcl}
\left\| x_i^{p+1} - u_i \right\|_i & = & \left\| F_i\left(x_1^{s_1(p)}, ..., x_\alpha^{s_\alpha(p)}\right) - F_i(u) \right\|_i \\
& \leq & \left\| F\left(x_1^{s_1(p)}, ..., x_\alpha^{s_\alpha(p)}\right) - F(u) \right\|_\infty \\
& \leq & \left\| \left(x_1^{s_1(p)}, ..., x_\alpha^{s_\alpha(p)}\right) - u \right\|_\infty \quad \text{(by } (h_3)\text{)} \\
& = & \left\| x_j^{s_j(p)} - u_j \right\|_j \quad (\text{for some } j, 1 \leq j \leq \alpha) \\
& \leq & \left\| x^{s_j(p)} - u \right\|_\infty \\
& \leq & \max\limits_{p-s \leq l \leq p} \left\| x^l - u \right\|_\infty \quad (\text{use } p - s \leq s_j(p) \leq p) \\
& = & z^p
\end{array}$$

then

$$\forall i \in \{1, ..., \alpha\}, \left\| x_i^{p+1} - u_i \right\|_i \leq z^p$$

that is

$$\left\| x^{p+1} - u \right\|_\infty \leq z^p$$

therefore

$$\begin{array}{rcl}
z^{p+1} & = & \max\limits_{0 \leq l \leq s} \left\| x^{p+1-l} - u \right\|_\infty \\
& = & Max\left\{ \max\limits_{0 \leq l \leq s-1} \left\| x^{p-l} - u \right\|_\infty \ , \ \left\| x^{p+l} - u \right\|_\infty \right\} \\
& \leq & z^p
\end{array}$$

which proves that the sequence $\{z^p\}_{p \in \mathbf{N}}$ is decreasing (positive) then it's convergent. It's limit is

$$\begin{array}{rcl}
\lim\limits_{p \to \infty} z^p & = & \lim\limits_{p \to \infty} \max\limits_{0 \leq l \leq s} \left\| x^{p-l} - u \right\|_\infty \\
& = & \lim\limits_{p \to \infty} \left\| x^{p-j(p)} - u \right\|_\infty \quad (0 \leq j(p) \leq s) \\
& = & \lim\limits_{p \to \infty} \| x^p - u \|_\infty
\end{array}$$

which proves that the sequence $\{\|x^p - u\|_\infty\}_{p \in \mathbf{N}}$ is convergent and so, the sequence $\{x^p\}_{p \in \mathbf{N}}$ is bounded.

(ii) As the sequence $\{x^{p_k}\}_{k\in\mathbf{N}}$ is bounded ($\{p_k\}_{k\in\mathbf{N}}$ is defined by (h_0)), it contains a subsequence noted also $\{x^{p_k}\}_{k\in\mathbf{N}}$ which is convergent in $\mathbf{R}^n$ to an x^*. We show that x^* is a fixed point of F. For this, we consider the sequence $\{y^p = x^p - F(x^p)\}_{p\in\mathbf{N}}$ and prove that $\lim\limits_{k\to\infty} y^{p_k} = 0$.

$$\begin{aligned}\left\|x^{pk}-u\right\|^2 &= \left\|y^{pk}+F\left(x^{pk}\right)-u\right\|^2\\ &= \left\|y^{pk}\right\|^2+\left\|F\left(x^{pk}\right)-u\right\|^2+2\left\langle F\left(x^{pk}\right)-u,y^{pk}\right\rangle\end{aligned}$$

then

$$\left\|y^{pk}\right\|^2=\left\|x^{pk}-u\right\|^2-\left\|F\left(x^{pk}\right)-u\right\|^2-2\left\langle F\left(x^{pk}\right)-u,y^{pk}\right\rangle$$

