

Stability of generalized Jensen functional equation on a set of measure zero

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Abstract

Let E is a complex vector space and F is real (or complex) Banach space. In this paper, we prove the Hyers-Ulam stability for the generalized Jensen functional equation

$$\sum_{k=0}^{m-1} f(x + b_k y) = m f(x), \quad x, y \in E,$$

where $f : E \rightarrow F$ and $b_k = \exp(\frac{2i\pi k}{m})$ for $0 \leq k \leq m-1$, on a set of Lebesgue measure 0.

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1. Introduction

The question concerning the stability of functional equations has been posed by S. M. Ulam in 1940 [29], and its solution given by H.D. Hyers [13] in 1941. Hyers theorem was generalized by T. Aoki [1] for additive mappings and by Th. M. Rassias [22] for linear mappings by considering an unbounded Cauchy difference. The paper [22] of Th. M. Rassias has provided a lot of influence in the development of what we now call Hyers-Ulam-Rassias stability of functional equations. During the last decades, many stability problems for various functional equations have been studied by numerous mathematicians, we refer, for example, to [12],[14],[15],[17].

The stability problems of several functional equations on a restricted domain have been extensively investigated by a number of authors, for example [[4], [9], [8], [10] [11], [17], [20], [26], [23]].

It is very natural to ask if the restricted domain $D = \{(x, y) \in \mathbf{E}^2 : \|x\| + \|y\| \geq d\}$ can be replaced by a much smaller subset $\Omega \subset D$ (e.g. a subset of measure 0 in a measure space E).

In 2013, J. Chung in [8] answered to this question by considering the stability of the Cauchy functional equation

$$(1.1) \quad f(x + y) = f(x) + f(y)$$

in a set $\Omega \subset \{(x, y) \in \mathbf{R}^2 : \|x\| + \|y\| \geq d\}$ where $m(\Omega) = 0$ and $f : \mathbf{R} \rightarrow \mathbf{R}$.

In 2014, J. Chung and J. M. Rassias [9] proved the stability of the quadratic functional equation in a set of measure zero.

Throughout we assume that E is \mathbf{C} vector space and F vector space over $K = \mathbf{R}$ or \mathbf{C} and $2 \leq m \in \mathbf{N}$. Our aim is to prove the Hyers-Ulam stability for the generalized Jensen functional equation

$$(1.2) \quad \sum_{k=0}^{m-1} f(x + b_k y) = m f(x), \quad x, y \in E,$$

where $b_k = \exp(\frac{2i\pi k}{m})$ for $0 \leq k \leq m-1$, on a set Lebesgue measure 0. These results are applied to study of an asymptotic behavior of this functional equation. The stability and solution of this equation and its generalizations were studied by numerous researchers, see for example [5, 2, 19, 28].

We also recall the following theorem which will be used in the sequel.

Theorem 1.1 ([6]). Suppose that E is a complex vector space and F is real (or complex) Banach space, $m \geq 2 \in \mathbf{N}$, $b_k = \exp(\frac{2ik\pi}{m})$ for $0 \leq k \leq m-1$ and $\delta \geq 0$. If $f : E \rightarrow F$ satisfies

$$(1.3) \quad \left\| \sum_{k=0}^{m-1} f(x + b_k y) - mf(x) \right\| \leq \delta$$

for all $x, y \in E$. Then there exists a unique generalized polynomial (GP) $q : E \rightarrow F$ at most $m-1$ such that $q(0) = 0$ and

$$\|f(x) - f(0) - q(x)\| \leq 3^{m-1}\delta \text{ for all } x \in E.$$

Moreover

$$\sum_{k=0}^{m-1} q(x + b_k y) - mq(x) = 0 \text{ for all } x, y \in E.$$

2. Main result

For given $x, y, t \in E$

$$P_{x,y,t} = \{(x; t), (x + b_k t; y), (x + b_k y; t), k = 0, \dots, m-1\}.$$

Let $\Omega \subset E^2$. Throughout this section, we assume that Ω satisfies the condition:

For given $x, y \in E$ there exists $t \in E$ such that

$$(C) \quad \{(x, t), (x + b_k t, y), (x + b_k y, t)\} \subset \Omega.$$

We prove the following stability theorem in Ω .

