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A matrix completion problem over integral domains: the case with $2n - 3$ prescribed blocks *

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Abstract

Let $\Lambda = \{\lambda_1, \ldots, \lambda_{nk}\}$ be a multiset of elements of an integral domain $\mathbf{R}$. Let P be a partially prescribed $n \times n$ block matrix such that each prescribed entry is a $k-$block (a $k \times k$ matrix over $\mathbf{R}$). If P has at most $2n-3$ prescribed entries then the unprescribed entries of P can be filled with $k-$blocks to obtain a matrix over $\mathbf{R}$ with spectrum Λ (some natural conditions on the prescribed entries are required). We describe an algorithm to construct such completion.

Keywords: *Matrix completions, inverse eigenvalue problems, matrices over integral domains.*

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1. Introduction

Inverse eigenvalue problems are problems of constructing a matrix with prescribed structural and spectral constraints. There are several questions that one can ask about any inverse eigenvalue problem: does there exist a solution matrix with the given constraints, is the solution unique, can one find an effective construction of a solution matrix when the problem is solvable. Inverse eigenvalue problems are classified into different types according to the specific constraints. For interested readers, we refer to the book by Chu and Golub [5] where an account of inverse eigenvalue problems with applications and exhaustive bibliography can be found.

A particular class of inverse eigenvalue problems are completion problems: given a matrix P with some of its entries specified, we would like to decide if and how we can choose unspecified entries of P in such a way that the completed matrix satisfies certain spectral properties. A survey on these type of problems is given by Ikramov and Chugunov in [8], where they are specially interested in the development of finite rational algorithms to construct a solution matrix completion. A different approach to the problem is given by Chu, Diele and Sgura in [4], where they consider gradient flow methods. An extensive list of results in completion problems is given in [1].

When presented with a partially prescribed matrix P of order n there are some situations in which we can immediately see that the completion to a matrix with a given spectrum $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ is not possible. For example, let P have a line (row or column) with all its elements prescribed, with all the off-diagonal entries in that line equal to 0 and the diagonal entry not in Λ. If such a line in a matrix P does not exist we will say that the lines of P are consistent with Λ. Another example, where the construction of a solution matrix completion is clearly impossible, is when we have all the diagonal elements of P prescribed and the sum of the diagonal elements is different to the sum of the elements in Λ. If this is not the case, we will say that the diagonal of P is consistent with Λ.

Our work was motivated by an interesting result of Hershkowitz [7]. He considered the case of a matrix of order n with prescribed spectrum, with at most $2n-3$ prescribed entries in arbitrary positions, and with the prescribed entries of the matrix and the prescribed eigenvalues lying in the same field. He showed that the two situations mentioned above are the only ones that we need to exclude if we want to find a completion with prescribed spectrum Λ.

Theorem 1.1. *([7]) (Herskowitz) For $n \geq 2$ let $\Lambda = \{\lambda_1, \dots, \lambda_n\}$ be a given multiset of elements in a field $\mathbf{F}$. Let P be a matrix of order n with at most $2n-3$ prescribed entries that belong to $\mathbf{F}$, and such that the lines and the diagonal of P are consistent with Λ. Then P can be completed with elements of $\mathbf{F}$ to obtain a matrix with spectrum Λ.*

Cravo and Silva in [6] extended Hershkowitz result to the case of $n \times n$ block matrices with $2n-3$ prescribed blocks. While this is not emphasised in the paper, the proofs in [6] are constructive and can be extended to integral domains. While matrix completion problems over fields have been extensively studied, little is known about completion problems over integral domains which include the important case of integers. In [3] we extended (whenever no line is fully prescribed) Theorem 1.1 to integral domains and we provided an algorithmic procedure to construct a solution matrix completion. The aim of this work is to extend our construction in [3] to block matrices, and so giving a different proof of Cravo and Silva result for integral domains whenever no line of blocks is fully prescribed.

In Section 2 we collect the notation that we use to work with for partially prescribed block matrices. In Section 3 we define the reduction of a partially prescribed block matrix of order n to a partially prescribed block matrix of order $n-1$, which will permit us to make an induction step. In Section 4 we provide an algorithmic procedure to construct a solution block matrix completion. In Section 5 we discuss possible extensions of our method to cover the case of $2n-3$ prescribed blocks with n of them forming a fully prescribed line. As a consequence we give an alternative proof of Theorem 1.1 without excluding the case of a fully prescribed line.

2. Comprehensive notation

Let $\mathbf{R}$ be an arbitrary integral domain. We will use the following notation throughout:

- $\mathcal{M}_{m\times n}$ is the set of $m \times n$ matrices over $\mathbf{R}$.
- $\overline{\mathcal{M}}_{m\times n}$ is the set of $m \times n$ matrices over $\mathbf{R} \cup \{\square\}$ ($\square$ corresponds to unprescribed entries).
- $\mathcal{M}^k_{m\times n}$ is the set of $m \times n$ block matrices over $\mathcal{M}_{k\times k}$ (each entry is a matrix in $\mathcal{M}_{k\times k}$).
- $\overline{\mathcal{M}}^k_{m\times n}$ is the set of $m \times n$ block matrices over $\mathcal{M}_{k\times k} \cup \{\square\}$ ($\square$ corresponds to unprescribed blocks).

