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PRODUCTS OF LF-TOPOLOGIES AND SEPARATION IN LF-TOP

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Abstract

For a GL-monoid L provided with an uniform structure, we build an LF-topology on the cartesian product of a family of LF-topological spaces. We also show that the product of an arbitrary family of Kolmogoroff (Hausdorff) LF-topological spaces is again a Kolmogoroff (Hausdorff) LF-topological space.

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0. Introduction

Over the years, a number of descriptions of products of fuzzy topological spaces have appeared. As mathematicians have attempted to develop and extend topics of general topology in various ways using the concept of fuzzy subsets of an ordinary set, it is not surprising that searches for such products were obtined with different degrees of success, depending on the structure of the underlying lattice L. The aim of this paper is to give a characterization of arbitrary products of LF-topological spaces, when the underlying lattice L is a GL - Monoid with some additional structures.

The paper is organized as follows: After some lattice-theoretical prerequisites, where we briefly recall the concept of a GL-monoid, we present the concept of uniform structures on GL-monoids in order to get conditions for the existence of arbitrary products of elements of a GL-monoid (section 2). Then, in section 3, we shall build the LF-topology product of a given family of LF-topological spaces. Finally in sections $4, 5$ and 6 we define Kolmogoroff and Hausdorff LF-topological spaces and we show that these properties are inherited by the product LF-topology from their factors, together with the separation concepts in this context.

1. GL - Monoids

The basic facts needed are presented in this section. We are mainly interested in the basic ideas about GL-monoids. Let $(L,)$ be a complete infinitely distributive lattice, i. e. $(L,)$ is a partially ordered set such that for every subset $A \subset L$ the join $\bigvee A$ and the meet $\bigwedge A$ are defined, and for every $\alpha \in L$ we have:

$$\left(\bigvee A\right) \wedge \alpha = \bigvee\{a \wedge \alpha \mid a \in A\}, \quad \left(\bigwedge A\right) \vee \alpha = \bigwedge\{a \vee \alpha \mid a \in A\}.$$

In particular, $\top := \bigvee L$ and $\bot := \bigwedge L$ are respectively the universal upper and the universal lower bounds in L. We also assume that $\bot \neq \top$, i.e. L has at least two elements. A $GL-$monoid (cf. [9]) is a complete lattice enriched with a further binary operation $\otimes$, i.e. a triple $(L, , \otimes)$ such that:

(1) $\otimes$ satisfies the isotonicity axiom i.e. for all $\alpha, \beta, \gamma \in L$, $\alpha\beta$ implies $\alpha \otimes \gamma\beta \otimes \gamma$;

(2) $\otimes$ is commutative, i.e. $\alpha \otimes \beta = \beta \otimes \alpha$, $\forall \alpha, \beta \in L$,

(3) $\otimes$ is associative, that is to say, $\alpha \otimes (\beta \otimes \gamma) = (\alpha \otimes \beta) \otimes \gamma$, $\forall \alpha, \beta, \gamma \in L$;

(4) $(L, , \otimes)$ is integral, i.e. the universal upper bound $\top$ is the unit with respect to $\otimes$: $\alpha \otimes \top = \alpha$, $\forall \alpha \in L$;

(5) $\perp$ is the zero element in $(L, , \otimes)$, that is to say, $\alpha \otimes \perp = \perp$, $\forall \alpha \in L$;

(6) $\otimes$ is distributive over arbitrary joins, this means that $\alpha \otimes (\bigvee_j \beta_j) = \bigvee_j (\alpha \otimes \beta_j)$, $\forall \alpha \in L$, $\forall \{\beta_j : j \in J\} \subset L$;

(7) $(L, , \otimes)$ is divisible, i.e. for every pair $(\alpha, \beta) \in L \times L$ with $\alpha\beta$ there exists $\gamma \in L$ such that $\alpha = \beta \otimes \gamma$.

On the other hand, every $GL-$ monoid is residuated, i.e. there exists an additional binary operation "$\longmapsto$" in L satisfying the condition:

$$\alpha \otimes \beta\gamma \Longleftrightarrow \alpha(\beta \longmapsto \gamma) \qquad \forall \alpha, \beta, \gamma \in L$$

for all $\alpha, \beta, \gamma \in L$. It is clear that:

$$\alpha \longmapsto \beta = \bigvee \{\lambda \in L \mid \alpha \otimes \lambda\beta\}.$$

The Heyting algebras and the MV-algebras are important examples of GL-monoids. A *Heyting algebra* (cf. [5]), is a GL-monoid of the kind $(L, , \wedge, \vee, \wedge)$ (i.e. in a Heyting algebra $\wedge = \otimes$). A GL-monoid is a *MV-algebra* if $(\alpha \longmapsto \perp) \longmapsto \perp = \alpha \quad \forall \alpha \in L$ (cf. [9]). Thus in an MV-algebra an order reversing involution ${}^c : L \to L$ can be naturally defined by setting $\alpha^c := \alpha \longmapsto \perp \quad \forall \alpha \in L$.

If X is a set and L is a GL-monoid, then the fuzzy powerset L^X in an obvious way can be pointwise endowed with a structure of a GL-monoid. In particular the L-sets 1_X and 0_X defined by $1_X(x) := \top$ and $0_X(x) := \perp$, $\forall x \in X$, are respectively the universal upper and lower bounds in L^X.

In the sequel L denotes an arbitrary GL-monoid.

2. Infinite Products in GL-monoids

In order to get arbitrary products of elements of a GL-monoid, let us begin with recalling the notion of infinite sums in commutative groups given by Bourbaki in [3]. We briefly sketch his construction below.

Infinite Sums in Topological Groups

Let us begin with the following data:

1. A Hausdorff commutative group $(G, +, \tau)$,
2. an index set I,
3. a family $(x_\lambda)_{\lambda \in I}$ of points of G, indexed by I.

If $\mathcal{P}_f(I)$ denotes the set of finite subsets of I, and with each $J \in \mathcal{P}_f(I)$ we associate the element $s_J := \sum_{i \in J} x_i$ of G, which we call the finite partial sum of the family $(x_\lambda)_{\lambda \in I}$ corresponding to the set J, we have thus a mapping

$\sum : \mathcal{P}_f(I) \longrightarrow G$

J $\longmapsto s_J$.