Theorem 2.1. Let $\delta \geq 0$. Suppose $f : E \rightarrow F$ satisfies the functional inequality

$$(2.1) \quad \left\| \sum_{k=0}^{m-1} f(x + b_k y) - mf(x) \right\| \leq \delta$$

for all $(x, y) \in \Omega$, where $b_k = \exp(\frac{2ik\pi}{m})$, $0 \leq k \leq m-1$. Then there exists a unique generalized polynomial (GP) $q : E \rightarrow F$ of degree at most $m-1$ such that $q(0) = 0$ and

$$(2.2) \quad \|f(x) - f(0) - q(x)\| \leq 3^m \delta \text{ for all } x \in E.$$

Moreover

$$\sum_{k=0}^{m-1} q(x + b_k y) - mq(x) = 0 \text{ for all } x, y \in E.$$

Proof. Let $D(x, y) = \sum_{k=0}^{m-1} f(x + b_k y) - mf(x)$. It is clear that

$$\begin{aligned}
 m \left[\sum_{k=0}^{m-1} (f(x + b_k y) - mf(x)) \right] &= \left[\sum_{k=0}^{m-1} (mf(x + b_k y) - \sum_{n=0}^{m-1} f(x + b_k y + b_n t)) \right] \\
 &+ \left[\sum_{k=0}^{m-1} (-mf(x + b_k t) + \sum_{n=0}^{m-1} f(x + b_k t + b_n y)) \right] \\
 &+ m \left[\sum_{k=0}^{m-1} f(x + b_k t) - mf(x) \right], \\
 &= - \sum_{k=0}^{m-1} D(x + b_k y, t) \\
 &+ \sum_{k=0}^{m-1} D(x + b_k t, y) + mD(x, t).
 \end{aligned}$$

Since Ω satisfies (C), for given $x, y \in E$, then there exists $t \in E$ such that $\|D(x + b_k y, t)\| \leq \delta$, $\|D(x + b_k t, y)\| \leq \delta$ and $\|D(x, t)\| \leq \delta$. Now by using the triangle inequality, we get

$$m \left\| \sum_{k=0}^{m-1} (f(x + b_k y) - mf(x)) \right\| \leq 3m\delta, \quad x, y \in E.$$

This implies that

$$\left\| \sum_{k=0}^{m-1} (f(x + b_k y) - mf(x)) \right\| \leq 3\delta, \quad x, y \in E.$$

Next, according to Theorem 1.1, there exists a unique generalized polynomial (GP) $q : E \rightarrow F$ of degree at most $m - 1$ such that $q(0) = 0$ and

$$\|f(x) - f(0) - q(x)\| \leq 3^m \delta \text{ for all } x \in E.$$

This completes the proof. \square

Now with $m = 2$. The condition (C) is reduced to the following: for given $x, y \in E$ there exists $t \in E$ such that

$$(2.3) \quad \{(x + y, t), (x - y, t), (x + t, y), (x - t, y), (x, t)\} \subset \Omega.$$

As a direct consequence of Theorem 2.1, we obtain the following corollary

Corollary 2.2. *Let $\delta \geq 0$. Suppose that $f : E \rightarrow F$ satisfies the functional inequality*

$$\|f(x+y) + f(x-y) - 2f(x)\| \leq \delta \text{ for all } x, y \in \Omega.$$

then there exists a unique additive mapping $A : E \rightarrow F$ such that

$$\|f(x) - f(0) - A(x)\| \leq 9\delta \text{ for all } x \in E.$$

3. The Ulam-Hyers stability in a set of Lebesgue measure zero

In this section, we prove the Ulam-Hyers stability in a set of Lebesgue measure zero for the functional inequality

$$(3.1) \quad \left\| \sum_{k=0}^{m-1} f(z + b_k \xi) - mf(z) \right\| \leq \delta \text{ for all } (z, \xi) \in \Omega$$

where $f : \mathbf{C} \rightarrow F$ and $\Omega \subset \mathbf{C}^2$ is of four-dimensional Lebesgue measure zero, and the Jensen functional inequality

$$(3.2) \quad \|f(x+y) + f(x-y) - 2f(x)\| \leq \delta \text{ for all } (x, y) \in \Omega \subset \mathbf{R}^2,$$

where $f : \mathbf{R} \rightarrow F$ and Ω is of two-dimensional Lebesgue measure zero.

Firstly, the inequality (3.1) is a particular case of (1.3) where $E = \mathbf{C}$. Now, the condition (C) reduces to the following:

For given $z, \xi \in \mathbf{C}$ there exists $\eta \in \mathbf{C}$ such that

$$(3.3) \quad \{(z + b_k \xi, \eta), (z + b_k \eta, \xi), (z, \eta) : k = 0, \dots, m-1\} \subset \Omega.$$

By virtue of Theorem 2.1, it suffice to construct a set $\Omega \subset \mathbf{C}^2$ of measure zero satisfying (3.3). It is known from [[21], Theorem 1.6] that there exists a set $K \subset \mathbf{R}$ of Lebesgue measure 0 such that $\mathbf{R} \setminus K$ is of first Baire category. That is, $\mathbf{R} \setminus K$ is a countable union of nowhere dense subsets of \mathbf{R} .

