Abstract

A new definition of almost fuzzy compactness is introduced in L-topological spaces by means of open L-sets and their inequality when L is a complete DeMorgan algebra. It can also be characterized by closed L-sets, regularly closed L-sets, regularly open L-sets and their inequalities. When L is a completely distributive DeMorgan algebra, its many characterizations are presented.

Keywords: L-topology, fuzzy compactness, almost fuzzy compactness, almost continuous, weakly continuous

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1. Introduction

Almost compactness has also been generalized to $L$-topological spaces by many authors (see [2, 3, 4, 6, 9, 10, 14, 15, 16]). These notions of almost fuzzy compactness rely on the structure of the basis lattice $L$, where $L = [0, 1]$ or $L$ is a completely distributive DeMorgan algebra. In [21], a new definition of fuzzy compactness was presented in $L$-fuzzy topological spaces by means of open $L$-sets and their inequality.

In this paper, based on [19, 21], we shall introduce a new definition of almost fuzzy compactness in $L$-topological spaces. When $L$ is a completely distributive DeMorgan algebra, its many characterizations are presented. From these characterizations we know that it is a generalization of the notion of almost fuzzy compactness in [3, 9].

2. Preliminaries

Throughout this paper, $(L, \lor, \land')$ is a complete DeMorgan algebra and $X$ is a nonempty set. $L^X$ is the set of all $L$-fuzzy sets (or $L$-sets for short) on $X$. The smallest element and the largest element in $L^X$ are denoted by $\bot_0$ and $\square_1$.

An element $a$ in $L$ is called a prime element if $a \geq b \land c$ implies $a \geq b$ or $a \geq c$. $a$ in $L$ is called a co-prime element if $a$ is a prime element [7].

The set of non-unit prime elements in $L$ is denoted by $P(L)$. The set of non-zero co-prime elements in $L$ is denoted by $M(L)$.

The binary relation $\prec$ in $L$ is defined as follows: for $a, b \in L$, $a \prec b$ if and only if for every subset $D \subseteq L$, the relation $b \leq \sup D$ always implies the existence of $d \in D$ with $a \leq d$ [5]. In a completely distributive DeMorgan algebra $L$, each element $b$ is a supremum of $\{a \in L \mid a \prec b\}$. In the sense of [11, 23], $\{a \in L \mid a \prec b\}$ is the greatest minimal family of $b$, denoted by $\beta(b)$. Moreover for $b \in L$, define $\alpha(b) = \{a \in L \mid a' \prec b'\}$ and $\alpha^*(b) = \alpha(b) \cap P(L)$.

For $a \in L$ and $A \in L^X$, we use the following notations in [20].

$A^{(a)} = \{x \in X \mid A(x) \leq a\}$, $A_{(a)} = \{x \in X \mid a \in \beta(A(x))\}$,

$A_{[a]} = \{x \in X \mid A(x) \geq a\}$.

An $L$-topological space (or $L$-space for short) is a pair $(X, T)$, where $T$ is a subfamily of $L^X$ which contains $\bot_0, \square_1$ and is closed for any suprema and finite infima. $T$ is called an $L$-topology on $X$. Each member of $T$ is called an open $L$-set and its quasi-complement is called a closed $L$-set.

**Definition 2.1** ([11, 23]). For a topological space $(X, \tau)$, let $\omega_L(\tau)$ denote the family of all the lower semi-continuous maps from $(X, \tau)$ to $L$, i.e.,
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ω_L(τ) = \{ A ∈ L^X | A^{(a)} ∈ τ, a ∈ L \}. Then ω_L(τ) is an L-topology on X, in this case, (X, ω_L(τ)) is called topologically generated by (X, τ).

Definition 2.2 ([11, 23]). An L-space (X, T) is called weakly induced if ∀a ∈ L, ∀A ∈ T, it follows that A^{(a)} ∈ [T], where [T] denotes the topology formed by all crisp sets in T.

It is obvious that (X, ω_L(τ)) is weakly induced.

Lemma 2.3 ([20]). Let (X, T) be a weakly induced L-space, a ∈ L, A ∈ T. Then A^{(a)} is an open L-set in [T].

For a subfamily Φ ⊆ L^X, 2^Φ denotes the set of all finite subfamilies of Φ.