Now, $\mathcal{P}_f(I)$ is a directed set (with respect to the inclusion relationship). Let Φ be the *section filter* of the directed set $\mathcal{P}_f(I)$:
For each $J \in \mathcal{P}_f(I)$, the *section* of $\mathcal{P}_f(I)$ relative to the element J is the set

$$S(J) = \{K \in \mathcal{P}_f(I) | J \subseteq K\}.$$

Then the set

$$\mathcal{S} = \{S(J) | J \in \mathcal{P}_f(I)\}$$

is a filter base. The filter Φ of sections of $\mathcal{P}_f(I)$ is the filter generated by $\mathcal{S}$.

The family $(x_\lambda)_{\lambda \in I}$ of points of $(G, +, \tau)$ is said to be summable if the mapping

$\sum : \mathcal{P}_f(I) \longrightarrow G$
$J \longmapsto s_J$
has a limit with respect to the section filter Φ. When such limit exists, it is denoted by $\sum_{i \in I} x_i$.

Infinite Products in GL-Monoids

Employing the method introduced in the previous section, and proceeding in the same way, we discuss the closely related notion of infinite products in GL-Monoids. Now, we need the following data:

1. A GL-monoid $(L, , \otimes)$,

2. an index set I,

3. a family $(x_\lambda)_{\lambda\in I}$ of elements of L.

If $\mathcal{P}_f(I)$ again denotes the set of finite subsets of I, and with each $J\in\mathcal{P}_f(I)$ we associate the element $\bigotimes_{i\in J} x_i$, of L, which we call the finite partial product of the family $(x_\lambda)_{\lambda\in I}$ corresponding to the set J, we have thus a mapping

$\prod:\mathcal{P}_f(I)\longrightarrow L$

$J\longmapsto\prod(J)=\bigotimes_{i\in J} x_i$.

We would like to define the "tensorial" product of the family $(x_\lambda)_{\lambda\in I}$ of points of L as the limit of the mapping

$\prod:\mathcal{P}_f(I)\longrightarrow L$

$J\longmapsto\bigotimes_{i\in J} x_i$.

with respect to the section filter Φ and some convergence structure on L. A fundamental approach to constructing such convergence is the uniform structure. Uniform spaces are the carriers of uniform convergence, uniform continuity and the like.

Uniform structures on GL-Monoids

In order to get conditions for the existence of "tensorial" product on GL-Monoids, we will now introduce a uniform structure on a GL-Monoids L, paraphrasing W. Kotzé in [6], Bourbaki in [3], and Willard in [10]:

Definition 2.1. *A mapping $f:L\to L$ is expansive if for each $a\in L$ we have that $af(a)$, i. e. $\Delta_L f$, where $\Delta_L:L\to L$ is the identity map of L. On the other hand we say that $f:L\to L$ commutes with arbitrary joins if*

$$f(\bigvee_\lambda x_\lambda)=\bigvee_\lambda f(x_\lambda).$$

for every family $(x_\lambda)_{\lambda\in I}$ of points of L.

We denote by $\widetilde{L^L}$ the set of all expansive mappings $f:L\to L$ that commute with arbitrary joins. Now we define for each $f\in\widetilde{L^L}$ the map $\hat{f}:L\to L$ by

$$\hat{f}(b)=\bigwedge\{a\in L\mid b[f(a\to\bot)\to\bot]\},\quad\forall b\in L.$$

Lemma 2.2. *If $f\in\widetilde{L^L}$ then $\hat{f}\in\widetilde{L^L}$.*

Proof. Let $f \in \widetilde{L^L}$, in order to show that $\hat{f}$ is expansive; we observe that

$$ba \Leftrightarrow a \longmapsto cb \longmapsto c \text{ and } a \longmapsto \perp f(a \longmapsto \perp).$$

The statement $bf(a \longmapsto \perp) \longmapsto \perp$ is equivalent to

$$a \longmapsto \perp f(a \longmapsto \perp) b \longmapsto \perp,$$

hence we have that ba, and therefore

$$b \bigwedge\{a \in L \mid b[f(a \to \perp) \to \perp]\} = \hat{f}(b).$$

Now, we wish to show that the mapping $\hat{f}$ commutes with arbitrary joins. Let $\{x_\lambda\}_{\lambda\in\Lambda}$ be a collection of elements of L and put:
B={y$\in L \mid \bigvee_{\lambda\in\Lambda} x_\lambda f(y \longmapsto \perp) \longmapsto \perp\}$
$A_\lambda = \{a \in L \mid x_\lambda f(a \longmapsto \perp) \longmapsto \perp\}, \ \forall\lambda \in \Lambda.$
Then $y \in B$ if and only if

$$f(y \longmapsto \perp) \bigwedge_{\lambda\in\Lambda} (x_\lambda \longmapsto \perp),$$

that is to say,

$$f(y \longmapsto \perp) x_\lambda \longmapsto \perp, \quad \forall\lambda \in \Lambda.$$

In other words, for each $\lambda \in \Lambda$ we have that $x_\lambda f(y \longmapsto \perp) \longmapsto \perp$ showing $y \in A_\lambda$, i.e. $B \subset A_\lambda$. We therefore have that
$\bigwedge_{\lambda\in\Lambda} A_\lambda \bigwedge B \Leftrightarrow \hat{f}(x_\lambda)\hat{f}(\bigvee_{\lambda\in\Lambda} x_\lambda)$
$\Leftrightarrow \bigvee_{\lambda\in\Lambda} \hat{f}(x_\lambda)\hat{f}(\bigvee_{\lambda\in\Lambda} x_\lambda).$
On the other hand, $\hat{f}(\bigvee_{\lambda\in\Lambda} x_\lambda) = \bigwedge\{c \in L \mid f(c \longmapsto \perp)(\bigvee_{\lambda\in\Lambda} x_\lambda) \longmapsto \perp\}$
$= \bigwedge\{c \in L \mid f(c \longmapsto \perp) \bigwedge_{\lambda\in\Lambda}(x_\lambda \longmapsto \perp)\}$
$\bigwedge\{c \in L \mid f(c \longmapsto \perp) x_\lambda \longmapsto \perp\} = \hat{f}(x_\lambda)$
$\bigvee_{\lambda\in\Lambda} \hat{f}(x_\lambda).$
This concludes the proof. □
Following W. Kotzé's paper (c.f. [6])

