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ON THE LOCAL CONVERGENCE OF A TWO-STEP STEFFENSEN-TYPE METHOD FOR SOLVING GENERALIZED EQUATIONS

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Abstract

We use a two–step Steffensen–type method [1], [2], [4], [6], [13]–[16] to solve a generalized equation in a Banach space setting under Hölder–type conditions introduced by us in [2], [6] for nonlinear equations. Using some ideas given in [4], [6] for nonlinear equations, we provide a local convergence analysis with the following advantages over related [13]–[16]: finer error bounds on the distances involved, and a larger radius of convergence. An application is also provided.

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1. Introduction

In this study we are concerned with the problem of approximating a locally unique solution x^* of the generalized equation

$$0 \in f(x) + G(x), \tag{1.1}$$

where f is a continuous function defined in a neighborhood V of the solution x^* included in a Banach space X with values in itself, and G is a set–valued map from X to its subsets with closed graph.
Many problems in mathematical programming, mathematical economics, variational inequalities and other fields can be formulated as in equation (1.1) [3], [5], [6], [8], [11], [12], [18]–[21] (see also the application at the end of the study).
We consider the two–step Steffensen–type method [1], [2], [4], [6], [13]–[16] for $x_0 \in V$ being the initial guess and all $k \geq 0$

$$\left\{ \begin{array}{l} 0 \in f(x_k) + [g_1(x_k), g_2(x_k); f]\ (y_k - x_k) + G(y_k) \\ 0 \in f(y_k) + [g_1(x_k), g_2(x_k); f]\ (x_{k+1} - y_k) + G(x_{k+1}), \end{array} \right. \tag{1.2}$$

where g_1 and g_2 are a continuous functions from V into X and $[x, y; f] \in L(X)$ (the space of bounded linear operator on X) is a divided difference of order one of f at the points x, y satisfying

$$[x, y; f]\ (y - x) = f(y) - f(x), \text{ for all } x \neq y. \tag{1.3}$$

Note that if f is Fréchet–differentiable at x, then $[x, x; f] = \nabla f(x)$.

For $G \equiv 0$, (1.2) reduces to methods studied in [1], [4], [6] for nonlinear equations.

Recently in [13] a local convergence analysis was provided for method (1.2) under Hölder–type conditions introduced by us in [4], [6] to solve nonlinear equations.

Motivated by optimization considerations, and using the ideas from [4], [6], [7] for nolinear equations we provide under less computational cost a new local convergence analysis for method (1.2) with the following advantages over the corresponding results in [13]–[16]: finer error bounds on the distances $\| x_k - x^* \|$ ($k \geq 0$), and a larger radius of convergence leading to fewer steps and a wider choice of initial guesses x_0.
This observation is very important in computational mathematics [1]–[22]. The study ends with an application.

2. Preliminaries and assumptions

In order to make the paper as self–contained as possible we reintroduce some results on fixed point theorem [6]–[9], [13]–[16].

We let $\mathcal{Z}$ be a metric space equipped with the metric ρ. For $A \subset \mathcal{Z}$, we denote by dist $(x, A) = \inf \{\rho(x, y),\ y \in A\}$ the distance from a point x to A. The excess e from A to the set $C \subset \mathcal{Z}$ is given by $e(A, C) = \sup \{\text{dist}\,(x, A),\ x \in C\}$. Let $\Lambda : XY$ be a set–valued map, we denote by $gph\,\Lambda = \{(x, y) \in X \times Y,\ y \in \Lambda(x)\}$ and $\Lambda^{-1}(y) = \{x \in X,\ y \in \Lambda(x)\}$ is the inverse of Λ. We call $B_r(x)$ the closed ball centered at x with radius r.