- Useful abbreviations: $\mathcal{M}_n = \mathcal{M}_{n\times n}$, $\overline{\mathcal{M}}_n = \overline{\mathcal{M}}_{n\times n}$, $\mathcal{M}_n^k = \mathcal{M}_{n\times n}^k$, and $\overline{\mathcal{M}}_n^k = \overline{\mathcal{M}}_{n\times n}^k$.

- If $P \in \overline{\mathcal{M}}_{m\times n}^k$, then $\#P$ denotes the number of prescribed blocks in P; $P_{(i)} \in \overline{\mathcal{M}}_{1\times n}^k$ denotes the $i-$th row of blocks of P; and $P^{(j)} \in \overline{\mathcal{M}}_{m\times 1}^k$ denotes the $j-$th column of blocks of P.

- We consider a special class of matrices within the set $\overline{\mathcal{M}}_n^k$

  $\widehat{\mathcal{M}}_n^k = \{P = (P_{ij})_{i,j=1}^n \in \overline{\mathcal{M}}_n^k : P_{i1} = \square \text{ or } P_{i2} = \square \quad \text{for all } i = 1, \dots, n\}$.

- Let $\mathcal{S}_n$ be the symmetric group on n elements. For $\tau \in \mathcal{S}_n$ and $P = (P_{ij})_{i,j=1}^n \in \overline{\mathcal{M}}_n^k$ we define

$$\tau(P) = \left(P_{\tau(i)\,\tau(j)}\right)_{i,j=1}^n .$$

- We can define an equivalence relation in $\overline{\mathcal{M}}_n^k$ as follows: $P, Q \in \overline{\mathcal{M}}_n^k$ are related if and only if $Q = \tau(P)$ or $Q = \tau(P^{\mathrm{T}})$ for some $\tau \in \mathcal{S}_n$. The equivalence class of P, denoted by $\mathcal{E}(P)$, is given by

$$\mathcal{E}(P) = \{\tau(P) : \tau \in \mathcal{S}_n\} \cup \{\tau(P^T) : \tau \in \mathcal{S}_n\}.$$

  Notice that if $P \in \mathcal{M}_n^k$ then all matrices in $\mathcal{E}(P)$ have the same spectrum.

3. Reductions and completions

Our method is based on the following lemma, which is a generalization for block matrices of a result that was presented in [9].

Lemma 3.1. *Let $A, X, Z \in \mathcal{M}_k$, $B \in \mathcal{M}_{1,n-2}^k$, $Y \in \mathcal{M}_{1,n-2}^k$, $C \in \mathcal{M}_{n-2,1}^k$ and $D \in \mathcal{M}_{n-2}^k$ and let*

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathcal{M}_{n-1}^k$$

and

$$L = \begin{pmatrix} X+Z & X & Y \\ A-X-Z & A-X & B-Y \\ C & C & D \end{pmatrix} \in \mathcal{M}_n^k \tag{3.1}$$

Then the spectrum of L is the spectrum of M together with the spectrum of Z.

Proof. We will show that the characteristic polynomial of L is equal to the product of the characteristic polynomial of M and the characteristic polynomial of Z:

$$\begin{aligned}
\det(\lambda I - L) &= \det\begin{pmatrix} \lambda I - X - Z & -X & -Y \\ -A + X + Z & \lambda I - A + X & -B + Y \\ -C & -C & \lambda I - D \end{pmatrix} \overrightarrow{R_1 + R_2 \to R_1} \\
&= \det\begin{pmatrix} \lambda I - A & \lambda I - A & -B \\ -A + X + Z & \lambda I - A + X & -B + Y \\ -C & -C & \lambda I - D \end{pmatrix} \overrightarrow{C_2 - C_1 \to C_2} \\
&= \det\begin{pmatrix} \lambda I - A & 0 & -B \\ -A + X + Z & \lambda I - Z & -B + Y \\ -C & 0 & \lambda I - D \end{pmatrix} \\
&= \det(\lambda I - Z) \det\begin{pmatrix} \lambda I - A & -B \\ -C & \lambda I - D \end{pmatrix} \\
&= \det(\lambda I - Z) \det(\lambda I - M).
\end{aligned}$$

The result stated in the lemma follows. ∎

Notice that the construction of the matrix L starting from the matrix M as given in (3.1) does not involve the inversion or the multiplication of matrices and it does not involve division. That is why it can be applied to the general setting of integral domains. We will use it to prove results in this paper by induction on the size of the partially prescribed matrix. In order to do that we need to reduce a completion problem for a matrix in $\widehat{\mathcal{M}}_n^k$ to a completion problem for a matrix in $\overline{\mathcal{M}}_{n-1}^k$. Next we give definitions that are needed to make this reduction.

Definition 3.1. *We introduce the following two operations between elements in $\overline{\mathcal{M}}_k$:*

1. *Given $R_1, R_2 \in \overline{\mathcal{M}}_k$ we define*

$$R_1 \oplus R_2 = \begin{cases} R_1 + R_2 & \text{if } R_1, R_2 \in \mathcal{M}_k \\ \square & \text{if } R_i = \square \text{ for some } i \end{cases}$$

2. *Given $S_1, S_2 \in \overline{\mathcal{M}}_k$ with at least one of the elements equal to $\square$, we define*

$$S_1 \odot S_2 = \begin{cases} \square & \text{if } S_1 = S_2 = \square \\ S_i & \text{if } S_i \in \mathcal{M}_k \text{ for some } i \end{cases}$$

(Operation $\odot$ is not defined if both S_1 and S_2 belong to $\mathcal{M}_k$.)