Lemma 3.1 (Lemma 2.4, [9]). *Let K be a subset of \mathbf{R} of measure 0 such that $K = \mathbf{R} \setminus K$ is of first Baire category. Then, for any countable subsets $U \subset \mathbf{R}, V \subset \mathbf{R} \setminus \{0\}$ and $M > 0$, there exists $t \geq M$ such that*

$$(3.4) \quad U + tV = \{u + tv : u \in U, v \in V\} \subset K.$$

From now on we identify \mathbf{C} with \mathbf{R}^2 .

Theorem 3.2. *Let K be the set defined in Lemma 3.1, R be the rotation*

$$\begin{pmatrix} -\cos \theta & 0 & \sin \theta & 0 \\ 0 & -\cos \theta & 0 & \sin \theta \\ -\sin \theta & 0 & -\cos \theta & 0 \\ 0 & -\sin \theta & 0 & -\cos \theta \end{pmatrix}$$

with $\theta \in \mathbf{R} \setminus \{\frac{\pi}{2} + k\pi, k\pi/k \in \mathbf{Z}\}$ and $\Omega = R^{-1}(K \times K \times K \times K)$. Then Ω satisfies (3.3) and has four-dimensional Lebesgue measure 0.

Proof. Our method in this proof is inspired by [10, Theorem 3.2]. Let

$$z = x + iy, \quad \xi = u + iv, \quad \eta = t + is \in \mathbf{C}, \quad k = 0, 1, \dots, m-1,$$

and let

$$\begin{aligned} P_{z,\xi,\eta,k} &= \{(x + u \cos \frac{2\pi k}{m} - v \sin \frac{2\pi k}{m}, y + u \sin \frac{2\pi k}{m} + v \cos \frac{2\pi k}{m}, t, s)\} \\ &\cup \{(x + t \cos \frac{2\pi k}{m} - s \sin \frac{2\pi k}{m}, y + t \sin \frac{2\pi k}{m} + s \cos \frac{2\pi k}{m}, u, v), \\ &\quad (x, y, t, s)\}. \end{aligned}$$

Then Ω satisfies (3.3) if and only if, for every $z = x + iy$ and $\xi = u + iv \in \mathbf{C}$, there exists $\eta = t + is \in \mathbf{C}$ such that

$$(3.5) \quad R\left(\bigcup_{k=0}^{m-1} P_{z,\xi,\eta,k}\right) \subset K \times K \times K \times K.$$

Hence the inclusion (3.5) is equivalent to

$$\begin{aligned} S_{z,\xi,\eta} &= \bigcup_{k=0}^{m-1} \{-\cos \theta p_1 + \sin \theta p_3, -\cos \theta p_2 + \sin \theta p_4, -\sin \theta p_1 - \cos \theta p_3, \\ &\quad -\sin \theta p_2 - \cos \theta p_4 : (p_1, p_2, p_3, p_4) \in P_{z,\xi,\eta,k}\} \subset K. \end{aligned}$$

Now, we can choose $\alpha \in \mathbf{R}(\alpha \neq 0)$ such that

$$\cos \frac{2\pi k}{m} - \alpha \sin \frac{2\pi k}{m} \neq 0, \sin \frac{2\pi k}{m} + \alpha \cos \frac{2\pi k}{m} \neq 0$$

for all $k = 0, 1, \dots, m-1$. Then it is easy to check that the set $S_{z,\xi,t+\alpha ti}$ is contained in the set of the form $U + tV$, where

$$\begin{aligned}
 U = & \bigcup_{k=0}^{m-1} \left\{ -\cos \theta \left(x + u \cos \frac{2\pi k}{m} - v \sin \frac{2\pi k}{m} \right), -\cos \theta \left(y + u \sin \frac{2\pi k}{m} + \right. \right. \\
 & \left. \left. v \cos \frac{2\pi k}{m} \right), -\sin \theta \left(x + u \cos \frac{2\pi k}{m} - v \sin \frac{2\pi k}{m} \right), -\sin \theta \left(y + u \sin \frac{2\pi k}{m} + \right. \right. \\
 & \left. \left. + v \cos \frac{2\pi k}{m} \right) \right\} \\
 \cup & \{ -x \cos \theta + u \sin \theta, -y \cos \theta + v \sin \theta, -(x \sin \theta + u \cos \theta), -(y \sin \theta \\
 & + v \cos \theta), -x \cos \theta, -y \cos \theta, -x \sin \theta, -y \sin \theta \}
 \end{aligned}$$

$$\begin{aligned}
 V = & \bigcup_{k=0}^{m-1} \left\{ \sin \theta, \alpha \sin \theta, -\cos \theta, -\alpha \cos \theta, -\cos \theta \left(\cos \frac{2\pi k}{m} - \alpha \sin \frac{2\pi k}{m} \right), \right. \\
 & \left. -\cos \theta \left(\sin \frac{2\pi k}{m} + \alpha \cos \frac{2\pi k}{m} \right), -\sin \theta \left(\cos \frac{2\pi k}{m} - \alpha \sin \frac{2\pi k}{m} \right), -\sin \theta \left(\sin \frac{2\pi k}{m} \right. \right. \\
 & \left. \left. + \alpha \cos \frac{2\pi k}{m} \right) \right\}.
 \end{aligned}$$