Definition 2.1. (see [8], [17], [20])

A set–valued Λ is said to be pseudo–Lipschitz around $(x_0, y_0) \in gph\,\Lambda$ with modulus M if there exist constants a and b such that

$$e(\Lambda(y') \cap B_a(y_0), \Lambda(y'')) \leq M \parallel y' - y'' \parallel, \quad \text{for all } y' \text{ and } y'' \text{ in } B_b(x_0). \tag{2.1}$$

Definition 2.2. ([6])

Let Ω be open subset of X, we say that the operator $[.,.; f]$ is $(\nu_0,\ \nu,\ p)$–Hölder continuous in Ω where $\nu_0 \geq 0$, $\nu \geq 0$ and $p \in [0, 1]$ if the following inequalities hold

$$\parallel [x, x^*; f] - [y, u; f] \parallel \leq\ \nu_0(\parallel x - y \parallel^p + \parallel x^* - u \parallel^p), \tag{2.2}$$

$$\begin{gathered}\parallel [x, y; f] - [u, v; f] \parallel \leq\ \nu(\parallel x - u \parallel^p + \parallel y - v \parallel^p),\\ \text{for all } x, y, u, v \in \Omega.\end{gathered} \tag{2.3}$$

We need the following fixed point theorems.

Lemma 2.3. (see [9]) *Let $(Z, \parallel . \parallel)$ be a Banach space, let ϕ a set–valued map from Z into the closed subsets of Z, let $\eta_0 \in Z$ and let r and λ be such that $0 \leq \lambda < 1$ and*
(a) dist $(\eta_0, \phi(\eta_0)) \leq r(1 - \lambda)$,
(b) $e(\phi(x_1) \cap B_r(\eta_0), \phi(x_2)) \leq \lambda \ \parallel x_1 - x_2 \parallel,\ \ \forall x_1, x_2 \in B_r(\eta_0)$,
then ϕ has a fixed–point in $B_r(\eta_0)$. That is, there exists $x \in B_r(\eta_0)$ such that $x \in \phi(x)$. If ϕ is single–valued, then x is the unique fixed point of ϕ in $B_r(\eta_0)$.

We suppose that, for every distinct points x and y in a open neighborhood V of x^*, there exists a first order divided difference of f at these points. We will make the following assumptions:

($\mathcal{H}\prime$) For $i = 1, 2$; the function g_i is α_i–center–Lipschitz from V into V, with $g_i(x^*) = x^*$, and $\alpha_i \in [0, 1)$. That is

$$(2.4)\quad \| g_1(x) - g_1(x^*) \| \leq \alpha_1 \| x - x^* \| \text{ and } \| g_2(x) - g_2(x^*) \| \leq \alpha_2 \| x - x^* \|, \text{ for all } x \in V;$$

($\mathcal{H}\infty$) $[., .; f]$ is (ν_0, ν, p)–Hölder continuous in V.
($\mathcal{H}\in$) The set–valued map $(f(x^*) + G)^{-1}$ is M–pseudo–Lipschitz around $(0, x^*)$.
($\mathcal{H}\ni$) For all $x, y \in V$, we have $||[x, y; f]|| \leq d$ with $M\, d < 1$, and $\| f(x) - f(x^*) \| \leq d_0 \| x - x^* \|$.

Before stating the main result on this study, we need to introduce some notations. First, for $k \in I\!N$ and (y_k), (x_k) defined in (1.2), let us define the set–valued mappings Q, ψ_k, ϕ_k : XX by the following

$$(2.5)\quad Q(.) := f(x^*) + G(.); \quad \psi_k(.) := Q^{-1}(Z_k(.)); \quad \phi_k(.) := Q^{-1}(W_k(.))$$

where Z_k and W_k are defined from X to X by

$$(2.6)\quad \begin{aligned} Z_k(x) &:= f(x^*) - f(y_k) - [g_1(x_k), g_2(x_k); f](x - y_k) \\ W_k(x) &:= f(x^*) - f(x_k) - [g_1(x_k), g_2(x_k); f](x - x_k) \end{aligned}$$

3. Local convergence analysis for method (1.2)

We show the main local convergence result for method (1.2):

Theorem 3.1. *We suppose that assumptions ($\mathcal{H}\prime$)–($\mathcal{H}\ni$) are satisfied. For every constant* $C > C_0 = \dfrac{M\, \nu_0\, ([1 + \alpha_1]^p + \alpha_2^p)}{1 - M\, d}$, *there exist* $\delta > 0$ *such that for every starting point* x_0 *in* $B_\delta(x^*)$ *(x_0 and x^* distincts), and a sequence* (x_k) *defined by (1.2) which satisfies*

$$(3.1)\quad \| x_{k+1} - x^* \| \leq C \| x_k - x^* \|^{p+1} .$$

The proof of Theorem 3.1 is by induction on k. We need to give two lemmas. In the first, we prove the existence of starting point y_0 for x_0 in V. In the second, we state a result which the starting point (x_0, y_0).