Definition 3.2. *Given a matrix*

$$Q = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} \in \widehat{\mathcal{M}}_2^k$$

and given $Z \in \mathcal{M}_k$, we define the $Z-$reduction of Q in the following way:

$$\Gamma_Z(Q) = \begin{cases} Q_{11} + Q_{21} & \text{if } Q_{11}, Q_{21} \in \mathcal{M}_k \\ Q_{12} + Q_{22} & \text{if } Q_{12}, Q_{22} \in \mathcal{M}_k \\ Q_{11} + Q_{22} - Z & \text{if } Q_{11}, Q_{22} \in \mathcal{M}_k \\ Q_{12} + Q_{21} + Z & \text{if } Q_{12}, Q_{21} \in \mathcal{M}_k \\ \square & \text{otherwise} \end{cases}$$

Definition 3.3. *For $n \geq 3$, given a matrix*

$$Q = \left(\begin{array}{cc|ccc} Q_{11} & Q_{12} & Q_{13} & \dots & Q_{1n} \\ Q_{21} & Q_{22} & Q_{23} & \dots & Q_{2n} \\ \hline Q_{31} & Q_{32} & Q_{33} & \dots & Q_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ Q_{n1} & Q_{n2} & Q_{n3} & \dots & Q_{nn} \end{array}\right) \in \widehat{\mathcal{M}}_n^k \tag{3.2}$$

and given $Z \in \mathcal{M}_k$, we define the $Z-$reduction of Q as the matrix

$$\Gamma_Z(Q) = \left(\begin{array}{c|ccc} \Gamma_Z \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} & Q_{13} \oplus Q_{23} & \dots & Q_{1n} \oplus Q_{2n} \\ \hline Q_{31} \odot Q_{32} & Q_{33} & \dots & Q_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ Q_{n1} \odot Q_{n2} & Q_{n3} & \dots & Q_{nn} \end{array}\right) \in \overline{\mathcal{M}}_{n-1}^k. \tag{3.3}$$

For $Q \in \widehat{\mathcal{M}}_n^k$ and $Z \in \mathcal{M}_k$, $\Gamma_Z(Q)$ is well defined. Moreover the following Lemma justifies the definitions above and will enable us to make an induction step in our proof.

Lemma 3.2. *Let $Q \in \widehat{\mathcal{M}}_n^k$ and $Z \in \mathcal{M}_k$. Then for every completion M of $\Gamma_Z(Q)$ one can construct a completion L of Q with the spectrum equal to the spectrum of M together with the spectrum of Z.*

Proof. Let

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathcal{M}_{n-1}^k \tag{3.4}$$

be a completion of $\Gamma_Z(Q)$, where $A \in \mathcal{M}_k$, $B \in \mathcal{M}^k_{1,n-2}$, $C \in \mathcal{M}^k_{n-2,1}$ and $D \in \mathcal{M}^k_{n-2}$.

We argue that one can choose $X \in \mathcal{M}_k$ and $Y \in \mathcal{M}^k_{1,n-2}$ so that

$$L = \begin{pmatrix} X+Z & X & Y \\ A-X-Z & A-X & B-Y \\ C & C & D \end{pmatrix} \tag{3.5}$$

is a completion of Q. The result then follows from Lemma 3.1.

Let us first consider the choice of X. From (3.5) we see that the entries in $\begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} \in \mathrm{M}_2^{\mathrm{k}}$ depend on $\begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} \in \widehat{\mathcal{M}}_2^{\mathrm{k}}$, Z and A as follows:

$\begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix}$	$\Gamma_Z(\begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix})$	X	$\begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}$
$\begin{pmatrix} \square & \square \\ \square & \square \end{pmatrix}$	$\square$	X	$\begin{pmatrix} X+Z & X \\ A-X-Z & A-X \end{pmatrix}$
$\begin{pmatrix} R & \square \\ \square & \square \end{pmatrix}$	$\square$	$R-Z$	$\begin{pmatrix} R & R-Z \\ A-R & A-R+Z \end{pmatrix}$
$\begin{pmatrix} \square & R \\ \square & \square \end{pmatrix}$	$\square$	R	$\begin{pmatrix} R+Z & R \\ A-R-Z & A-R \end{pmatrix}$
$\begin{pmatrix} \square & \square \\ R & \square \end{pmatrix}$	$\square$	$A-Z-R$	$\begin{pmatrix} A-R & A-R-Z \\ R & R+Z \end{pmatrix}$
$\begin{pmatrix} \square & \square \\ \square & R \end{pmatrix}$	$\square$	$A-R$	$\begin{pmatrix} A-R+Z & A-R \\ R-Z & R \end{pmatrix}$
$\begin{pmatrix} R & \square \\ S & \square \end{pmatrix}$	$R+S$	$R-Z$	$\begin{pmatrix} R & R-Z \\ S & S+Z \end{pmatrix}$
$\begin{pmatrix} \square & R \\ \square & S \end{pmatrix}$	$R+S$	R	$\begin{pmatrix} R+Z & R \\ S-Z & S \end{pmatrix}$
$\begin{pmatrix} R & \square \\ \square & S \end{pmatrix}$	$R+S-Z$	$R-Z$	$\begin{pmatrix} R & R-Z \\ S-Z & S \end{pmatrix}$
$\begin{pmatrix} \square & R \\ S & \square \end{pmatrix}$	$R+S+Z$	R	$\begin{pmatrix} R+Z & R \\ S & S+Z \end{pmatrix}$

In the table above $R, S, X \in \mathcal{M}_k$, and in the first case X can be chosen to be arbitrary. In the cases where

$$\Gamma_Z\left(\begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix}\right) = \square,$$

A denotes its completion. Note that $Q \in \widehat{\mathcal{M}}^k_n$ implies $\#(Q_{i1}\ Q_{i2}) \leq 1$ for $i = 1, 2$ which explains why the first column in the previous table considers all possibilities.