By (3.4), for given $z = x + iy$, $\xi = u + iv \in \mathbf{C}$ and $M > 0$ there exists $t \geq M$ such that

$$S_{z,\xi,t+\alpha ti} \subset U + tV \subset K.$$

Thus, Ω satisfies (3.3). \square

Theorem 3.3. *There exists a set $\Omega \subset \mathbf{C}^2$ of Lebesgue measure zero such that if $f : \mathbf{C} \rightarrow F$ satisfies the inequality*

$$\left\| \sum_{k=0}^{m-1} f(z + b_k \xi) - m f(z) \right\| \leq \delta \text{ for all } (z, \xi) \in \Omega$$

for all $(z, \xi) \in \Omega$, then there exists a unique mapping $q : \mathbf{C} \rightarrow F$ satisfying

$$\sum_{k=0}^{m-1} q(z + b_k \xi) - m q(z) = 0 \text{ for all } z, \xi \in \mathbf{C}.$$

such that

$$\|f(z) - f(0) - q(z)\| \leq 3^m \delta \text{ for all } z \in \mathbf{C}.$$

Proof. The proof follows immediately from Theorems 3.2 and 2.1. \square

Remark 3.4. It is easy to see that the set $\Omega_d = \{(z, \xi) \in \mathbf{C}^2 : |z| + |\xi| \geq d\}$ also satisfies (3.3). Thus, the result of Theorem (3.3) holds when Ω is replaced by Ω_d . Thus, as consequence of Theorem (3.3) with the above remark, we obtain a strong version of asymptotic behavior of f satisfies

$$(3.6) \quad \left\| \sum_{k=0}^{m-1} f(z + b_k \xi) - mf(z) \right\| \rightarrow 0, \text{ as } |z| + |\xi| \rightarrow \infty, (z, \xi) \in \Omega.$$

Corollary 3.5. Suppose that $f : \mathbf{C} \rightarrow F$ satisfies (3.6). Then f satisfies the functional equation

$$\sum_{k=0}^{m-1} f(z + b_k \xi) = mf(z), \quad z, \xi \in \mathbf{C}.$$

Secondly, we consider (3.2) in view of Corollary 2.2, it suffice to construct a set $\Omega \subset \mathbf{R}^2$ of measure zero satisfying (2.3).

Theorem 3.6. Let $\Omega = \exp(-i\theta)K \times K$ be the rotation of $K \times K$ by $-\theta$. Then Ω satisfies (2.3) and has two-dimensional Lebesgue measure 0.

Proof. The proof is similar to the proof of Theorem (3.2). However, we prove the completeness. Let $x, y, t \in \mathbf{R}$, and let

$$P_{x,y,t} = \{(x+y, t), (x-y, t), (x+t, y), (x-t, y), (x, t)\}.$$

Then Ω satisfies (2.3) if and only if, for every $x, y \in \mathbf{R}$ such that

$$(3.7) \quad \exp(i\theta)P_{x,y,t} \subset K \times K.$$

The inclusion (3.7) is equivalent to

$$S_{x,y,t} := \{-\cos \theta u + \sin \theta v, -\sin \theta u - \cos \theta v : (u, v) \in P_{x,y,t}\} \subset K.$$

It is easy to check that the set $S_{x,y,t}$ is contained in a set of form $U + tV$, where

$$\begin{aligned} U &= \{\cos \theta(x+y), \sin \theta(x+y), \cos \theta(x-y), \sin \theta(x-y), x \cos \theta \\ &\quad - y \sin \theta, y \cos \theta + x \sin \theta, x \cos \theta, x \sin \theta\} \\ V &= \{\pm \sin \theta, \pm \cos \theta\} \end{aligned}$$

By Lemma 3.1, for given $x, y \in \mathbf{R}$ and $M > 0$ there exists $t \geq M$ such that

$$S_{x,y,t} = U + tV \subset K.$$

Thus, Ω satisfies (2.3). \square

By Corollary 2.2 and Theorem (3.6) we have the following result.

Theorem 3.7. *Let $\delta \geq 0$. There exists a set $\Omega \subset \mathbf{R}^2$ of Lebesgue measure zero such that if $f : \mathbf{R} \rightarrow F$ satisfies the inequality*

$$\|f(x+y) + f(x-y) - 2f(x)\| \leq \delta.$$

only for all $x, y \in \Omega$, then there exists a unique additive mapping $A : \mathbf{R} \rightarrow F$ such that

$$\|f(x) - f(0) - A(x)\| \leq 9\delta \text{ for all } x \in \mathbf{R}.$$

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