Let us mention that y_0 and x_1 are a fixed points of ϕ_0 and ψ_0 respectively if and only if $0 \in f(x_0) + [g_1(x_0), g_2(x_0); f](y_0 - x_0) + G(y_0)$ and $0 \in f(y_0) + [g_1(x_0), g_2(x_0); f](x_1 - y_0) + G(x_1)$ respectively.

Proposition 3.2. *Under the assumptions of Theorem 3.1, there exists $\delta > 0$ such that for every starting point x_0 in $B_\delta(x^*)$ (x_0 and x^* distincts), the set–valued map ϕ_0 has a fixed point y_0 in $B_\delta(x^*)$, and satisfying*

$$\| y_0 - x^* \| \leq C \| x_0 - x^* \|^{p+1} . \tag{3.2}$$

Proof of the Proposition 3.2.

By hypothesis ($\mathcal{H}\in$) there exist positive numbers M, a and b such that

$$e(Q^{-1}(y') \cap B_a(x^*), Q^{-1}(y'')) \leq M \| y' - y'' \|, \ \forall y', y'' \in B_b(0). \tag{3.3}$$

Fix $\delta > 0$ such that

$$\delta < \delta_0 = \min \left\{ a \ ; \ \sqrt[p+1]{\frac{b}{4 \ \nu \ ([1+\alpha_1]^p + ([1+\alpha_2]^p)}} \ ; \ \frac{1}{\sqrt[p]{C}} \ ; \ \frac{b}{2\, d_0} \right\}. \tag{3.4}$$

The main idea of the proof of Proposition 3.2 is to show that both assertions (a) and (b) of Lemma 2.3 hold; where $\eta_0 := x^*$, ϕ is the function ϕ_0 defined in (2.5) and where r and λ are numbers to be set. According to the definition of the excess e, we have

$$\text{dist}\,(x^*, \phi_0(x^*)) \leq e\Bigg(Q^{-1}(0) \cap B_\delta(x^*), \phi_0(x^*) \Bigg). \tag{3.5}$$

Moreover, for all point x_0 in $B_\delta(x^*)$ (x_0 and x^* distincts) we have

$$\| W_0(x^*) \| = \| f(x^*) - f(x_0) - [g_1(x_0), g_2(x_0); f](x^* - x_0) \| .$$

Note that for $x \in B_\delta(x^*)$ we get (since $\alpha_i \in [0,1)$)

$$\| g_i(x) - x^* \| \leq \| g_i(x) - g_i(x^*) \| \leq \| x - x^* \| \leq \delta,$$

which implies that $g_i(x) \in B_\delta(x^*)$.

In view of assumptions ($\mathcal{H}\prime$)–($\mathcal{H}\infty$) we obtain

$$\begin{array}{rcl} \| W_0(x^*) \| & = & \| \Big([x_0, x^*; f] - [g_1(x_0), g_2(x_0); f] \Big)(x^* - x_0) \| \\ & \leq & \| [x_0, x^*; f] - [g_1(x_0), g_2(x_0); f] \| \ \| x^* - x_0 \| \\ & \leq & \nu_0 \ (\| x_0 - g_1(x_0) \|^p + \| x^* - g_2(x_0) \|^p) \ \ \| x^* - x_0 \| \\ & \leq & \nu_0 \ ([1+\alpha_1]^p + \alpha_2^p) \ \ \| x^* - x_0 \|^{p+1} \end{array} \tag{3.6}$$

Then (3.4) yields, $W_0(x^*) \in B_b(0)$.