Now let us consider the choice of Y. Appropriate choice of Y can fix for $j \in \{3, \ldots, n\}$ the values in either $L_{1j} \in \mathcal{M}_k$ or in $L_{2j} \in \mathcal{M}_k$. If for all $j \in \{3, \ldots, n\}$ L_{1j} or L_{2j} is unprescribed, then the completion is possible. If for some $j \in \{3, \ldots, n\}$ both L_{1j} and L_{2j} are prescribed, then $M_{1,j-1} = L_{1j} + L_{2j}$, and the choice $Y_{1,j-2} = L_{1j}$ gives the desired completion.

From the definition of $\odot$ and $\Gamma_Z(Q)$ we can now conclude that, for the choices of X and Y above, L in (3.5) is a completion of Q. ∎

4. No full line of blocks is prescribed

In order to be able to find a solution, using our method, to the completion problem when $2n-3$ blocks are prescribed over integral domains we need to assume that no full line of blocks is prescribed. We deal with the case when a full line of blocks is prescribed in Section 5.

Let us assume that $P = (P_{ij})_{i,j=1}^n \in \overline{\mathcal{M}}_n^k$ satisfies $\#P \leq 2n-3$ and that each line of blocks of P has at least one unprescribed block. These types of matrices P may be divided further in two complementary sets: the ones with at least one unprescribed block on the diagonal, and the ones with all the blocks on the diagonal prescribed. Those cases are explicit in the theorems below and the rest of this section is dedicated to their proofs.

Theorem 4.1. *For $n \geq 2$ let $\Lambda = \{\lambda_1, \ldots, \lambda_{nk}\}$ be a multiset of elements in an integral domain $\mathbf{R}$. Let $P \in \overline{\mathcal{M}_n^k}$ with at most $2n-3$ prescribed blocks, and such that the block diagonal and each line of blocks of P has at least one unprescribed block. Then P can be completed to obtain a matrix with spectrum Λ.*

Theorem 4.2. *For $n \geq 3$ let $\Lambda = \{\lambda_1, \ldots, \lambda_{nk}\}$ be a multiset of elements in an integral domain $\mathbf{R}$. Let $P \in \overline{\mathcal{M}_n^k}$ with at most $2n-3$ prescribed blocks, and such that all blocks on the diagonal are prescribed. Then P can be completed to obtain a matrix with spectrum Λ if and only if the diagonal of P is consistent with Λ.*

For each of the theorems above we state and prove a lemma that will enable us to make an induction step. First lemma will be used in the proof of Theorem 4.1.

Lemma 4.1. *For $n \geq 3$ let $P \in \overline{\mathcal{M}}_n^k$ with $\#P \leq 2n-3$, and such that the diagonal and each line of blocks of P has at least one unprescribed block. Then there exists a matrix $Q \in \widehat{\mathcal{M}}_n^k$ in the equivalence class $\mathcal{E}(P)$ such that, for any matrix $Z \in \mathcal{M}_k$, $\Gamma_Z(Q) \in \overline{\mathcal{M}}_{n-1}^k$ satisfies $\#\Gamma_Z(Q) \leq 2(n-1)-3$ and the diagonal and each line of blocks of $\Gamma_Z(Q)$ has at least one unprescribed block.*

Proof. The proof of this lemma can be reconstructed from the proof of Lemma 3.3 of [3]. The only difference is that the role of entries in that proof is played here by blocks. ∎

Now we consider the case in which all the blocks on the diagonal are prescribed. Note that the condition that all blocks on the diagonal are prescribed implies that no line may have all its blocks prescribed.

Lemma 4.2. *For $n \geq 4$ let $P \in \overline{\mathcal{M}}_n^k$ with $\#P \leq 2n-3$ and such that all the diagonal blocks are prescribed. Then there exists a matrix $Q \in \widehat{\mathcal{M}}_n^k$ in the equivalence class $\mathcal{E}(P)$ such that, for every $Z \in \mathcal{M}_k$, we have $\Gamma_Z(Q) \in \overline{\mathcal{M}}_{n-1}^k$ with $\#\Gamma_Z(Q) \leq 2(n-1)-3$ and all blocks in the diagonal of $\Gamma_Z(Q)$ are prescribed.*

Proof. Since there are n prescribed blocks on the diagonal, then there are at most $n-3$ prescribed blocks out of the diagonal. Therefore there are at least 3 columns for which the only prescribed block is the one on the diagonal. So there exists $Q \in \mathcal{E}(P)$ of the following form:

$$Q = \left(\begin{array}{cc|c} Q_{11} & \square & \\ \square & Q_{22} & R \\ \hline \square & \square & \\ \vdots & \vdots & S \\ \square & \square & \end{array}\right) \in \widehat{\mathcal{M}}_n^k,$$

where $R \in \overline{\mathcal{M}}_{2,n-2}^k$ and $S \in \overline{\mathcal{M}}_{n-2}^k$. We have two possibilities:

1. If $\#R \geq 1$ then $\#\Gamma_Z(Q) \leq \#Q - 2 \leq 2(n-1) - 3$, and clearly all diagonal blocks of $\Gamma_Z(Q)$ will be prescribed.