however

$$\begin{aligned}\langle F(x^{p_k})-u,y^{p_k}\rangle &= \langle F(x^{p_k})-F(u),x^{p_k}-F(x^{p_k})\rangle\\ &= \langle F(x^{p_k})-F(u),[x^{p_k}-F(u)]-[F(x^{p_k})-F(u)]\rangle\\ &= \langle F(x^{p_k})-F(u),x^{p_k}-u\rangle-\|F(x^{p_k})-F(u)\|^2\\ &\geq 0\ (by\ (h_4))\end{aligned}$$

so,

$$\begin{aligned}\|y^{p_k}\|^2 &\leq \|x^{p_k}-u\|^2-\|F(x^{p_k})-u\|^2\\ &= \|x^{p_k}-u\|^2-\left\|x^{p_k+1}-u\right\|^2\ (by\ (h_0))\end{aligned}$$

However, by (i) the sequence $\{\|x^p-u\|_\infty\}_{p\in\mathbf{N}}$ is convergent, then the sequence $\{\|x^p-u\|_\infty\}_{p\in\mathbf{N}}$ is also convergent with limit

$$\begin{aligned}\lim_{p\to\infty}\|x^p-u\| &= \lim_{k\to\infty}\|x^{p_k}-u\|\\ &= \lim_{k\to\infty}\left\|x^{p_k+1}-u\right\|\\ &= \|x^*-u\|\end{aligned}$$

and so

$$\lim_{k\to\infty}\|y^{p_k}\|=0$$

which implies that

$$\lim_{k\to\infty}y^{p_k}=0$$

and so

$$x^* - F(x^*) = 0$$

that is x^* is a fixed point of F.

(iii) We prove as in (i) that the sequence $\{\|x^p - x^*\|_\infty\}_{p\in\mathbf{N}}$ is convergent, so

$$\lim_{p\to\infty} \|x^p - x^*\|_\infty = \lim_{k\to\infty} \left\|x^{pk} - x^*\right\|_\infty = 0$$

Which proves that $x^p \to x^*$ with respect to the uniform norm $\|..\|_\infty$.
■

Remark 3.2. *The hypothesis* (h_0) *means that the processors are synchronized and all the components are infinitely updated at the same iteration. This subsequence can be chosen by the programmer (Bahi [2]).*

Remark 3.3. *The hypothesis* (h_1) *means that the delays dues to the communications between processors and to the calculus are bounded, which means that after* $(s+1)$ *iterations, all the processors are supposed to have update their own data (Bahi [2]).*

Remark 3.4. *The hypothesis* (h_4) *is verified by a large class of operators. For example, the resolvent* $F_\lambda = (I + \lambda T)^{-1}$ *(where* $\lambda > 0$*) associated with a maximal monotone operator* T *(see Lemma 4.3 below). Again, the metric projection* p_c *of a Hilbert space* H *onto a nonempty closed convex set* C*; that is, for* $x \in H$*,* $p_c(x)$ *is the unique element of* C *which satisfies*

$$\|x - p_c(x)\| = \inf_{y\in C} \|x - y\|$$

see for proof, Phelps [10], Examples 1.2.(f). In the linear case, take for example a linear operator which is symmetric, positive semi-definite (or simply positive) and nonexpansive, as shown in the following proposition:

Proposition 3.5. *Let* A *be a linear symmetric positive and nonexpansive operator in* $\mathbf{R}^n$*. Then* A *verify the hypothesis* (h_4)*.*

Proof. Recall that an operator A is said to be symmetric if for all x, y $\in \mathbf{R}^n$, $\langle Ax, y\rangle = \langle x, Ay\rangle$.

(i) The operator $B = I - A$ is symmetric. Indeed, $\forall x, y \in \mathbf{R}^n$

$$\begin{aligned} \langle Bx, y\rangle &= \langle x - Ax, y\rangle = \langle x, y\rangle - \langle Ax, y\rangle \\ &= \langle x, y\rangle - \langle x, Ay\rangle = \langle x, y - Ay\rangle \\ &= \langle x, By\rangle \end{aligned}$$

(ii) The operator B est positive. Indeed, $\forall x \in \mathbf{R}^n$

$$\langle Bx, x\rangle = \langle x - Ax, x\rangle = \|x\|^2 - \langle Ax, x\rangle \geq 0$$

since $\quad \langle Ax, x\rangle \leq \|Ax\|\,\|x\| \leq \|x\|^2$.