Definition 2.4 ([19, 21]). Let (X, T) be an L-space. G ∈ L^X is called fuzzy compact if for every family U ⊆ T, it follows that

\[ \bigwedge_{x ∈ X} \left( G'(x) ∨ \bigvee_{A ∈ U} A(x) \right) ≤ \bigvee_{y ∈ 2(U)} \bigwedge_{x ∈ X} \left( G'(x) ∨ \bigvee_{A ∈ V} A(x) \right). \]

Lemma 2.5 ([19, 21]). Let L be a complete Heyting algebra, f : X → Y be a map, f^-L : L^X → L^Y is the extension of f, then for any family P ⊆ L^Y, we have:

\[ \bigvee_{y ∈ Y} \left( f^-L(G)(y) ∧ \bigwedge_{B ∈ P} B(y) \right) = \bigvee_{x ∈ X} \left( G(x) ∧ \bigwedge_{B ∈ P} f^-L(B)(x) \right). \]

Definition 2.6 ([1]). Let (X, T_1) and (Y, T_2) be two L-spaces. A map f : (X, T_1) → (Y, T_2) is called

1) almost continuous if f^-L(G) ∈ T_1 for all regularly open L-set G in (Y, T_2);
2) weakly continuous if f^-L(G) ≤ \text{int}(f^-L(\text{cl}(G))) for every open L-set G in (Y, T_2).

Lemma 2.7 ([1]). Let (X, T_1) and (Y, T_2) be two L-spaces. A map f : (X, T_1) → (Y, T_2) is:

1) almost continuous if and only if f^-L(G) is closed in (X, T_1) for all regularly closed L-set G in (Y, T_2);
2) weakly continuous if and only if f^-L(G) ≥ \text{cl}(f^-L(\text{int}(G))) for every closed L-set G in (Y, T_2).
Lemma 2.8 ([1]). The closure of an open $L$-set is regularly closed and the interior of a closed $L$-set is regularly open.

Definition 2.9 ([8]). An $L$-space $(X, T)$ is said to be regular if every open $L$-set $G$ is a supremum of open $L$-sets whose closure is less than $G$.

3. Definition and characterizations of almost fuzzy compactness

Definition 3.1. Let $(X, T)$ be an $L$-space. $G \in L^X$ is called almost fuzzy compact if for every family $\mathcal{U} \subseteq T$, it follows that

$$\bigwedge_{x \in X} \left( G'(x) \vee \bigvee_{A \in \mathcal{U}} A(x) \right) \leq \bigvee_{\mathcal{V} \in \mathcal{P}(\mathcal{U})} \left( G'(x) \vee \bigvee_{A \in \mathcal{V}} \text{cl}(A)(x) \right).$$

Definition 3.2. Let $(X, T)$ be an $L$-space. $G \in L^X$ is called almost countably fuzzy compact if for every countable family $\mathcal{U} \subseteq T$, it follows that

$$\bigwedge_{x \in X} \left( G'(x) \vee \bigvee_{A \in \mathcal{U}} A(x) \right) \leq \bigvee_{\mathcal{V} \in \mathcal{P}(\mathcal{U})} \left( G'(x) \vee \bigvee_{A \in \mathcal{V}} \text{cl}(A)(x) \right).$$

For an open $L$-set $A$, by $A \leq \text{int}(\text{cl}(A))$ we can obtain the following theorem.

Theorem 3.3. Fuzzy compactness $\Rightarrow$ almost fuzzy compactness $\Rightarrow$ almost countable fuzzy compactness.

From Definition 3.1 and Definition 3.2 we can obtain the following theorem by using quasi-complement.

Theorem 3.4. Let $(X, T)$ be an $L$-space. $G \in L^X$ is almost (countably) fuzzy compact if and only if for every (countable) family $\mathcal{P} \subseteq T'$, it follows that

$$\bigvee_{x \in X} \left( G(x) \wedge \bigwedge_{A \in \mathcal{P}} A(x) \right) \geq \bigwedge_{\mathcal{F} \in \mathcal{P}(\mathcal{P})} \bigvee_{x \in X} \left( G(x) \wedge \bigwedge_{A \in \mathcal{F}} \text{int}(A)(x) \right).$$

Definition 3.5 ([21]). Let $(X, T)$ be an $L$-space, $a \in L \setminus \{1\}$ and $G \in L^X$. A family $\mathcal{A} \subseteq L^X$ is said to be
(1) an $a$-shading of $G$ if for any $x \in X$, it follows that
\[ G'(x) \lor \bigvee_{A \in \mathcal{U}} A(x) \not\leq a. \]