2. Now we deal with the case $\#R = 0$. If all the rows in Q have the only prescribed block on the diagonal then, $\#Q = n$ and $\#\Gamma_Z(Q) = n-1 \leq 2(n-1)-3$ for $n \geq 4$. Otherwise there exists an index $i \geq 3$ such that the i-th row has at least 2 prescribed blocks. Consider the matrix $\tau(Q)$ where $\tau \in \mathcal{S}_\setminus$ is the transposition of 1 and i:

$$\tau(Q) = \left(\begin{array}{cc|ccc} Q_{ii} & \square & ? & \dots & ? \\ \square & Q_{22} & \square & \dots & \square \\ \hline ? & \square & ? & \dots & ? \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ ? & \square & ? & \dots & ? \end{array}\right) \in \widehat{\mathcal{M}}_n^k.$$

It is clear that all the blocks on the diagonal of $\Gamma_Z(\tau(Q))$ are prescribed and we have $\#\Gamma_Z(\tau(Q)) \leq \#\tau(Q) - 2 \leq 2(n-1) - 3$, completing the proof.

∎

In the next two lemma's we prove the first step for the induction argument to prove Theorems 4.1 and 4.2.

Lemma 4.3. *Theorem 4.1 holds for $n = 2$.*

Proof. Assume $P \in \overline{\mathcal{M}}_2^k$ is a matrix that satisfies the conditions of Theorem 4.1, i.e. P has at most one fully prescribed block. If the prescribed block is off the diagonal, then we can choose the unprescribed off-diagonal block to be equal to zero and we are free to choose the diagonal blocks to be any matrices with the desired spectrum. If the prescribed block is on the diagonal we can, without loss of generality, assume that it is the first diagonal block and P is of the form:

$$P = \begin{pmatrix} R & \square \\ \square & \square \end{pmatrix}.$$

From Lemma 3.2 it follows that

$$\begin{pmatrix} R & R - L_2 \\ L_1 - R & L_1 + L_2 - R \end{pmatrix}$$

is a completion of P with spectrum $\sigma(L_1) \cup \sigma(L_2)$. We finish the proof if we choose L_1 to be the diagonal matrix with entries $\lambda_1, \ldots, \lambda_k$, and L_2 to be the diagonal matrix with entries $\lambda_{k+1}, \ldots, \lambda_{2k}$. ∎

Lemma 4.4. *Let*

$$P = \begin{pmatrix} P_{11} & \square & \square \\ \square & P_{22} & \square \\ \square & \square & P_{33} \end{pmatrix} \in \widehat{\mathcal{M}}_3^k$$

and let $\lambda_1, \ldots, \lambda_{3k} \in R$. Then P can be completed to a matrix of $\mathcal{M}_3^k$ with spectrum $\{\lambda_1, \ldots, \lambda_{3k}\}$ if and only if the trace condition is satisfied.

Proof. This result is a special case of Theorem 5.1 in [2]. ∎

Now we can prove Theorems 4.1 and 4.2 at the same time.

Proof. (of Theorems 4.1 and 4.2) Let $P \in \overline{\mathcal{M}_n^k}$ satisfy conditions of Theorem 4.1 or the conditions of Theorem 4.2, and let $\{\lambda_1, \ldots, \lambda_{nk}\}$ be a multiset of elements that belong to $\mathbf{R}$. Let L_i be the diagonal matrix with entries $\lambda_{k(i-1)+1}, \ldots, \lambda_{ki}$ for $i = 1, \ldots, n$. In Lemmas 4.2 and 4.1 we showed how to find in the equivalence class $\mathcal{E}(P)$ a matrix $Q \in \widehat{\mathcal{M}_n^k}$ such that $\Gamma_{L_n}(Q) \in \overline{\mathcal{M}}_{n-1}^k$ is a matrix that again satisfies conditions of Theorem 4.1 or Theorem 4.2, depending on our starting matrix. By induction hypothesis, $\Gamma_{L_n}(Q)$ can be completed to a matrix $M \in \mathcal{M}_{n-1}^k$ with spectrum $\{\lambda_1, \ldots, \lambda_{k(n-1)}\}$. In Section 3 we showed how to construct a matrix $L \in \mathcal{M}_n^k$ with spectrum equal to the spectrum of M with the spectrum of L_n adjoined (i.e., $\{\lambda_{k(n-1)+1}, \ldots, \lambda_{kn}\}$), and such that L is a completion of Q.