Definition 2.3. *An L-uniformity is a map* $\mathcal{U} : \widetilde{L^L} \to L$ *satisfying the following axioms:*

(lu0) $\mathcal{U}(1_L) = \top.$

(lu1) fg *implies* $\mathcal{U}(f)\mathcal{U}(g).$

(lu2) $\mathcal{U}(f) \otimes \mathcal{U}(g) \leq \mathcal{U}(f \otimes g)$, *for all* $f, g \in \widetilde{L^L}$.

(lu3) $\mathcal{U}(f)\mathcal{U}(\hat{f})$, *for each* $f \in \widetilde{L^L}$.

(lu4) For each $f \in \widetilde{L^L}$ *there exists* $g \in \widetilde{L^L}$ *such that* $g \circ gf$ *and* $\perp\mathcal{U}(g)\mathcal{U}(f)$.

Now we note the set $\{x \in L \mid x \neq \perp\}$ with L^0; and the foregoing definition is reworded from [3] and[4]:

Definition 2.4. *Let* $B : L \to L^L$ *be a map; then for each* $p \in L$ *the image of* p *under* B *is denoted by* $B_p : L \to L$. B *is an L-neighborhood system on* L *iff* B *satisfies the following axioms*

(lv0) $B_p(\top) = \top$.

(lv1) ab *implies* $B_p(a)B_p(b)$.

(lv2) For all $a, b \in L$, $B_p(a) \otimes B_p(b)B_p(a \otimes b)$.

(lv3) $B_p(a) \in L^0$ *implies* pa,

(lv4) If $B_p(a) \in L^0$ *then there exists* $b \in L$ *such that* $B_p(a)B_p(b)$, *and* $B_q(a) \in L^0$, *for all* qb.

Theorem 2.5. *Let* $\mathcal{U} : \widetilde{L^L} \to L$ *Be an L-uniformity and let* $p \in L$. *Then* $B_p : L \to L$ *given by*

$$B_p(x) = \{\mathcal{U}\left(\bigvee\{g \in \widetilde{L^L} \mid g(p) = x\}\right), \quad \text{if } \{g \in \widetilde{L^L} \mid g(p) = x\} \neq \emptyset, \perp, \text{elsewhere.}$$

is an L-neighborhood of p *on* L

Proof. $(lv0)$. Since $1_L(p) = \top$ and $\mathcal{U}(1_L) = \top$, it follows that $B_p(\top) = \top$

$(lv1)$. Let $a, b \in L$ such that $a \leq b$. We distinguish the following cases:
Case 1: $a \neq f(p)$ and $b \neq f(p)$ for all $f \in \widetilde{L^L}$; then

$$B_p(a) = \perp = B_p(b).$$

Case 2: $a \neq f(p)$ for all $f \in \widetilde{L^L}$ and $b = g(p)$ for some $g \in \widetilde{L^L}$, then

$$B_p(a) = \perp < B_p(b) = \mathcal{U}\left(\bigvee\{g \in \widetilde{L^L} \mid g(p) = b\}\right).$$

Case 3: There exists $f \in \widetilde{L^L}$ such that $f(p) = a$. Construct a map $g : L \to L$ defined by $g(x) = f(x) \vee b$, for each $x \in L$. We now verify that $g \in \widetilde{L^L}$, in fact we have that

1. g is expansive since

$$x \leq f(x) \leq f(x) \vee b = g(x), \quad \text{for each } x \in L$$

2. g commutes with arbitrary joins:

$$g(\bigvee_{\lambda} x_{\lambda}) = f(\bigvee_{\lambda} x_{\lambda}) \vee b = (\bigvee_{\lambda} f(x_{\lambda})) \vee b = (\bigvee_{\lambda} f(x_{\lambda}) \vee b) = \bigvee_{\lambda} g(x_{\lambda}).$$

Finally, $a = f(p)g(p) = f(p) \vee b = a \vee b = b$, and therefore $B_p(a)B_p(b)$.
$(lv2)$. For each $x \in L$, consider the set $S_x := \{f \in \widetilde{L^L} \mid f(p) = x\}$, and let $f_0 = \bigvee S_a$, $g_0 = \bigvee S_b$ and $h_0 = \bigvee S_{a\otimes b}$. By $(lu2)$ of definition 2.3, we have that $\mathcal{U}(f_0) \otimes \mathcal{U}(g_0)\mathcal{U}(f_0 \otimes g_0)$ then

$$B_p(a) \otimes B_p(b) = \mathcal{U}(f_0) \otimes \mathcal{U}(g_0)\mathcal{U}(f_0 \otimes g_0)\mathcal{U}(h_0) = B_p(a \otimes b),$$

because $f_0 \otimes g_0 \in S_{a\otimes b}$.

$(lv3)$. Since the elements of $\widetilde{L^L}$ are expansive mappings, the conclusion is obvious.