(iii) A and B are commuting operators. Indeed,

$$AB = A(I - A) = A - A^2 = (I - A)A = BA$$

(iv) AB is a symmetric operator. Indeed, $\forall x, y \in \mathbf{R}^n$

$$\langle ABx, y\rangle = \langle Bx, Ay\rangle = \langle x, BAy\rangle = \langle x, ABy\rangle$$

(v) AB is a positive operator (see proof in [7], Theorem 10.7).

(vi) The operator A verify the hypothesis (h_4). Indeed, $\forall x \in \mathbf{R}^n$

$$\langle Ax, x\rangle - \|Ax\|^2 = \langle Ax, x - Ax\rangle = \langle Ax, Bx\rangle = \langle ABx, x\rangle \geq 0$$

■

4. Applications

4.1. Solutions of maximal strongly monotone operators

In this section, we apply the parallel asynchronous algorithm with bounded delays to the proximal mapping $F = (I + cT)^{-1}$ $(c > 0)$ associated with the maximal monotone operator cT. We say that a multifunction T from $D(T) \subseteq \mathbf{R}^n \to \mathbf{R}^n$ is monotone if

$$\forall x, x' \in D(T),\ \langle x - x', y - y'\rangle \geq 0,\ \forall y \in Tx,\ \forall y' \in Tx'.$$

It is said to be maximal monotone if, in addition, the graph

$$G(T) = \{(x, y) : x \in D(T) \textit{ and } y \in Tx\}$$

is not properly contained in the graph of any other monotone operator $T' : D(T) \subseteq \mathbf{R}^n \to \mathbf{R}^n$. It is said to be strongly monotone with modulus a $(a > 0)$ or a-strongly monotone if

$$\forall x, x' \in D(T),\ \langle x - x', y - y' \rangle \geq a \left\| x - x' \right\|^2,\ \forall y \in Tx,\ \forall y' \in Tx'.$$

Let T be a multivalued maximal monotone operator defined from $\mathbf{R}^n$ to $\mathbf{R}^n$. A fundamental problem is to determine an x^* in $\mathbf{R}^n$ satisfying $0 \in Tx^*$ which will be called a solution of the operator T. The following Theorem gives a general result concerning the solution of a maximal strongly monotone operator.

Theorem 4.1. *Let T be a multivalued maximal a-strongly monotone operator in $\mathbf{R}^n$ $(a > 0)$. Then*

1. *T has a unique solution x^*.*
2. *Any parallel asynchronous algorithm with bounded delays associated with the single-valued mapping $F = (I + cT)^{-1}$ where $c \geq \frac{\sqrt{\alpha}-1}{a}$ converges in $\mathbf{R}^n$ to the solution x^* of the problem $0 \in Tx$.*

Proof. We give the proof in the form of Lemmas. The two Lemmas 4.2 and 4.3 were shown in [1] by Addou and Benahmed.

Lemma 4.2. *(Addou-Benahmed [1], Theorem 4) Let T be a maximal monotone operator in $\mathbf{R}^n$ and $F = (I + cT)^{-1}$, $(c > 0)$. Then the solutions of T are exactly the fixed points of F in $\mathbf{R}^n$.*

Proof of Lemma 4.2.

$$\begin{aligned} 0 \in Tx &\iff x \in (I + cT)x \\ &\iff x = (I + cT)^{-1}x \\ &\iff x = Fx \end{aligned}$$

■

Lemma 4.3. *(Addou-Benahmed [1], Theorem 4) Let T be a maximal monotone operator and $F = (I + cT)^{-1}$ $(c > 0)$. Then F satisfy the hypothesis (h_4).*