Any matrix in the equivalence class $\mathcal{E}(L)$ has spectrum $\{\lambda_1, \ldots, \lambda_{kn}\}$. Since $Q \in \mathcal{E}(P)$ then there exists some permutation τ of $\{1, \ldots, n\}$ such that $Q = \tau(P)$ or $Q = \tau(P^T)$, therefore we conclude that $\tau^{-1}(L)$ or $\tau^{-1}(L^T)$ is a desired completion of matrix P. ∎

5. Extension to a full prescribed line

Let $P \in \overline{\mathcal{M}_n^k}$ with $n \geq 3$, with $\#P \leq 2n-3$ and with a full prescribed line. Let $\Lambda = \{\lambda_1, \ldots, \lambda_{nk}\}$ be a multiset of elements in an integral domain $\mathbf{R}$. We consider the problem of the completion of P (replacing the unprescribed entries of P by $k-$blocks) so that we obtain a matrix of order nk whose spectrum is Λ. The aim in this section will be to reduce this problem to the problem of the completion of a 2×2 block matrix with two prescribed blocks allocated in its last column and whose spectrum is a subset of Λ with $2k$ elements. We will show that this reduction can always be done, however on one side the reduction is not unique, and on the other side a completion of the resulting 2×2 matrix with the desired spectrum may be possible only in some cases. An example of an irresolvable completion is when the full line of P has the off-diagonal $k-$blocks equal to zero and the spectrum of the diagonal block is not contained in Λ. Moreover, the case $n = 2$ and $k = 1$ already shows us that the completion may not be possible over integral domains even if this case is excluded.

Example 5.1. *Over the integers, the completion with spectrum $\Lambda = \{\lambda_1, \lambda_2\}$ of the matrix*

$$P = \begin{pmatrix} \square & p_{12} \\ \square & p_{22} \end{pmatrix}$$

exists if either: (i) $p_{12} = 0$ and $p_{22} \in \Lambda$; or (ii) $p_{12} \neq 0$ and $\frac{(\lambda_1 - p_{22})(p_{22} - \lambda_2)}{p_{12}}$ is an integer. In the last case, the completion is unique and it is equal to

$$\begin{pmatrix} \lambda_1 + \lambda_2 - p_{22} & p_{12} \\ \frac{(\lambda_1 - p_{22})(p_{22} - \lambda_2)}{p_{12}} & p_{22} \end{pmatrix}.$$

Lemma 5.1. *Let $P \in \overline{\mathcal{M}}_n^k$ with $n \geq 3$, with $\#P \leq 2n - 3$ and with a full prescribed line. Then there exists in $\mathcal{E}(P)$ (the equivalence class of P) some $Q \in \widehat{\mathcal{M}}_n^k$ such that for all $Z \in \mathcal{M}_k$ the matrix $\Gamma_Z(Q)$ has a full prescribed line and $\#\Gamma_Z(Q) \leq \max\{n-1, 2(n-1)-3\}$.*

Proof. We divide the proof in two cases:

1. If $\#P = n$ then there exists $Q \in \mathcal{E}(P)$ with $\#Q^{(n)} = n$. So $\#\Gamma_Z(Q)^{(n-1)} = n - 1$ and $\#\Gamma_Z(Q) = n - 1$.
2. If $n < \#P \leq 2n - 3$ then there exists $Q \in \mathcal{E}(P)$ with $\#Q^{(1)} = 0$, $\#Q^{(n)} = n$, and $\#Q_{(2)} \geq \cdots \geq \#Q_{(n-1)}$. Two possibilities appear:
   - (i) if $\#Q_{(1)} + \#Q_{(2)} \geq 3$ then $\#\Gamma_Z(Q) \leq 2(n-1) - 3$ and the last column of $\Gamma_Z(Q)$ is fully prescribed;
   - (ii) if $\#Q_{(1)} + \#Q_{(2)} = 2$ consider $\tau \in \mathcal{S}_n$ to be the transposition of 2 and n, then $\#\Gamma_Z(\tau(Q)) = n - 1$ and the first column of $\Gamma_Z(\tau(Q))$ is fully prescribed.

∎

We observe that if we start from any $P \in \overline{\mathcal{M}}_n^k$ with $n \geq 3$, with $\#P \leq 2n - 3$ and with a full line of n prescribed k-blocks, then we will arrive, after a repeated application of the procedure of the proof of Lemma 5.1, at a 2×2 block matrix with two prescribed $k-$blocks allocated in its last column. Now we ask about how are these two prescribed $k-$blocks related to the original prescribed blocks. Clearly each one of them will be equal to the sum of several of the $k-$block of the full prescribed line of P. Indeed, about the components of these sums, all we can say at present is that the

$k-$block of the full prescribed line which is on the diagonal of P will finish in the prescribed diagonal block of the 2×2 block matrix.

In what follows we will assume, without loss of generality, that the full prescribed line of P is its last column.

Corollary 5.1. *Let $P \in \overline{\mathcal{M}}_n^k$ with $n \geq 3$, with $\#P \leq 2n-3$ and with $P_{1n}, \ldots, P_{nn}$ prescribed. Assume that a series of successive application of Lemma 5.1 reduces P to*

$$M = \begin{pmatrix} \square & \sum_{i \in S_1} P_{in} \\ \square & \sum_{i \in S_2} P_{in} \end{pmatrix}, \tag{5.1}$$

where $\{S_1, S_2\}$ is a partition of $\{1, \ldots, n\}$. Then the following are satisfied:

(i) $n \in S_2$.

(ii) If there exists a $\sigma_1 \subset \sigma$ with $2k$ elements so that there exists a completion of M with spectrum σ_1, then there exists a completion of P with spectrum σ.