$(lv4)$. Suppose $B_p(a) \in L^0$ and, as in $(lv2)$, let $f_0 = \bigvee S_a$. In virtue of $(lu4)$ of definition 2.3, there exists $g \in \widetilde{L^L}$ such that $g \circ g f_0$ and $\mathcal{U}(g) \in L^0$. Since the elements of $\widetilde{L^L}$ are expansive mappings and preserve arbitrary joins, we get

$$xg(x)g(g(x))f_0(x), \quad \forall x \in L.$$

Let $b = g(p)$. It remains to show that $B_q(a) \in L^0$ for all qb. Take $h : L \to L$ defined by $h(x) = g(x) \vee a$, as in the proof of $(lv1)$ (case 3), and note that

$$qb \Rightarrow g(q)g(b)a,$$

and so

$$a = g(q) \vee a = h(q).$$

Therefore $h \in \{k \in \widetilde{L^L} \mid k(q) = a\}$. Consequently

$$\mathcal{U}(g)\mathcal{U}(h)B_q(a),$$

proving $(lv4)$. $\square$

Now, we return to the existence of arbitrary product of elements of a GL-monoid $(L, , \otimes)$:

Definition 2.6. *Let $(x_\lambda)_{\lambda\in I}$ be an arbitrary family of points of L, let Φ be the section filter of the directed set $\mathcal{P}_f(I)$, and let $\mathcal{U} : \widetilde{L^L} \to L$ an L-uniformity. A point $p \in L$ is said to be a limit of the mapping*

$\prod : \mathcal{P}_f(I) \longrightarrow L$

$J \longmapsto \otimes_{i\in J} x_i.$

with respect to the section filter Φ and with respect to the L-uniformity $\mathcal{U}$ if $\prod^{-1}(a) \in \Phi$ for each $a \in L$ such that $B_p(a) \in L^0$. When such limit exists, it is denoted by $\bigotimes_{i\in I} x_i$.

Some examples

Example 2.7. *Let $(G, , +, \tau)$ be a conditionally complete Hausdorff commutative topological l-group (i.e. $(G, , +)$ is a patrially ordered commutative group in which every bounded subset has a supremum and infimum (c.f. [2]). Further let u be an element of the positive cone $G^+ \{0\}$ of G. Then*

$$(L,), \quad \text{where } L = \{g \in G \mid 0gu\},$$

is a complete lattice. On L we consider the binary operation $\otimes$ defined by

$$x \otimes y = (x + y - u) \vee 0 \quad \forall x, y \in L.$$

Then $(L, , \otimes)$ is a complete MV-algebra (c.f. [4]).
Let I be an index set and let $(x_i)_{i\in I}$ be a family of point of L indexed by I. Then $\bigotimes_{i\in I} x_i$ exists whenever the family $(x_\lambda)_{\lambda\in I}$ of points of $(L, +, \tau)$ is summable in $(G, , +, \tau)$, (c.f. [3]) and $\sum_{i\in I} x_i 2u$.

Example 2.8. *Let $(I, , Prod, \tau)$ be the real unit interval provided with the usual order, the usual multiplication, and the (uniform) topology of subspace of the real line (R, τ_u). It is easy to see that $(I, , Prod)$ is a GL-monoid (c.f. [2]).*

Let Λ be an index set and let $(x_\lambda)_{\lambda\in\Lambda}$ be a family of point of I indexed by Λ. Then $\bigotimes_{\lambda\in\Lambda} x_\lambda$ exists whenever the family $(x_\lambda)_{\lambda\in\Lambda}$ of points of $(I, Prod, \tau)$ is multipliable (c.f. [3]).

Example 2.9. *In a GL-monoid $(L, , \otimes)$ the product of the collection $\{x_i\}_{i\in I}$, where $x_i = \top$ for each $i \in I$, is $\top$, since for each $J \in \mathcal{P}_f(I)$ one has that*

$$\bigotimes_{i\in J} x_i = \underbrace{\top \otimes \top \otimes \ldots \otimes \top}_{\text{finite factors}} = \top.$$

(This example is useful for working with products of L-Topological Spaces).

3. Product of LF-Topological Spaces

In this section we build the product of an arbitrary collection of LF-topological spaces, for a GL-Monoid $(L, , \otimes)$ (which we always assume to be equipped with arbitrary "tensorial" products).
Given a family $\{(X_\lambda, \tau_\lambda) \mid \lambda \in \Lambda\}$ of LF-topological spaces, we want to build the LF-topology product on the cartesian product

$$_{\Lambda} := \prod_{\lambda\in\Lambda} X_\lambda = \{\phi : \Lambda \to \bigcup_{\lambda\in\Lambda} X_\lambda \mid \phi_\lambda \in X_\lambda,\ \lambda \in \Lambda\}$$

associate with the LF-topologies τ_λ, $\lambda \in \Lambda$. (ϕ_λ denotes the λth component of ϕ, i.e. $\phi(\lambda)$).

Preliminary discussion

Each projection

$$p_\alpha :_{\Lambda} \longrightarrow X_\alpha,\ \alpha \in \Lambda,\ \text{defined by}\ \ p_\alpha(\phi) = \phi_\alpha$$

induces the powerset operator

$$p_\alpha^{\leftarrow} : L^{X_\alpha} \longrightarrow L^{\Lambda}$$

defined by $p_\alpha^{\leftarrow}(g) = g \circ p_\alpha$, for all $g \in L^{X_\alpha}$ (c. f. [8]).

Now we need to build a map such that:

1. $\circ p_\alpha^{\leftarrow} = \tau_\alpha$,
2. is a LF-topology.
3. is universal in the following sense: If $\eta : L^{\Lambda} \to L$ is such that $\eta \circ p_\alpha^{\leftarrow} = \tau_\alpha$ then $\Xi\eta$ (c.f. [1]).

In order to get such a mapping, we proceed as follows:
For each $f \in L^{\Lambda}$ let us consider

$$\Gamma_f = \{\mu \in \prod_{\lambda\in\Lambda} L^{X_\lambda} \mid \mu_\alpha = 1_{X_\alpha} \text{for all but finitely many indices } \alpha, \text{ and} \bigotimes_{\alpha\in\Lambda} p_\alpha^{\leftarrow}(\mu_\alpha) f\}.$$

Lemma 3.1. *Let* $1_\Delta \in \prod_{\lambda\in\Lambda} L^{X_\lambda}$ *defined by* $(1_\Delta)_\alpha = 1_{X_\alpha}$, *for each* $\alpha \in \Lambda$. *Then* $1_\Delta \in \Gamma_{1_\Lambda}$.