(2) a strong $a$-shading of $G$ if \( \bigwedge_{x \in X} \left( G'(x) \lor \bigvee_{A \in \mathcal{U}} A(x) \right) \not\leq a. \)

(3) an $a$-remote family of $G$ if for any $x \in X$, it follows that
\[ G(x) \land \bigwedge_{B \in \mathcal{P}} B(x) \not\geq a. \]

(4) a strong $a$-remote family of $G$ if \( \bigvee_{x \in X} \left( G(x) \land \bigwedge_{B \in \mathcal{P}} B(x) \right) \not\geq a. \)

From Definition 3.1, Definition 3.2, Theorem 3.4 and Theorem 3.5 we immediately obtain the following result.

**Theorem 3.6.** Let $(X, T)$ be an $L$-space and $G \in L^X$. Then the following conditions are equivalent:

(1) $G$ is almost (countably) fuzzy compact.

(2) For any $a \in L \setminus \{1\}$, each (countable) open strong $a$-shading $\mathcal{U}$ of $G$ has a finite subfamily $\mathcal{V}$ such that $\mathcal{V}^-$ is a strong $a$-shading of $G$, where $\mathcal{V}^- = \{ \text{cl}(A) \mid A \in \mathcal{V} \}$.

(3) For any $a \in L \setminus \{0\}$, each (countable) closed strong $a$-remote family $\mathcal{P}$ of $G$ has a finite subfamily $\mathcal{F}$ such that $\mathcal{F}^o$ is a strong $a$-remote family of $G$, where $\mathcal{F}^o = \{ \text{int}(A) \mid A \in \mathcal{F} \}$.

Moreover by means of regularly open $L$-sets and regularly closed $L$-sets, we can give the following characterizations of almost (countable) fuzzy compactness.

**Theorem 3.7.** Let $(X, T)$ be an $L$-space and $G \in L^X$. Then the following conditions are equivalent:

(1) $G$ is almost (countably) fuzzy compact.

(2) For each (countable) family $\mathcal{U}$ of regularly open $L$-sets, it follows that
\[ \bigwedge_{x \in X} G'(x) \lor \bigvee_{A \in \mathcal{U}} A(x) \leq \bigvee_{\mathcal{V} \in 2^{\mathcal{U}}} \left( G'(x) \lor \bigvee_{A \in \mathcal{V}} \text{cl}(A)(x) \right). \]

(3) For each (countable) family $\mathcal{U}$ of regularly closed $L$-sets, it follows that
\[ \bigvee_{x \in X} G(x) \land \bigwedge_{A \in \mathcal{U}} A(x) \geq \bigwedge_{\mathcal{V} \in 2^{\mathcal{U}}} \bigvee_{x \in X} \left( G(x) \land \bigwedge_{A \in \mathcal{V}} \text{int}(A)(x) \right). \]
**Proof.** (2) ⇔ (3) is obvious. Because a regularly open $L$-set is open, we easily obtain (1) ⇒ (2). Now we prove (2) ⇒ (1). Suppose that $\mathcal{U}$ is a family of open $L$-sets. From Lemma 2.8 we know that $\text{int(cl}(A))$ is a regularly open $L$-set for each $A \in \mathcal{U}$. Hence by (2) we obtain

$$
\bigwedge_{x \in X} \left( G'(x) \lor \bigvee_{A \in \mathcal{U}} A(x) \right)
$$

$$
= \bigwedge_{x \in X} \left( G'(x) \lor \bigvee_{A \in \mathcal{U}} \text{int}(A)(x) \right)
$$

$$
\leq \bigwedge_{x \in X} \left( G'(x) \lor \bigvee_{A \in \mathcal{U}} \text{int}(\text{cl}(A))(x) \right)
$$

$$
\leq \bigvee_{V \in 2(\mathcal{U})} \bigwedge_{x \in X} \left( G'(x) \lor \bigvee_{A \in V} \text{int}(\text{cl}(A))(x) \right)
$$

$$
= \bigvee_{V \in 2(\mathcal{U})} \bigwedge_{x \in X} \left( G'(x) \lor \bigvee_{A \in V} \text{cl}(A)(x) \right).
$$

This shows that (1) is true.