Proof. (i) Let $Q = (Q_{ij})_{i,j=1}^n \in \widehat{\mathcal{M}}_n^k$ be a partially prescribed matrix with a full prescribed column $Q_{1j}, \ldots, Q_{nj}$ of $k-$blocks. Note that $\Gamma_Z(Q)$ will again have a full prescribed column $Q_{1j}+Q_{2j}, Q_{3j}, \ldots, Q_{nj}$ of $k-$blocks. In particular, if $j \in \{1,2\}$ then $\#\Gamma_Z(Q)^{(1)} = n-1$ and if $j \in \{3,4,\ldots,n\}$ then $\#\Gamma_Z(Q)^{(j-1)} = n-1$. This implies that the $k-$block on the diagonal of the fully prescribed column of $\Gamma_Z(Q)$ is either the sum of Q_{jj} and some other prescribed block in the j^{th} column of Q or it is equal to Q_{jj}.

(ii) We start with a completion of M with spectrum σ_1. Then we repeatedly apply Lemma 3.2. In each step we incorporate k different elements of σ till we arrive at a completion of P with spectrum σ. ∎

Corollary 5.1 tells us that the exact values of prescribed blocks outside fully prescribed column do not play a role when attacking the problem with our method, only their position is important. The partition $\{S_1, S_2\}$ of $\{1, \ldots, n\}$ obtained in the reduced matrix (5.1) of Corollary5.1 is not unique, but not all partitions are allowed. If we start from a situation where $\#P = n$, then all possible partitions with $n \in S_2$ can be obtained. In the case when $n < \#P \leq 2n-3$ then, according to the proof of Lemma 5.1, we look for $Q \in E(P)$ so that $\#Q^{(1)} = 0$ and $\#Q_{(1)} + \#Q_{(2)} \geq 3$. Neither of these two assumptions is necessary. While $\#Q^{(1)} = 0$ is sufficient to guarantee $Q \in \widehat{\mathcal{M}}_n^k$, it is not necessary. And assumption $\#Q_{(1)} + \#Q_{(2)} \geq 3$ assures that we reduce the number of prescribed elements by 2, however, we can have situations where we have reduction of only one in the number of prescribed $k-$blocks at a certain step at the expense of a reduction of more that 2 in the number of prescribed $k-$blocks at some other step. In what follows we give a discussion of what partitions can occur. Next we state an straightforward necessary condition on S_1 and S_2.

Proposition 5.1. *Let $P \in \overline{\mathcal{M}}_n^k$ with $n \geq 3$, with $\#P \leq 2n-3$ and with $P_{1n}, \ldots, P_{nn}$ prescribed. And let $P_{i_1 j}, \ldots, P_{i_t j}$ be prescribed for some $j \neq n$. If we can achieve a partition $\{S_1, S_2\}$ of $\{1, \ldots, n\}$ as in Corollary 5.1, then $S_i \neq \{i_1, \ldots, i_t\}$ for $i = 1, 2$.*

Proof. Acoording to Definition 3.3, to completely remove the prescribed blocks $P_{i_1 j}, \ldots, P_{i_t j}$ by successive reductions it is necessary that at least once one of the first two rows must be a row that has an unprescribed element in position j. ∎

Proposition 5.1 suggests that it is not easy to describe all allowed partitions in Corollary 5.1 without additional assumptions on the pattern of prescribed blocks. Even if P has its last column fully prescribed and some of its off-diagonal $k-$blocks are nonzero, we can sometimes end up in a 2×2 reduced matrix (5.1) with $\sum_{i \in S_1} P_{in} = 0$ and with $\sum_{i \in S_2} P_{in}$ having an spectrum which is not a subset of Λ. We will show next that this can be avoided by carefully choosing an adequate matrix in the equivalence class at each step before applying Γ_Z.

Lemma 5.2. *Let $P \in \overline{\mathcal{M}}_n^k$ with $n \geq 3$, with $\#P \leq 2n-3$ and with $P_{1n}, \ldots, P_{nn}$ prescribed. If at least one of the $k-$blocks $P_{1n}, \ldots, P_{n-1n}$ is nonzero, then we can achieve a partition $\{S_1, S_2\}$ of $\{1, \ldots, n\}$ as in Corollary 5.1 so that $\sum_{i \in S_1} P_{in} \neq 0$.*

Proof. It will be sufficient to show that for all $Z \in M_k$ the equivalence class $E(P)$ contains some $Q \in \widehat{\mathcal{M}}_n^k$ such that $\Gamma_Z(Q)$ has a full prescribed line with a nonzero off-diagonal k-block and

$$\#\Gamma_Z(Q) \leq \max\{n-1, 2(n-1)-3\}.$$

We consider two cases:

1. There exists $j \in \{1, \ldots, n-1\}$ such that $\#P^{(j)} = 0$ and $\#P_{(j)} \geq 2$.

   Therefore $n \geq 4$ and $\mathcal{E}(P)$ contains a matrix

$$Q = \begin{pmatrix} \square & ? & \cdots & ? & Q_{1n} \\ \square & ? & \cdots & ? & Q_{2n} \\ \vdots & \vdots & & \vdots & \vdots \\ \square & ? & \cdots & ? & Q_{nn} \end{pmatrix} \in \widehat{\mathcal{M}}_n$$

   with $\#Q_{(1)} \geq 2$; with $Q_{2n} = \ldots = Q_{n-1,n} = 0$ if $Q_{n-1,n} = 0$; and with at least one nonzero off-diagonal $k-$block in $Q^{(n)}$.