Proof of Lemma 4.3. Consider x, x' in $\mathbf{R}^n$ and show that

$$\|F(x) - F(x')\|^2 \leq \langle F(x) - F(x'), x - x' \rangle$$

Put $y = F(x)$ and $y' = F(x')$ then,

$$\begin{cases} x \in y + cTy \\ x' \in y' + cTy' \end{cases}$$

i.e.

$$\begin{cases} x \in y + cTy \\ x' \in y' + cTy' \end{cases}$$

As $c > 0$, the operator cT is monotone and so,

$$\langle (x - y) - (x' - y'), y - y' \rangle \geq 0$$

therefore

$$\langle x - x', y - y' \rangle - \|y - y'\|^2 \geq 0$$

which implies

$$\|F(x) - F(x')\|^2 \leq \langle F(x) - F(x'), x - x' \rangle$$

■

We complete the proof of Theorem by the Lemma,

Lemma 4.4. Let T be a maximal a-strongly monotone operator in $\mathbf{R}^n$ $(a > 0)$ and $F = (I + cT)^{-1}$ $(c > 0)$. Then

(a) F has a unique fixed point x^*

(b) For $c \geq \frac{\sqrt{\alpha}-1}{a}$, the mapping F is nonexpansive with respect to the norm $\|..\|_\infty$ in $\mathbf{R}^n$

Proof of Lemma 4.4.

(a) Consider $T' = T - aI$, $\beta = \frac{1}{1+ac}$ and $F' = (I + \beta c T')^{-1}$. Take $x \in \mathbf{R}^n$,

$$\begin{aligned} y = F(x) \quad &\Longleftrightarrow \quad x \in y + cTy \\ &\Longleftrightarrow \quad x \in (1+ac)y + c(T - aI)y \\ &\Longleftrightarrow \quad \beta x \in \beta(1+ac)y + \beta c T' y \\ &\Longleftrightarrow \quad \beta x \in (I + \beta c T')y \\ &\Longleftrightarrow \quad y = F'(\beta x) \end{aligned}$$

i.e., $\forall x \in \mathbf{R}^n$, $F(x) = F'(\beta x)$.

As T is a-strongly monotone (maximal), the operator T' is maximal monotone and then the map $F' = (I + \beta c T')^{-1}$ is nonexpansive in $\mathbf{R}^n$, so $\forall x, x' \in \mathbf{R}^n$

$$\|F(x) - F(x')\| = \|F'(\beta x) - F'(\beta x')\| \leq \beta \|x - x'\|$$

As $\beta = \frac{1}{1+ac} < 1$, the application F is contractive in $\mathbf{R}^n$ and then has a unique fixed point x^* (Banach's fixed point Theorem) which will be the solution of the operator T by Lemma 4.2.

(b) The Euclidean norm and the uniform norm are equivalents in $\mathbf{R}^n$ by the relation:

$$\|x\|_\infty \leq \|x\| \leq \sqrt{\alpha}\, \|x\|_\infty \ , \quad \forall\, x \in \mathbf{R}^n$$

Then, $\forall x, x' \in \mathbf{R}^n$

$$\|F(x) - F(x')\| \leq \beta \|x - x'\|$$

implies

$$\begin{aligned} \|F(x) - F(x')\|_\infty &\leq \beta \sqrt{\alpha}\, \|x - x'\|_\infty \\ &= \tfrac{\sqrt{\alpha}}{1+ac} \|x - x'\|_\infty \end{aligned}$$

It is sufficient to take c such that

$$\frac{\sqrt{\alpha}}{1+ac} \leq 1$$

i.e.

$$c \geq \frac{\sqrt{\alpha} - 1}{a} \tag{4.1}$$

■

So, the theorem is entirely shown.