Analogous to Theorem 3.6 we have the following result.

**Theorem 3.8.** Let $(X, T)$ be an $L$-space and $G \in L^X$. Then the following conditions are equivalent:

1. $G$ is almost (countably) fuzzy compact.
2. For any $a \in L \setminus \{1\}$, each (countable) regularly open strong $a$-shading $\mathcal{U}$ of $G$ has a finite subfamily $\mathcal{V}$ such that $\mathcal{V}^{-}$ is a strong $a$-shading of $G$.
3. For any $a \in L \setminus \{0\}$, each (countable) regularly closed strong $a$-remote family $\mathcal{P}$ of $G$ has a finite subfamily $\mathcal{F}$ such that $\mathcal{F}^{\circ}$ is a strong $a$-remote family of $G$.

**Theorem 3.9.** Let $(X, T)$ be a regular $L$-space and $G \in L^X$. Then $G$ is fuzzy compact if and only if it is almost fuzzy compact.

**Proof.** The necessity is obvious. Now we prove the sufficiency. Let $\{A_i\}_{i \in \Omega}$ be a family of open $L$-sets. By regularity of $(X, T)$, we know that for each $i \in \Omega$, there exists a family $\{B_{ij} \mid j \in \Delta_i\}$ of open $L$-sets such that $A_i = \bigvee_{j \in \Delta_i} B_{ij}$ and $\text{cl}(B_{ij}) \leq A_i$. By almost fuzzy compactness of $G$,
we know

\[ \bigwedge_{x \in X} \left( G'(x) \lor \bigvee_{i \in \Omega} A_i(x) \right) = \bigwedge_{x \in X} \left( G'(x) \lor \bigvee_{i \in \Omega} B_{ij}(x) \right) \]

\[ \leq \bigvee_{\Gamma \in 2^{(\Omega)}} \bigvee_{\Theta \in 2^{(\Delta_i)}} \bigwedge_{x \in X} \left( G'(x) \lor \bigvee_{i \in \Gamma} \bigvee_{j \in \Theta} \text{cl}(B_{ij})(x) \right) \]

\[ \leq \bigvee_{\Gamma \in 2^{(\Omega)}} \bigwedge_{x \in X} \left( G'(x) \lor \bigvee_{i \in \Gamma} A_i(x) \right). \]

Therefore \( G \) is fuzzy compact.

4. Some properties of almost fuzzy compactness

**Theorem 4.1.** Let \( L \) be a complete Heyting algebra. If both \( G \) and \( H \) are almost (countably) fuzzy compact, then \( G \lor H \) is almost (countably) fuzzy compact.

**Proof.** For any family \( \mathcal{P} \) of closed \( L \)-sets, by Theorem 3.4 we have

\[
\bigvee_{x \in X} \left( (G \lor H)(x) \land \bigwedge_{B \in \mathcal{P}} B(x) \right) = \left\{ \bigvee_{x \in X} \left( G(x) \land \bigwedge_{B \in \mathcal{P}} B(x) \right) \right\} \lor \left\{ \bigvee_{x \in X} \left( H(x) \land \bigwedge_{B \in \mathcal{P}} B(x) \right) \right\}
\]

\[
\geq \left\{ \bigwedge_{F \in 2^{(\mathcal{P})}} \bigvee_{x \in X} \left( G(x) \land \bigwedge_{B \in F} \text{int}(B)(x) \right) \right\} \lor \left\{ \bigwedge_{F \in 2^{(\mathcal{P})}} \bigvee_{x \in X} \left( H(x) \land \bigwedge_{B \in F} \text{int}(B)(x) \right) \right\}
\]

\[
= \bigwedge_{F \in 2^{(\mathcal{P})}} \bigvee_{x \in X} \left( (G \lor H)(x) \land \bigwedge_{B \in F} \text{int}(B)(x) \right).
\]

This shows that \( G \lor H \) is almost fuzzy compact. \( \Box \)

**Theorem 4.2.** If \( G \) is almost (countably) fuzzy compact, and \( H \) is clopen, then \( G \land H \) is almost (countably) fuzzy compact.