Proof. It follows from the fact that

$$\bigotimes_{\lambda\in\Lambda}(1_\Delta)_\alpha \circ p_\lambda 1_{_\Lambda}$$

□

Lemma 3.2. *Let $\mu \in \prod_{\lambda\in\Lambda} L^{X_\lambda}$ and suppose that $X_\lambda = X$ for all $\lambda \in \Lambda$. Then*

$$\bigotimes_{\lambda\in\Lambda}(\mu_\lambda \circ p_\lambda) = (\bigotimes_{\lambda\in\Lambda} \mu_\lambda) \circ p_\lambda.$$

Proof. Since $(\bigotimes_{\lambda\in\Lambda} \mu_\lambda)(t) = \bigotimes_{\lambda\in\Lambda}(\mu_\lambda(t))$, for each $t \in X$, and $(\mu \circ p_\lambda)(r) = \mu(r_\lambda)$, for all $r \in_\Lambda$, we have that $[(\bigotimes_{\lambda\in\Lambda} \mu_\lambda) \circ p_\lambda](r) = (\bigotimes_{\lambda\in\Lambda} \mu_\lambda)(r_\lambda)$
$= \bigotimes_{\lambda\in\Lambda}(\mu_\lambda(r_\lambda))$
$= \bigotimes_{\lambda\in\Lambda}(\mu_\lambda(p_\lambda(r)))$
$= \bigotimes_{\lambda\in\Lambda}(\mu_\lambda \circ p_\lambda)(r)$.

Thus

$$\bigotimes_{\lambda\in\Lambda}(\mu_\lambda \circ p_\lambda) = (\bigotimes_{\lambda\in\Lambda} \mu_\lambda) \circ p_\lambda.$$

□

Lemma 3.3. *If $\mu \in \Gamma_f$ and $\eta \in \Gamma_g$ then $\bigotimes_{\lambda\in\Lambda} p_\lambda^{\leftarrow}(\mu_\lambda \otimes \eta_\lambda) f \otimes g$.*

Proof. Since

$$\bigotimes_{\alpha\in\Lambda}(\mu_\alpha \circ p_\alpha) f \quad \text{and} \quad \bigotimes_{\lambda\in\Lambda}(\eta_\lambda \circ p_\lambda) g,$$

then

$$\bigotimes_{\alpha\in\Lambda}(\mu_\alpha \circ p_\alpha) \otimes \bigotimes_{\lambda\in\Lambda}(\eta_\lambda \circ p_\lambda) f \otimes g.$$

On the other hand, since $\otimes$ is associative and commutative,

$$\bigotimes_{\lambda\in\Lambda}[(\mu_\lambda \circ p_\lambda) \otimes (\eta_\lambda \circ p_\lambda)] f \otimes g.$$

Applying lemma 3.2, yields

$$\bigotimes_{\lambda\in\Lambda}[(\mu_\lambda \otimes \eta_\lambda) \circ p_\lambda] f \otimes g$$

and the proof is complete. □

Now we have

Theorem 3.4. *The map* $: L^{\Lambda} \to L$ *defined by*

$$(f) = \bigvee\{\bigotimes_{\lambda\in\Lambda} \tau_\lambda(\mu_\lambda) \mid \mu \in \Gamma_f\}.$$

is an LF-topology on $_\Lambda$.

Proof.

1. Using lemma 3.1 we check the first fuzzy topological axiom:
$$(1_{_\Lambda}) = \bigvee\{\bigotimes_{\lambda\in\Lambda} \tau_\lambda(\mu_\lambda) \mid \mu \in \Gamma_{1_{_\Lambda}}\} = \top.$$

2. Let $f, g \in L^{\Lambda}$, we must verify the second fuzzy topological axiom: $(f) \otimes (g)(f \otimes g)$. Since
$$(f) = \bigvee\{\bigotimes_{\lambda\in\Lambda} \tau_\lambda(\mu_\lambda) \mid \mu \in \Gamma_f\}\ ,\ (g) = \bigvee\{\bigotimes_{\lambda\in\Lambda} \tau_\lambda(\nu_\lambda) \mid \nu \in \Gamma_g\}$$
and $\otimes$ commutes with arbitrary joins, from lemma 3.3 it follows,
(f)$\otimes$(g) $= \bigvee\{\bigotimes_{\lambda\in\Lambda} \tau_\lambda(\mu_\lambda) \otimes \bigotimes_{\alpha\in\Lambda} \tau_\alpha(\nu_\alpha) \mid \mu \in \Gamma_f,\ \nu \in \Gamma_g\}$
$\bigvee\{\tau_\lambda(\bigotimes_{\lambda\in\Lambda} \mu_\lambda) \otimes \tau_\alpha(\bigotimes_{\alpha\in\Lambda} \nu_\alpha) \mid \mu \in \Gamma_f,\ \nu \in \Gamma_g\}$
$\bigvee\{\bigotimes_{\lambda\in\Lambda} \tau_\lambda(r_\lambda) \mid \Gamma_{f\otimes g}\}$
$= (f \otimes g)$.

3. Let $\{f_j \mid j \in J\} \subseteq L^{\Lambda}$. As follows, we check the third fuzzy topological axiom:
$$\bigwedge_{j\in J} (f_j)(\bigvee_{j\in J} f_j).$$
We have $\bigwedge_{j\in J}(f_j) = \bigwedge_{j\in J} \bigvee\{\bigotimes_{\lambda\in\Lambda} \tau_\lambda((\mu_j)_\lambda) \mid \mu_j \in \Gamma_{f_j}\}$
$\bigvee[\bigwedge_{j\in J}\{\bigotimes_{\lambda\in\Lambda} \tau_\lambda((\mu_j)_\lambda) \mid \mu_j \in \Gamma_{f_j}\}]$
$\bigvee\{\bigotimes_{\lambda\in\Lambda} \tau_\lambda(\bigwedge_{j\in J}((\mu_j)_\lambda)) \mid \mu_j \in \Gamma_{f_j}\}]$.
On the other hand, since
$$(\bigvee_{j\in J} f_j) = \bigvee\{\bigotimes_{\lambda\in\Lambda} \tau_\lambda(\rho_\lambda) \mid \rho \in \Gamma_{\bigvee_{j\in J} f_j}\},$$
the result follows.