4.2. Minimization of functional

Let's begin by this proposition that provides a characterization of strongly convex functions. Recall that a function $f : \mathbf{R}^n \to \mathbf{R}\cup\{+\infty\}$ is said to be strongly convexe with modulus a ($a > 0$) or a-strongly convex if for all $x, x' \in \mathbf{R}^n$ and $t \in]0, 1[$ one has

$$f(tx + (1-t)x') \leq tf(x) + (1-t)f(x') - \frac{1}{2}at(1-t)\left\|x - x'\right\|^2$$

The subdifferential of a proper (i.e not identically $+\infty$) convex function f on $\mathbf{R}^n$ is the (generally multivalued) mapping $\partial f : \mathbf{R}^n \to \mathbf{R}^n$ defined by

$$\partial f(x) = \{y \in \mathbf{R}^n \ f(x') \geq f(x) + \langle y, x' - x\rangle, \ \forall x' \in \mathbf{R}^n\}$$

which is a maximal monotone operator if in addition f is a lower semicontinuous function (lsc).

Proposition 4.5 (Rockafellar [13, Proposition 6] Let $f : \mathbf{R}^n \to \mathbf{R}\cup\{+\infty\}$ be convex proper and lower semicontinuous. Then the following conditions are equivalent:

(a) f is a-strongly convex

(b) ∂f is a-strongly monotone

(c) whenever $y \in \partial f(x)$ one has for all $x' \in \mathbf{R}^n$:

$$f(x') \geq f(x) + \langle y, x' - x\rangle + \frac{1}{2}a\left\|x - x'\right\|^2$$

Corollary 4.6. Let $f : \mathbf{R}^n \to \mathbf{R}\cup\{+\infty\}$ be a lower semicontinuous a-strongly convex function which is proper. Then

(1) f has a unique minimizer x^*

(2) Any asynchronous parallel algorithm with bounded delays associated with the single-valued mapping $F = (I + c\partial f)^{-1}$ where $c \geq \frac{\sqrt{\alpha}-1}{a}$ converges to the minimizer x^* of f in $\mathbf{R}^n$.

Proof. Remark that

$$\begin{aligned} 0 \in \partial f(x) \quad &\Longleftrightarrow \quad f(x') \geq f(x) \ \forall x' \in \mathbf{R}^n \\ &\Longleftrightarrow \quad f(x) = \min_{x' \in \mathbf{R}^n} f(x') \end{aligned}$$

so, the solutions of the operator ∂f are exactly the minimizer of f.
The subdifferential ∂f is maximal and a-strongly monotone (Proposition 4.5). We apply then Theorem 4.1 to the operator ∂f. ∎

4.3. Saddle point

In this paragraph, we apply Theorem 4.1 to calculate a saddle point of functional $L : \mathbf{R}^n \times \mathbf{R}^m \to [-\infty, +\infty]$. Recall that a saddle point of L is an element (x^*, y^*) of $\mathbf{R}^n \times \mathbf{R}^m$ satisfying

$$L(x^*, y) \leq L(x^*, y^*) \leq L(x, y^*),\ \forall (x, y) \in \mathbf{R}^n \times \mathbf{R}^m$$

which is equivalent to

$$L(x^*, y^*) = \inf_{x \in \mathbf{R}^n} L(x, y^*) = \sup_{y \in \mathbf{R}^m} L(x^*, y)$$

Suppose that $L(x, y)$ is proper and convex lower semicontinuous in $x \in \mathbf{R}^n$, concave upper semicontinuous in $y \in \mathbf{R}^m$, then L is a proper closed saddle function in the terminology of Rockafellar [10]. Let the subdifferential of L at $(x, y) \in \mathbf{R}^n \times \mathbf{R}^m$, $\partial L(x, y)$, be defined as the set of vectors $(z, t) \in \mathbf{R}^n \times \mathbf{R}^m$ satisfying

$$\forall (x', y') \in \mathbf{R}^n \times \mathbf{R}^m \quad L(x, y') - \langle y' - y, t \rangle \leq L(x, y) \leq L(x', y) - \langle x' - x, z \rangle$$

then the multifunction T_L defined in $\mathbf{R}^n \times \mathbf{R}^m$ by

$$T_L(x, y) = \{(z, t) \in \mathbf{R}^n \times \mathbf{R}^m : (z, -t) \in \partial L(x, y)\}$$

is a maximal monotone operator; see Rockafellar [10], [11]. In this case the global saddle points of L (with respect to minimizing in x and maximizing in y) are the elements (x, y) solutions of the problem $(0, 0) \in T_L(x, y)$. That is

$$(0, 0) \in T_L(x^*, y^*) \Longleftrightarrow (x^*, y^*) = \arg \min_{x \in \mathbf{R}^n} \max_{y \in \mathbf{R}^m} L(x, y)$$