Using (3.3) we have

$$\begin{aligned} e\Big(Q^{-1}(0)\cap B_\delta(x^*), \phi_0(x^*)\Big) &= e\Big(Q^{-1}(0)\cap B_\delta(x^*), Q^{-1}[W_0(x^*)]\Big) \\ &\le M\ \nu_0\ ([1+\alpha_1]^p + \alpha_2^p)\ \parallel x^* - x_0 \parallel^{p+1} \end{aligned} \tag{3.7}$$

By the inequality (3.5), we get

$$\text{dist}\,(x^*, \phi_0(x^*)) \le M\ \nu_0\ ([1+\alpha_1]^p + \alpha_2^p)\ \parallel x^* - x_0 \parallel^{p+1}. \tag{3.8}$$

Since $C(1 - M\,d) > M\ \nu_0\ ([1+\alpha_1]^p + \alpha_2^p)$, there exists $\lambda \in [M\,d, 1[$ such that $C(1-\lambda) \ge M\ \nu_0\ ([1+\alpha_1]^p + \alpha_2^p)$ and

$$\text{dist}\,(x^*, \phi_0(x^*)) \le C\ (1-\lambda)\ \parallel x_0 - x^* \parallel^{p+1}. \tag{3.9}$$

By setting $r := r_0 = C\ \parallel x_0 - x^* \parallel^{p+1}$ we can deduce from the inequality (3.9) that the assertion (a) in Lemma 2.3 is satisfied.

Now, we show that condition (b) of Lemma 2.3 is satisfied.
By (3.4) we have $r_0 \le \delta \le a$ and moreover for $x \in B_\delta(x^*)$ we have

$$\begin{aligned} \parallel W_0(x) \parallel &= \parallel f(x^*) - f(x_0) - [g_1(x_0), g_2(x_0); f](x - x_0) \parallel \\ &\le \parallel f(x^*) - f(x) \parallel + \parallel f(x) - f(x_0) - [g_1(x_0), g_2(x_0); f](x - x_0) \parallel \\ &\le \parallel f(x^*) - f(x) \parallel + \parallel [x_0, x; f] - [g_1(x_0), g_2(x_0); f] \parallel\ \parallel x - x_0 \parallel \end{aligned} \tag{3.10}$$

Using assumptions ($\mathcal{H}\prime$)–($\mathcal{H}\infty$), and ($\mathcal{H}\ni$), we get

$$\begin{aligned} \parallel W_0(x) \parallel &\le d_0\ \parallel x^* - x \parallel + \nu\ (\parallel x_0 - g_1(x_0) \parallel^p + \parallel x - g_2(x_0) \parallel^p)\ \parallel x - x_0 \parallel \\ &\le d_0\ \parallel x^* - x \parallel + \nu \Big((\parallel x_0 - x^* \parallel + \parallel x^* - g_1(x_0) \parallel)^p + \\ &\qquad (\parallel x - x^* \parallel + \parallel x^* - g_2(x_0) \parallel)^p \Big)\ \parallel x - x_0 \parallel \\ &\le d_0\ \delta + \ \nu\ ([1+\alpha_1]^p + ([1+\alpha_2]^p)\ \delta^p\ (2\delta) \\ &= d_0\ \delta + 2\ \nu\ ([1+\alpha_1]^p + ([1+\alpha_2]^p)\ \delta^{p+1} \end{aligned} \tag{3.11}$$

Then by (3.4) we deduce that for all $x \in B_\delta(x^*)$ we have $W_0(x) \in B_b(0)$. Then it follows that for all $x', x'' \in B_{r_0}(x^*)$, we have

$$e(\psi_0(x') \cap B_{r_0}(x^*), \phi_0(x'')) \leq e(\phi_0(x') \cap B_\delta(x^*), \phi_0(x'')),$$

which yields by (3.3)

$$\begin{aligned} e(\phi_0(x') \cap B_{r_0}(x^*), \phi_0(x'')) &\leq M \parallel W_0(x') - W_0(x'') \parallel \\ &\leq M \parallel [g_1(x_0), g_2(x_0); f] \parallel \parallel x'' - x' \parallel \end{aligned} \tag{3.12}$$

Using $(\mathcal{H}\ni)$ and the fact that $\lambda \geq M\, d$, we obtain

$$e(\phi_0(x') \cap B_{r_0}(x^*), \phi_0(x'')) \leq M\ d \parallel x'' - x' \parallel \leq \lambda \parallel x'' - x' \parallel \tag{3.13}$$

and thus condition (b) of Lemma 2.3 is satisfied. Since both conditions of Lemma 2.3 are fulfilled, we can deduce the existence of a fixed point $y_0 \in B_{r_0}(x^*)$ for the map ϕ_0. This finishes the proof of Proposition 3.2.