**Proof.** Since \( G \) is almost fuzzy compact, for any family \( \mathcal{P} \) of closed
\(L\)-sets, by Theorem 3.4 we have

\[
\bigvee_{x \in X} \left( (G \land H)(x) \land \bigwedge_{B \in \mathcal{P}} B(x) \right)
\]

\[
= \bigvee_{x \in X} \left( G(x) \land \bigwedge_{B \in \mathcal{P}} B(x) \right)
\]

\[
\geq \bigwedge_{\mathcal{F} \in 2^{\mathcal{P} \cup \{H\}}} \bigwedge_{x \in X} \left( G(x) \land \bigwedge_{B \in \mathcal{F}} \text{int}(B)(x) \right)
\]

\[
= \bigwedge_{\mathcal{F} \in 2^{\mathcal{P}}} \bigwedge_{x \in X} \left( G(x) \land \bigwedge_{B \in \mathcal{F}} \text{int}(B)(x) \right)
\]

\[
\land \bigwedge_{\mathcal{F} \in 2^{\mathcal{P}}} \bigwedge_{x \in X} \left( G(x) \land \text{int}(H)(x) \land \bigwedge_{B \in \mathcal{F}} \text{int}(B)(x) \right)
\]

\[
= \bigwedge_{\mathcal{F} \in 2^{\mathcal{P}}} \bigwedge_{x \in X} \left( (G \land H)(x) \land \bigwedge_{B \in \mathcal{F}} \text{int}(B)(x) \right)
\].

This shows that \(G \land H\) is almost fuzzy compact. \(\square\)

**Theorem 4.3.** Let \(L\) be a complete Heyting algebra, and let \(f : (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2)\) be almost continuous. If \(G\) is almost (countably) fuzzy compact in \((X, \mathcal{T}_1)\), then so is \(f_{L}^{-}(G)\) in \((Y, \mathcal{T}_2)\).

**Proof.** Suppose that \(\mathcal{P}\) be a family of regularly closed \(L\)-sets, by Lemma 2.5 and almost fuzzy compactness of \(G\), we have

\[
\bigvee_{y \in Y} \left( f_{L}^{-}(G)(y) \land \bigwedge_{B \in \mathcal{P}} B(y) \right)
\]

\[
= \bigvee_{x \in X} \left( G(x) \land \bigwedge_{B \in \mathcal{P}} f_{L}^{-}(B)(x) \right)
\]

\[
\geq \bigwedge_{\mathcal{F} \in 2^{\mathcal{P}}} \bigwedge_{x \in X} \left( G(x) \land \bigwedge_{B \in \mathcal{F}} \text{int}(f_{L}^{-}(B))(x) \right)
\]

\[
\geq \bigwedge_{\mathcal{F} \in 2^{\mathcal{P}}} \bigwedge_{x \in X} \left( G(x) \land \bigwedge_{B \in \mathcal{F}} f_{L}^{-}(\text{int}(B))(x) \right)
\]

\[
= \bigwedge_{\mathcal{F} \in 2^{\mathcal{P}}} \bigwedge_{y \in Y} \left( f_{L}^{-}(G)(y) \land \bigwedge_{B \in \mathcal{F}} \text{int}(B)(y) \right).
\]

Therefore \(f_{L}^{-}(G)\) is almost fuzzy compact.
Theorem 4.4. Let $L$ be a complete Heyting algebra, and let $f : (X, T_1) \rightarrow (Y, T_2)$ be weakly continuous. If $G$ is (countably) fuzzy compact in $(X, T_1)$, then $f^{-\ast}_L(G)$ is almost (countably) fuzzy compact in $(Y, T_2)$.

Proof. Let $\mathcal{P}$ be a family of regularly closed $L$-sets, by Lemma 2.5 and fuzzy compactness of $G$, we have

$$\bigvee_{y \in Y} \left( f^{-\ast}_L(G)(y) \land \bigwedge_{B \in \mathcal{P}} B(y) \right)$$

$$= \bigvee_{x \in X} \left( G(x) \land \bigwedge_{B \in \mathcal{P}} f^{-\ast}_L(B)(x) \right)$$

$$\geq \bigvee_{x \in X} \left( G(x) \land \bigwedge_{B \in \mathcal{P}} \text{cl}(f^{-\ast}_L(\text{int}(B)))(x) \right)$$

$$\geq \bigwedge_{\mathcal{F} \in 2^{|\mathcal{P}|}} \bigvee_{x \in X} \left( G(x) \land \bigwedge_{B \in \mathcal{F}} f^{-\ast}_L(\text{int}(B))(x) \right)$$

$$= \bigwedge_{\mathcal{F} \in 2^{|\mathcal{P}|}} \bigvee_{y \in Y} \left( f^{-\ast}_L(G)(y) \land \bigwedge_{B \in \mathcal{F}} \text{int}(B)(y) \right).$$

Therefore $f^{-\ast}_L(G)$ is almost fuzzy compact.