Definition 4.7. The functional L from $\mathbf{R}^n \times \mathbf{R}^m$ to $[-\infty, +\infty]$ is said to be strongly convex-concave with modulus a $(a > 0)$ or a-strongly convex-concave if $L(x, y)$ is a-strongly convex in x and a-strongly concave in y.

Lemma 4.8. If the functional L is a-strongly convex-concave, then the multifunction T_L is an a-strongly monotone operator.

Proof. Define the inner product and the norm in $\mathbf{R}^n \times \mathbf{R}^m$ as follows: For $(x, y), (x', y') \in \mathbf{R}^n \times \mathbf{R}^m$:

$$\begin{cases} \langle (x, y), (x', y') \rangle_{\mathbf{R}^n \times \mathbf{R}^m} = \langle x, x' \rangle_{\mathbf{R}^n} + \langle y, y' \rangle_{\mathbf{R}^n} \\ \|(x, y)\|_{\mathbf{R}^n \times \mathbf{R}^m} = \sqrt{\|x\|^2_{\mathbf{R}^n} + \|y\|^2_{\mathbf{R}^m}} \end{cases}$$

we write simply as

$$\begin{cases} \langle (x,y)\,,(x',y')\rangle = \langle x,x'\rangle + \langle y,y'\rangle \\ \|(x,y)\| = \sqrt{\|x\|^2 + \|y\|^2} \end{cases}$$

Consider $(x,y),(x',y')$ in $\mathbf{R}^n\times\,\mathbf{R}^m$, $(z,t)\in T_L(x,y)$ and $(z',t')\in T_L(x',y')$ and show that

$$\langle (x,y)-(x',y'),(z,t)-(z',t')\rangle \geq a\,\|(x,y)-(x'y')\|^2$$

The function $L(x,y)$ is a-strongly convex in x. Proposition 4.5 implies that the operator $\partial_x L$ is a-strongly monotone in x. As $z\in\partial_x L(x,y)$ and $z'\in\partial_x L(x',y')$ we obtain,

$$\langle z-z',x-x'\rangle \geq a\,\|x-x'\|^2$$

In the same way, $-t\in\partial_y L(x,y)$, $-t'\in\partial_y L(x',y')$ and $\partial_y(-L)$ is a-strongly monotone in y (use proposition 4.5 with $f(y)=-L(x,y)$) we obtain,

$$\langle (-t)-(-t'),y-y'\rangle \leq -a\,\|y-y'\|^2$$

thus is

$$\langle t-t',y-y'\rangle \geq a\,\|y-y'\|^2$$

therefore

$$\langle z-z',x-x'\rangle + \langle t-t',y-y'\rangle \geq a\left(\|x-x'\|^2 + \|y-y'\|^2\right)$$

i.e.

$$\langle (z,t)-(z',t'),(x,y)-(x',y')\rangle \geq a\,\|(x,y)-(x',y')\|^2$$

which proves that T_L is a-strongly monotone in $\mathbf{R}^n\times\,\mathbf{R}^m$. ■

If L is a a-strongly convex-concave function and proper closed then T_L is maximal (see Rockafellar [11]) a-strongly monotone (Lemma 4.8). We can then apply Theorem 4.1 to the operator T_L so,

Corollary 4.9. Let L be a proper closed a-strongly convex-concave function from $\mathbf{R}^n \times \mathbf{R}^m$ into $[-\infty, +\infty]$. Then

1. L has a unique saddle point (x^*, y^*).

2. Any parallel asynchronous algorithm with bounded delays associated with the single-valued mapping $F = (I + cT_L)^{-1}$ where $c \geq \frac{\sqrt{\alpha}-1}{a}$ from $\mathbf{R}^n \times \mathbf{R}^m$ into $\mathbf{R}^n \times \mathbf{R}^m$ converges to the saddle point (x^*, y^*) of L.