Proposition 3.3. *Under the assumptions of Theorem 3.1, there exist $\delta > 0$ such that for every starting point x_0 in $B_\delta(x^*)$ and y_0 given by Proposition 3.2 (x_0 and x^* distincts), and the set–valued map ψ_0 has a fixed point x_1 in $B_\delta(x^*)$ satisfying*

$$\parallel x_1 - x^* \parallel \leq C \parallel x_0 - x^* \parallel^{p+1} . \tag{3.14}$$

Idea of the proof of Proposition 3.3.

The proof of Proposition 3.3 is the same one as that of Proposition 3.2. The choise of δ is the same one given by (3.4).

The inequality (3.5) is valid if we replace ϕ_0 by ψ_0.

Moreover, for all point x_0 in $B_\delta(x^*)$ (x_0 and x^* distincts), we have

$$\parallel Z_0(x^*) \parallel = \parallel f(x^*) - f(y_0) - [g_1(x_0), g_2(x_0); f](x^* - y_0) \parallel .$$

In view of assumptions $(\mathcal{H}\prime)$–$(\mathcal{H}\infty)$ we get

$$\begin{aligned} \parallel Z_0(x^*) \parallel &= \parallel \Big([y_0, x^*; f] - [g_1(x_0), g_2(x_0); f]\Big)(x^* - y_0) \parallel \\ &\leq \parallel [y_0, x^*; f] - [g_1(x_0), g_2(x_0); f] \parallel \parallel x^* - y_0 \parallel \\ &\leq \nu_0 \left(\parallel y_0 - g_1(x_0) \parallel^p + \parallel x^* - g_2(x_0) \parallel^p\right) \parallel x^* - y_0 \parallel \end{aligned} \tag{3.15}$$

By Proposition 3.2 and (3.4) we have

$$\begin{aligned} \| Z_0(x^*) \| &\leq C\ \nu_0 \left((C \| x_0 - x^* \|^{p+1} + \alpha_1 \ \| x_0 - x^* \|)^p + \alpha_2^p \right) \ \| x^* - x_0 \|^{p+1} \\ &\leq \nu_0 \ ([1+\alpha_1]^p + \alpha_2^p) \ \| x^* - x_0 \|^{p+1} . \end{aligned} \tag{3.16}$$

Then (3.4) yields, $Z_0(x^*) \in B_b(0)$.

Setting $r := r_0 = C \ \| x_0 - x^* \|^{p+1}$, we can deduce from the assertion (a) in Lemma 2.3 is satisfied.

By (3.4) we have $r_0 \leq \delta \leq a$ and moreover for $x \in B_\delta(x^*)$ we have

$$\begin{aligned} \| Z_0(x) \| &= \| f(x^*) - f(y_0) - [g_1(x_0), g_2(x_0); f](x - y_0) \| \\ &\leq \| f(x^*) - f(x) \| + \| f(x) - f(y_0) - [g_1(x_0), g_2(x_0); f](x - y_0) \| \\ &\leq \| f(x^*) - f(x) \| + \| [y_0, x; f] - [g_1(x_0), g_2(x_0); f] \| \ \| x - y_0 \| \end{aligned} \tag{3.17}$$

Using the assumptions $(\mathcal{H}\prime)$–$(\mathcal{H}\infty)$ and $(\mathcal{H}\ni)$, Proposition 3.2 and (3.4) we obtain

$$\| Z_0(x) \| \ \leq \ d_0\ \delta + 2\ \nu\ ([1+\alpha_1]^p + ([1+\alpha_2]^p)\ \delta^{p+1} \tag{3.18}$$

A slight change in the end of proof of Proposition 3.2 shows that the condition (b) of Lemma 2.3 is satisfied. The existence of a fixed point $x_1 \in B_{r_0}(x^*)$ for the map ψ_0 is ensured. This finishes the proof of Proposition 3.3.