5. Further characterizations of almost fuzzy compactness

In this section, we assume that $L$ is a completely distributive DeMorgan algebra.

Definition 5.1 ([21]). Let $(X, T)$ be an $L$-space, $a \in L \setminus \{0\}$ and $G \in L^X$. A family $\mathcal{U} \subseteq L^X$ is called a $\beta_a$-cover of $G$ if for any $x \in X$, it follows that $a \in \beta \left( G'(x) \vee \bigvee_{A \in \mathcal{U}} A(x) \right)$. $\mathcal{U}$ is called a strong $\beta_a$-cover of $G$ if $a \in \beta \left( \bigwedge_{x \in X} \left( G'(x) \vee \bigvee_{A \in \mathcal{U}} A(x) \right) \right)$.

Definition 5.2 ([21]). Let $(X, T)$ be an $L$-space, $a \in L \setminus \{0\}$ and $G \in L^X$. A family $\mathcal{U} \subseteq L^X$ is called a $Q_a$-cover of $G$ if for any $x \in X$, it follows that $G'(x) \vee \bigvee_{A \in \mathcal{U}} A(x) \geq a$.

Analogous to [21] we can obtain the following theorem.
**Theorem 5.3.** Let \((X,T)\) be an \(L\)-space and \(G \subseteq L^X\). Then the following conditions are equivalent.

1. \(G\) is almost (countably) fuzzy compact.
2. For any \(a \in L \setminus \{0\}\) (or \(a \in M(L)\)), each (countable) closed strong \(a\)-remote family \(P\) of \(G\) has a finite subfamily \(F\) such that \(F^o\) is an (a strong) \(a\)-remote family of \(G\).
3. For any \(a \in L \setminus \{0\}\) (or \(a \in M(L)\)) and any (countable) closed strong \(a\)-remote family \(P\) of \(G\), there exist a finite subfamily \(F\) of \(P\) and \(b \in \beta(a)\) (or \(b \in \beta^*(a)\)) such that \(F^o\) is a (strong) \(b\)-remote family of \(G\).
4. For any \(a \in L \setminus \{1\}\) (or \(a \in P(L)\), each (countable) open strong \(a\)-shading \(U\) of \(G\) has a finite subfamily \(V\) such that \(V^-\) is an (a strong) \(a\)-shading of \(G\).
5. For any \(a \in L \setminus \{1\}\) (or \(a \in P(L)\)) and any (countable) open strong \(a\)-shading \(U\) of \(G\), there exist a finite subfamily \(V\) of \(U\) and \(b \in \alpha(a)\) (or \(b \in \alpha^*(a)\)) such that \(V^-\) is a (strong) \(b\)-shading of \(G\).
6. For any \(a \in L \setminus \{0\}\) (or \(a \in M(L)\)), each (countable) open strong \(b_a\)-cover \(U\) of \(G\) has a finite subfamily \(V\) such that \(V^-\) is a (strong) \(b_a\)-cover of \(G\).
7. For any \(a \in L \setminus \{0\}\) (or \(a \in M(L)\)) and any (countable) open strong \(b_a\)-cover \(U\) of \(G\), there exist a finite subfamily \(V\) of \(U\) and \(b \in L\) (or \(b \in M(L)\)) with \(a \in \beta(b)\) such that \(V^-\) is a (strong) \(b_b\)-cover of \(G\).
8. For any \(a \in L \setminus \{0\}\) (or \(a \in M(L)\)) and any \(b \in \beta(a)\setminus \{0\}\), each (countable) open \(Q_a\)-cover of \(G\) has a finite subfamily \(V\) such that \(V^-\) is a \(Q_b\)-cover of \(G\).
9. For any \(a \in L \setminus \{0\}\) (or \(a \in M(L)\)) and any \(b \in \beta(a)\setminus \{0\}\) (or \(b \in \beta^*(a)\)), each (countable) open \(Q_a\)-cover of \(G\) has a finite subfamily \(V\) such that \(V^-\) is a (strong) \(b_b\)-cover of \(G\).