□

Theorem 3.5. *The* β*th projection map* $p_\beta :_\Lambda \longrightarrow X_\beta$ *is continuous.*

Proof. Let $g_\beta \in L^{X_\beta}$. We need to check that $\tau_\beta(g_\beta)(g_\beta \circ p_\beta)$. Define $g :_\Lambda \longrightarrow L$ by

$$g_\mu = \{ g_{\ \beta}, \ \text{ if } \mu = \beta 1_{X_\mu}, \ \text{ if } \mu \neq \beta.$$

It follows that $\mu \in \Gamma_{g_\beta \circ p_\beta}$, and thus that $(g_\beta \circ p_\beta)\tau_\beta(g_\beta)$. $\square$

Theorem 3.6. *$: L^\Lambda \to L$ is the weakest LF-topology on $_\Lambda$ for which each projection map $p_\beta :_\Lambda \longrightarrow X_\beta$ is continuous.*

Proof. Let $: L^\Lambda \to L$ be an LF-topology for which each projection p_β is continuous, i.e. $\tau_\beta(g_\beta)(g_\beta \circ p_\beta)$, $\forall\ g_\beta \in L^{X_\beta}$. We need to check that $(g_\beta \circ p_\beta)(g_\beta \circ p_\beta)$.

For each $\beta \in \Lambda$, and for each $g_\beta \in L^{X_\beta}$,

$$\bigvee \{ \bigotimes_{\beta \in \Lambda} \tau_\beta(\rho_\beta) \mid \rho \in \Gamma_{g_\beta \circ p_\beta} \} (g_\beta \circ p_\beta).$$

Thus $(g_\beta \circ p_\beta)(g_\beta \circ p_\beta)$. $\square$

4. Products of Kolmogoroff and Hausdorff LF-topological Spaces

Kolmogoroff L-topological Spaces have been considered by U. Höhle, A. Šostak in [4]. In this section we shall define the notion of Kolmogoroff LF-topological space. We shall show then that the Kolmogoroff property is inherited by the product LF-topological Space from the coordenate LF-topological Spaces.

Definition 4.1. *Let (X, τ) be an LF-topological space. (X, τ) is a Kolmogoroff LF-space (i.e. fulfills the T_0 axiom)if for every pair $(p, q) \in X \times X$ with $p \neq q$, there exists $g \in L^X$ such that*

- $\tau(g) \in L^0 := L - \perp$,
- $g(p) \neq g(q)$.

Theorem 4.2. *Let $(X_\lambda, \tau_\lambda)_{\lambda \in \Lambda}$ be a nonempty family of Kolmogoroff LF-topological spaces. Then $(_\Lambda,)$ is also a Kolmogoroff LF-topological space.*

Proof. If $\alpha,\ \beta \in_\Lambda$ with $\alpha \neq \beta$, then $\alpha_\lambda \neq \beta_\lambda$ for some $\lambda \in \Lambda$. By hypothesis there exists $g_\lambda \in L^{X_\lambda}$ such that

- $\tau_\lambda(g_\lambda) \in L^0$,
- $g_\lambda(\alpha_\lambda) \neq g_\lambda(\beta_\lambda)$,

in other words,

$$(g_\lambda \circ p_\lambda)(\alpha) = g_\lambda(\alpha_\lambda) \neq g_\lambda(\beta_\lambda) = (g_\lambda \circ p_\lambda)(\beta).$$

Thus, for

$$\hat{g} := g_\lambda \circ p_\lambda \in L^\Lambda$$

define $g :_\Lambda \longrightarrow L$ by

$$g_\mu := \{\, g\,_\lambda\,, \quad \text{if } \mu = \lambda 1_{X_\lambda}, \quad \text{if } \mu \neq \lambda.$$

It follows that $g \in \Gamma_{\hat{g}}$, because $\lambda \neq \mu$ implies

$$(g_\mu \circ p_\mu) \otimes (1_{X_\lambda} \circ p_\lambda) g_\lambda \circ p_\lambda.$$

Therefore $(\hat{g}) = \bigvee\{\bigotimes_{\nu\in\Lambda} \tau_\nu(h_\nu) \mid h \in \Gamma_{\hat{g}}\}$
$\geq \left(\bigotimes_{\nu\neq\lambda} \tau_\nu(1_{X_\nu})\right) \otimes \tau_\lambda(g_\lambda)$
$= \top \otimes \tau_\lambda(g_\lambda)$
$= \tau_\lambda(g_\lambda) \in L^0$, showing

$$(\hat{g}) \in L^0.$$

□

Hausdorff LF-topological Spaces

Hausdorff L-topological Spaces were considered in [4]. In this section we seek generalizations of their results to LF-topological Spaces. Finally, we shall show that the Hausdorff property is inherited by the product LF-topological Space from their factors.

Let (X, τ) be an LF-topological space. For each $g \in L^X$, such that $\tau(g) \in L^0$, define $g^* \in L^X$ by

$$g^* := \bigvee\{h \in L^X \mid \tau(h) \in L^0 \ \text{ y } \ h \otimes g = 0_X\}.$$

Definition 4.3. *(X, τ) is a Hausdorff LF-topological Space (i.e. fulfills the T_2 axiom) iff whenever p and q are distint points of X, there exists $g \in L^X$ satisfying:*

- $\tau(g) \in L^0$,
- $\tau(g^*) \in L^0$, *and,*
- $g(q) \otimes g^*(p) \neq \bot$.

Theorem 4.4. *Let* $(X_\lambda, \tau_\lambda)_{\lambda\in\Lambda}$ *be a nonempty family of Hausdorff LF-topological spaces. Then* $(_\Lambda,)$ *is also a Hausdorff LF-topological space.*

Proof. If $x = (x_\lambda)_{\lambda\in\Lambda}$ and $y = (y_\lambda)_{\lambda\in\Lambda}$ are distint points of $_\Lambda$, then there exists $i \in \Lambda$ such that $x_i \neq y_i$. Since (X_i, τ_i) is a Hausdorff *LF*-topological space, there exists $g_i \in L^{X_i}$ satisfying:

- $\tau_i(g_i) \in L^0$,
- $\tau_i(g_i^*) \in L^0$, y,
- $g_i(y_i) \otimes g_i^*(x_i) \neq \top$.