4.4. Variational inequality

Let C be a nonempty closed convex set in $\mathbf{R}^n$ and A a multivalued maximal monotone operator in $\mathbf{R}^n$ such that $D(A) = C$. The variational inequality problem in its general form consists of finding $x^* \in C$ satisfying

$$\exists\, y^* \in Ax^*,\ \langle y^*, x - x^* \rangle \geq 0,\ \forall x \in C. \tag{4.2}$$

For $x \in \mathbf{R}^n$, let $N_c(x)$ be the normal cone to C at x defined by

$$N_c(x) = \{y \in \mathbf{R}^n : \langle y, x - z \rangle \geq 0,\ \forall z \in C\}.$$

The multifunction T defined in $\mathbf{R}^n$ by,

$$Tx = \begin{cases} Ax + N_c(x) & if\ x \in C \\ \emptyset & if\ x \notin C \end{cases} \tag{4.3}$$

is a maximal monotone operator (Rockafellar [12]).

Lemma 4.10 If A is a-strongly monotone then T is an a-strongly monotone operator.

Proof. Consider $x, x' \in D(T) = C$, $y \in Tx$ and $y' \in Tx'$ then

$$\begin{cases} y = y_1 + y_2,\ y_1 \in Ax,\ y_2 \in N_c(x) \\ y' = y_1' + y_2',\ y_1' \in Ax',\ y_2' \in N_c(x') \end{cases}$$

however

$$\begin{cases} y_2 \in N_c(x) \Rightarrow \langle y_2, x - z \rangle \geq 0,\ \forall z \in C \\ y_2' \in N_c(x') \Rightarrow \langle y_2', x' - z \rangle \geq 0,\ \forall z \in C \end{cases}$$

therefore

$$\begin{aligned} \langle y - y', x - x' \rangle &= \langle y_1 - y_1', x - x' \rangle + \langle y_2 - y_2', x - x' \rangle \\ &= \underbrace{\langle y_1 - y_1', x - x' \rangle}_{\geq a\|x - x'\|^2} + \underbrace{\langle y_2, x - x' \rangle}_{\geq 0} + \underbrace{\langle y_2' - y_1', x' - x \rangle}_{\geq 0} \\ &\geq a \|x - x'\|^2 \end{aligned}$$

■

Lemma 4.11 The solutions of the operator T are exactly the solutions of the variational inequality problem (4.2).

Proof.

$$\begin{aligned} 0 \in Tx^* &\iff 0 \in Ax^* + N_c(x^*) \\ &\iff \exists y^* \in Ax^* : 0 \in y^* + N_c(x^*) \\ &\iff \exists y^* \in Ax^* : -y^* \in N_c(x^*) \\ &\iff \exists y^* \in Ax^* : \langle -y^*, x^* - z \rangle \geq 0 \ \forall z \in C \\ &\iff \exists y^* \in Ax^* : \langle y^*, z - x^* \rangle \geq 0 \ \forall z \in C \\ &\iff x^* \ \textit{is solution of} \ (4.2) \end{aligned}$$

■

By applying the Lemma 4.10, Lemma 4.11 and Theorem 4.1, we can write

Corollary 4.12. Let C be a nonempty closed convex set in $\mathbf{R}^n$ and A a multivalued maximal a-strongly monotone operator in $\mathbf{R}^n$ such that $D(A) = C$. Then

1. The variational inequality problem (4.2) has a unique solution x^*.
2. Any parallel asynchronous algorithm with bounded delays associated with the single-valued mapping $F = (I + cT)^{-1}$ where T defined by (4.3) and $c \geq \frac{\sqrt{\alpha}-1}{a}$ converges to the solution x^* of the problem (4.2).

Acknowledgments: The author would like to thank the anonymous referees for their careful reading and useful remarks.

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