**Remark 5.4.** In Theorem 5.3, ‘open’ can be replaced by ‘regularly open’, and ‘closed’ can be replaced by ‘regularly closed’.

**Remark 5.5.** From (2) of Theorem 5.3 we know that our notion of almost fuzzy compactness is a generalization of almost \(F\)-compactness in [3, 9].

The following theorem shows that almost (countable) fuzzy compactness is a good extension.

**Theorem 5.6.** Let \((X,\tau)\) be a topological space and \((X,\omega(\tau))\) be generated topologically by \((X,\tau)\). Then \((X,\omega(\tau))\) is almost (countably) fuzzy compact if and only if \((X,\tau)\) is almost (countably) compact.
Proof. (Necessity) Let $A$ be an open cover of $(X, \tau)$. Then $\{\chi_A \mid A \in A\}$ is a family of open $L$-sets in $(X, \omega(\tau))$ with $\bigwedge_{x \in X} \bigvee_{A \in A} \chi_A(x) = 1$. From almost fuzzy compactness of $(X, \omega(\tau))$ we know that

$$\bigvee_{V \in 2^{\mathcal{U}}} \bigwedge_{x \in X} \left( \bigvee_{A \in V} \chi_{cl(A)}(x) \right) = \bigvee_{V \in 2^{\mathcal{U}}} \bigwedge_{x \in X} \left( \bigvee_{A \in V} cl(A)(x) \right) = 1.$$ 

This implies that there exists $V \in 2^{\mathcal{U}}$ such that $\bigwedge_{x \in X} \left( \bigvee_{A \in V} \chi_{cl(A)}(x) \right) = 1$. Hence $\{cl(A) \mid A \in V\}$ is a cover of $(X, \tau)$. Therefore $(X, \tau)$ is almost compact.

(Sufficiency) Let $\mathcal{U}$ be a family of open $L$-sets in $(X, \omega(\tau))$ and let $\bigwedge_{x \in X} \left( \bigvee_{B \in \mathcal{U}} B(x) \right) = a$. If $a = 0$, then obviously we have

$$\bigwedge_{x \in X} \left( \bigvee_{B \in \mathcal{U}} B(x) \right) \leq \bigvee_{V \in 2^{\mathcal{U}}} \bigwedge_{x \in X} \left( \bigvee_{A \in V} cl(B)(x) \right).$$

Now we suppose that $a \neq 0$. In this case, for any $b \in \beta(a) \setminus \{0\}$ we have

$$b \in \beta \left( \bigwedge_{x \in X} \left( \bigvee_{B \in \mathcal{U}} B(x) \right) \right) \subseteq \bigcap_{x \in X} \beta \left( \bigvee_{B \in \mathcal{U}} B(x) \right) = \bigcap_{x \in X} \bigcup_{B \in \mathcal{U}} \beta(B(x)).$$

From Lemma 2.3 this implies that $\{B(b) \mid B \in \mathcal{U}\}$ is an open cover of $(X, \tau)$. From almost fuzzy compactness of $(X, \tau)$ we know that there exists $V \in 2^{\mathcal{U}}$ such that $\{cl(B(b)) \mid B \in V\}$ is a cover of $(X, \tau)$. From [17] we can obtain that $cl(B(b)) \subseteq cl(B)[b]$. This shows that $\{cl(B)[b] \mid B \in V\}$ is a cover of $(X, \tau)$. Hence $b \leq \bigwedge_{x \in X} \left( \bigvee_{B \in V} cl(B)(x) \right)$. Further we have

$$b \leq \bigwedge_{x \in X} \left( \bigvee_{B \in V} cl(B)(x) \right) \leq \bigvee_{V \in 2^{\mathcal{U}}} \bigwedge_{x \in X} \left( \bigvee_{B \in V} cl(B)(x) \right).$$

This implies

$$\bigwedge_{x \in X} \left( \bigvee_{B \in \mathcal{U}} B(x) \right) = a = \bigvee \{b \mid b \in \beta(a)\} \leq \bigvee_{V \in 2^{\mathcal{U}}} \bigwedge_{x \in X} \left( \bigvee_{B \in V} cl(B)(x) \right).$$

Therefore $(X, \omega(\tau))$ is almost fuzzy compact.
References


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