Proof of Theorem 3.1.

Keeping $\eta_0 = x^*$ and setting $r := r_k = C \ \| x^* - x_k \|^{p+1}$, the application of Proposition 3.2 and Proposition 3.3 to the map ϕ_k and ψ_k respectively gives the existence of a fixed points y_k and x_{k+1} for ϕ_k and ψ_k respectively which is an elements of $B_{r_k}(x^*)$. This last fact implies the inequality (3.1), which is the desired conclusion.

Remark 3.4. *The sequence (y_n) given by algorithm (1.2) is also super–linearly convergent to a solution x^* of (1.1).*

Remark 3.5. In order for us to compare our results with corresponding ones in [13], let us introduce assumptions:

$(\mathcal{H}\prime)$' For $i = 1, 2$; there exist parameters α_3, $\alpha_4 \in [0, 1)$ such that

$$\| g_1(x) - g_1(y) \| \leq \alpha_3 \ \| x - y \|, \tag{3.19}$$

$$\| g_2(x) - g_2(y) \| \leq \alpha_4 \| x - y \|, \quad \text{for all } x, y \in V, \tag{3.20}$$

and

$$g_i(x^*) = x^*.$$

$(\mathcal{H}\infty)$' $[.,.;f]$ is (ν, p)–Hölder continuous in V.

$(\mathcal{H}\ni)$' For all $x, y \in V$, we have $||[x, y; f]|| \leq d$, and $M\, d < 1$.

Using **(H0)**', **(H1)**', **(H2)**, **(H3)'**, similar result was shown in [13]. Let us define

$$C_0' = \frac{M\ \nu\,[(1+\alpha_3)^2 + \alpha_4^2]}{1 - M\,d}, \tag{3.21}$$

and

$$\delta_0' = \min\left\{ a\ ;\ \sqrt[p+1]{\frac{b}{4\ \nu\ ([1+\alpha_3]^p + ([1+\alpha_4]^p)}}\ ;\ \frac{1}{\sqrt[p]{C}}\ ;\ \frac{b}{2\,d}. \right\}. \tag{3.22}$$

Assumption **(H0)** is weaker than **(H0)**'. Note also that in general

$$\nu_0 \leq \nu, \tag{3.23}$$

$$d_0 \leq d, \tag{3.24}$$

$$\alpha_1 \leq \alpha_3, \tag{3.25}$$

and

$$\alpha_2 \leq \alpha_4 \tag{3.26}$$

hold, and $\frac{\nu}{\nu_0}$, $\frac{d}{d_0}$, $\frac{\alpha_3}{\alpha_1}$ and $\frac{\alpha_4}{\alpha_2}$ can be arbitrarily large [4], [6]. Hence, if strict inequality hold in any of (3.23)–(3.26) and δ_0 is not equal to a or $\frac{1}{\sqrt[p]{C}}$, then we conclude:

$$C_0 \leq C_0', \tag{3.27}$$

and

$$\delta_0' \leq \delta_0, \tag{3.28}$$

which justify the advantages of our analysis over the corresponding ones in [13] mentioned in the introduction. Similar improvements can immediately follow the same way with the works in [9]–[21].

Application 3.6. (see [18])
Let K be a convex set in $I\!R^n$, P is a topological space and φ is a function from $P \times K$ to $I\!R^n$, the "perturbed" variational inequality problem consists of seeking k_0 in K such that

$$\text{For each } k \in K, \quad (\varphi(p, k_0); k - k_0) \geq 0 \tag{3.29}$$

where $(.;.)$ is the usual scalar product on $I\!R^n$ and p is fixed parameter in P. Let $\mathcal{I}_K$ be a convex indicator function of K and ∂ denotes the subdifferential operator. Then the problem (3.29) is equivalent to problem

$$0 \in \varphi(p, k_0) + \mathcal{H}(k_0) \tag{3.30}$$

with $\mathcal{H} = \partial \mathcal{I}_K$. $\mathcal{H}$ is also called the normal cone of K. The "perturbed" variational inequality problem (3.29) is equivalent to (3.30) which is a generalized equation in the form (1.1). Consequently, we can approximate the solution k_0 of (3.29) using our method (1.2).

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