Consider the element

$$g = \bigotimes_{\lambda\in\Lambda} h_\lambda \circ p_\lambda \quad \text{in} \quad L^{\Lambda}$$

where

$$h_\lambda = \{\, 1_{X_\lambda}, \text{if } \lambda \neq i g_i, \text{if } \lambda = i.$$

Since

$$\Gamma_g = \{f \in_\Lambda \;\mid\; \bigotimes_{\lambda\in\Lambda} f_\lambda \circ p_\lambda g\}$$

we have that (g)$= \bigvee\{\bigotimes_{\lambda\in\Lambda} \tau_\lambda(f_\lambda) \mid f \in \Gamma_g\}$
$\geq \left(\bigotimes_{\lambda\neq i} \tau_\lambda(1_{X_\lambda})\right) \otimes \tau_i(g_i)$
$= \tau_i(g_i) \neq \bot$
i.e, $(g) \in L^0$.

On the other hand, g$^* = \bigvee\{f \in L^{\Lambda} \mid (f) \in L^0 \;\text{ y }\; f \otimes g0_X \otimes 1_\Lambda\}$
$= \bigvee\{f \in L^{\Lambda} \mid (f) \in L^0 \;\text{ y }\; f \otimes (\bigotimes_{\lambda\in\Lambda} h_\lambda \circ p_\lambda) = 0_X\}$
$= \bigvee\{f \in L^{\Lambda} \mid (f) \in L^0 \;\text{ y }\; f \otimes (g_i \circ p_i) = 0_X\}$ and

$$\Gamma_g^* = \{f \in_\Lambda \;\mid\; \bigotimes_{\lambda\in\Lambda} f_\lambda \circ p_\lambda g^*\}$$

therefore, (g*) $= \bigvee\{\bigotimes_{\lambda\in\Lambda} \tau_\lambda(f_\lambda) \mid f \in \Gamma_g^*\}$
$\geq \left(\bigotimes_{\lambda\neq i} \tau_\lambda(1_{X_\lambda})\right) \otimes \tau_i(g_i^*)$
$= \tau_i(g_i^*) \neq \bot$, showing, $(g^*) \in L^0$.

Finally, g(y)=$\bigotimes_{\lambda\in\Lambda} h_\lambda \circ p_\lambda(y)$
$= \left(\bigotimes_{\lambda\neq i} 1_{X_\lambda}(y_\lambda)\right) \otimes g_i(y_i)$
$= \top \otimes g_i(y_i)$
$= g_i(y_i)$
and
g$^*(x) = \left(\bigvee\{f \in L^\Lambda \mid (f) \in L^0 \text{ y } f \otimes (g_i \circ p_i) = 0_X\}\right)(x)$
$\geq (g_i^* \circ p_i)(x)$
$= g_i^*(x_i)$.
It follows that

$$g^*(x) \otimes g(y) \geq g_i^*(x_i) \otimes g_i(y_i) \neq \perp.$$

Hence $(_\Lambda,)$ is a Hausdorff LF-topological space. □

5. From the Quasi-coincident Neigborhoods

Let $x \in X$ and $\lambda \in L$ be, the L-point x_λ is the L-set $x_\lambda : X \to L$ defined as

$$x_\lambda(y) = \{ \lambda \text{ if } y = x \perp \text{ if } y \neq x$$

We note the set of L-points of X with $pt(L^X)$.

We say that x_λ quasi-coincides with $f \in L^X$ or say that x_λ is quasi-coincident with f (cf [7], [11]) when

1. $\lambda \vee f(x) = \top$ and
2. $\lambda \wedge f(x) > \perp$,

if x_λ quasi-coincides with f, we denote this $x_\lambda q f$; relation x_λ does not quasi-coincide with f or x_λ is not quasi-coincident with f is denoted by $x_\lambda \neg q f$.

Let (X, τ) be an LF-topological space and $x_\lambda \in pt(L^X)$, and define $Q_{x_\lambda} : L^X \to L$ by

$$Q_{x_\lambda}(f) = \{ \bigvee_{x_\lambda q g g \leq f} \tau(g), \text{ si } x_\lambda q f \perp, \text{ si } x_\lambda \neg q f$$

The set

$$\mathcal{Q} = \{Q_{x_\lambda} \mid x_\lambda \in pt(L^X)\}$$

is called the LF-quasi-coincident neigborhood system of τ. Certainly we can think $Q_{x_\lambda}(f)$ as degree to which f is a quasi-coincident neigborhood of x_λ.

Proposition 5.1. *Let (X, τ) be a LF-topological space, then*

1. $Q_{x_\lambda}(1_X) = \top$ *for all* $x_\lambda \in pt(L^X)$,
2. $Q_{x_\lambda}(0_X) = \bot$ *for all* $x_\lambda \in pt(L^X)$,
3. *If* $Q_{x_\lambda}(f) > \bot$ *then* $x_\lambda q f$,
4. $Q_{x_\lambda}(f \wedge g) = Q_{x_\lambda}(f) \wedge Q_{x_\lambda}(g)$ *for all* $x_\lambda \in pt(L^X)$ *and for all pair* $f, g \in L^X$,
5. *For each* $x_\lambda \in pt(L^X)$ *and for all* $f \in L^X$,
$$Q_{x_\lambda}(f) = \bigvee_{x_\lambda q g g \leq f} \bigwedge_{y_\mu q g} Q_{y_\mu}(g),$$
6. *For each* $f \in L^X$,
$$\tau(f) = \bigwedge_{x_\lambda q f} Q_{x_\lambda}(f).$$

Proof. Let (X, τ) an LF-topological space and $x_\lambda \in pt(L^X)$, we have that,

1. From $1_X(x) = \top$ and $\lambda \vee \top = \top$ we obtain that $x_\lambda q 1_X$; on the other hand, for each $f \in L^X$ we have that $f \leq 1_X$ and we obtain
$$Q_{x_\lambda}(1_X) = \bigvee_{x_\lambda q f} \tau(f) = \tau(1_X) = \top.$$
2. From $0_X(x) = \bot$, $\lambda \vee \bot = \lambda$, $\lambda \wedge \bot = \bot$ and $0_X \leq f$ we obtain:
$$Q_{x_\lambda}(0_X) = \bot.$$
3. If $x_\lambda \neg q f$ then $Q_{x_\lambda}(f) = \bot$ consequently, $Q_{x_\lambda}(f) > \bot$ implies $x_\lambda q f$.
4. Let $f, g \in L^X$ by,
$$Q_{x_\lambda}(f \wedge g) = \bigvee_{x_\lambda q h h \leq f \wedge g} \tau(h) \leq \bigvee_{x_\lambda q h h \leq f} \tau(h) = Q_{x_\lambda}(f)$$
also, $Q_{x_\lambda}(f \wedge g) \leq Q_{x_\lambda}(g)$; then
$$Q_{x_\lambda}(f \wedge g) \leq Q_{x_\lambda}(f) \wedge Q_{x_\lambda}(g);$$

on the other hand, $Q_{x_\lambda}(f) \wedge Q_{x_\lambda}(g) = \Big(\bigvee_{x_\lambda q h h \leq f} \tau(h) \Big) \wedge \Big(\bigvee_{x_\lambda q k k \leq g} \tau(k) \Big) =$
$\bigvee_{x_\lambda q h;\ h \leq f x_\lambda q k;\ k \leq g} \Big(\tau(h) \wedge \tau(k) \Big)$
$\leq \bigvee_{x_\lambda q h;\ h \leq f x_\lambda q k;\ k \leq g} \tau(h \wedge k) = \bigvee_{x_\lambda q h \wedge k h \wedge k \leq f \wedge g} \tau(h \wedge k)$
$= Q_{x_\lambda}(f \wedge g)$, i. e., $Q_{x_\lambda}(f \wedge g) = Q_{x_\lambda}(f) \wedge Q_{x_\lambda}(g)$.

5. For each $x_\lambda \in pt(L^X)$ and for all $f \in L^X$,

$$Q_{x_\lambda}(f) = \bigvee_{g \leq f; x_\lambda q g} \bigwedge_{y_\mu q g} Q_{y_\mu}(g),$$

6. For each $f \in L^X$,

$$\tau(f) = \bigwedge_{x_\lambda q f} Q_{x_\lambda}(f).$$

□

6. Separation Degrees

In contrast with the classical topology, we shall introduce a kind of separation where the topological spaces have separation degrees; these topics are due to how many or how much two L-points are separated, this question is naturally extended to the LF-topological space ambience. These ideas are inspired in [11] where the development of theoretical elements is applied on the lattice I, the unitary interval.

Let (X, τ) be a LF-topological space,

1. Given $x_\lambda,\ x_\mu \in pt(L^X)$, i. e. L-points with the same support; the degree in which the points $x_\lambda,\ x_\mu$ are quasi-T_0 is

$$q - T_0(x_\lambda, x_\mu) = \Big(\bigvee_{x_\lambda \neg q f} Q_{x_\mu}(f) \Big) \vee \Big(\bigvee_{x_\mu \neg q g} Q_{x_\lambda}(g) \Big)$$

2. The degree to which (X, τ) is quasi-T_0 is

$$q - T_0(X, \tau) = \bigwedge \{ q - T_0(x_\lambda, x_\mu) \mid x \in X,\ \lambda \neq \mu \}$$

We emphasize that the degree quasi-T_0 is defined on L-points with the same support.

3. Given x_λ, y_μ L-points with different support, i. e. $x \neq y$, the degree for which, x_λ, y_μ are T_0 is
$$T_0(x_\lambda, y_\mu) = \Big(\bigvee_{x_\lambda \neg q f} Q_{y_\mu}(f)\Big) \vee \Big(\bigvee_{y_\mu \neg q g} Q_{x_\lambda}(g)\Big)$$
4. Now, the degree to which (X, τ) is T_0 is
$$T_0((X,\tau)) = \bigwedge\{T_0(x_\lambda, y_\mu) \mid x_\lambda, y_\mu \in pt(L^X), x \neq y\}.$$
5. The degree to which $x_\lambda,\ y_\mu \in pt(L^X)$ with $x \neq y$ are T_1 is
$$T_1(x_\lambda, y_\mu) = \Big(\bigvee_{x_\lambda \neg q f} Q_{y_\mu}(f)\Big) \wedge \Big(\bigvee_{y_\mu \neg q g} Q_{x_\lambda}(g)\Big)$$
6. The degree to which (X, τ) is T_1 is
$$T_1((X,\tau)) = \bigwedge\{T_1(x_\lambda, y_\mu) \mid x_\lambda, y_\mu \in pt(L^X), x \neq y\}.$$
7. The degree to which $x_\lambda,\ y_\mu \in pt(L^X)$ with $x \neq y$ are T_2 is
$$T_2(x_\lambda, y_\mu) = \bigvee_{f \wedge g = 0_X} \Big(Q_{y_\mu}(f) \wedge Q_{x_\lambda}(g)\Big),$$
8. The degree to which (X, τ) is T_2 is
$$T_2((X,\tau)) = \bigwedge\{T_2(x_\lambda, y_\mu) \mid x_\lambda, y_\mu \in pt(L^X), x \neq y\}.$$

Proposition 6.1. *For each LF-topological space (X, τ) we have that*
$$T_0((X,\tau)) \geq T_1((X,\tau)) \geq T_2((X,\tau)).$$

7. Concluding Remarks

One of the most pervasive and widely applicable constructions in mathematics is that of products. We hope that the results outlined in this paper have exhibited the main properties of products of LF-topological spaces. Clearly, there is much work remaining to be done in this area. Here are some things that might deserve further attention:

1. Describe the relation between products of LF-topological spaces and compact of LF-topological spaces (Tychonoff Theorem).
2. Describe the products of variable-basis fuzzy topological spaces.
3. Examine the relation between products of LF-topological spaces and further separation axioms.

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