   Then $\Gamma_Z(Q)^{(n-1)} = n-1$ and

$$\#\Gamma_Z(Q) \leq \#Q - \#Q_{(1)} \leq \#Q - 2 \leq 2(n-1) - 3.$$

   It remains to prove that $\Gamma_Z(Q)^{(n-1)}$ has a nonzero off-diagonal k-block. Two possibilities appear:

   1. $Q_{n-1,n} \neq 0$. Then $\Gamma_Z(Q)_{n-2,n-1} = Q_{n-1,n} \neq 0$.
   2. $Q_{n-1,n} = 0$. Then $Q_{2n} = \ldots = Q_{n-1,n} = 0$ and $\Gamma_Z(Q)_{1,n-1} = Q_{1n} \neq 0$.

2. For all $j \in \{1, \ldots, n-1\}$ such that $\#P^{(j)} = 0$ we have $\#P_{(j)} = 1$.

   Since $\#P^{(1)} + \cdots + \#P^{(n-1)} \leq n-3$ then P has two empty columns. So $\mathcal{E}(P)$ contains a matrix

$$Q = \begin{pmatrix} \square & \square & \cdots & \square & \square & Q_{1n} \\ \square & ? & \cdots & ? & \square & Q_{2n} \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ \square & ? & \cdots & ? & \square & Q_{n-2,n} \\ \square & \square & \cdots & \square & \square & Q_{n-1,n} \\ \square & ? & \cdots & ? & \square & Q_{nn} \end{pmatrix} \in \widehat{\mathcal{M}}_n$$

   with $\#Q^{(1)} = \#Q^{(n-1)} = 0$; with $Q_{1n} = 0$ if $Q_{n-1,n} = 0$ (only possible when $n \geq 4$); with at least one nonzero off-diagonal $k-$block in $Q^{(n)}$; and with $\#Q_{(2)} \geq \ldots \geq \#Q_{(n-2)}$ (only possible when $n \geq 4$).

   Two possibilities appear:

   1. $\#Q_{(2)} \geq 2$. Then $\#\Gamma_Z(Q)^{(n-1)} = n-1$ and

$$\#\Gamma_Z(Q) \leq \#Q - 2 \leq 2(n-1) - 3.$$

      It remains to prove that $\Gamma_Z(Q)^{(n-1)}$ has a nonzero off-diagonal k-block. Three possibilities appear:

1. $Q_{n-1,n} \neq 0$. Then $\Gamma_Z(Q)_{n-2,n-1} = Q_{n-1,n} \neq 0$.
2. $Q_{n-1,n} = 0$ and $Q_{2n} \neq 0$. Then $\Gamma_Z(Q)_{1,n-1} = Q_{1n} + Q_{2n} = Q_{2n} \neq 0$.
3. $Q_{n-1,n} = 0$ and $Q_{2n} = 0$. Then $Q_{1n} = 0$ and $Q_{in} \neq 0$ for some $i \in \{3, \dots, n-2\}$. So $\Gamma_Z(Q)_{i-1,n-1} = Q_{in} \neq 0$.

2. $\#Q_{(2)} = 1$. In the equivalence class of Q we have the matrix

$$Q' = \begin{pmatrix} Q_{nn} & \square & ? & \cdots & ? & \square \\ Q_{1n} & \square & \square & \cdots & \square & \square \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ Q_{n-1,n} & \square & \square & \cdots & \square & \square \end{pmatrix}$$

Then $\#\Gamma_Z(Q')^{(1)} = \#\Gamma_Z(Q') = n-1$. It remains to prove that $\Gamma_Z(Q')^{(1)}$ has a nonzero off-diagonal k-block. Two possibilities appear:

1. $Q_{n-1,n} \neq 0$. Then $\Gamma_Z(Q')_{n-1,1} = Q_{n-1,n} \neq 0$.
2. $Q_{n-1,n} = 0$. Then $Q_{1n} = 0$ and $Q_{in} \neq 0$ for some $i \in \{2, \dots, n-2\}$. So $\Gamma_Z(Q')_{i1} = Q_{in} \neq 0$.

∎

Now we have all the necessary tools to give an alternative proof of Theorem 1.1 which besides provides an algorithmic procedure to construct a solution matrix completion.

Proof. (of Theorem 1.1) The case where P has no full prescribed line was proved in [3]. So, assume that P is a partially prescribe matrix over a field $\mathbf{F}$ with at most $2n-3$ prescribed entries and with a full prescribed line. Without loss of generality we assume that $p_{1n}, p_{2n}, \dots, p_{nn}$ are prescribed. If $p_{1n} = p_{2n} = \cdots = p_{n-1,n} = 0$, then p_{nn} has to be contained in Λ in order for the line to be consistent with Λ. This case is then naturally reduced to the case of partially prescribed matrices of order $n-1$ with at most $n-3$ prescribed elements (so, without full prescribed lines).

Now we assume that $p_{in} \neq 0$ for some $i \in \{1, 2, \dots, n-1\}$. By Lemma 5.2, we can choose S_1 in Corollary 5.1 in such a way that $\sum_{i \in S_1} p_{in} \neq 0$. This reduces our problem to the 2×2 case of the form:

$$\begin{pmatrix} \square & s_{12} \\ \square & s_{22} \end{pmatrix},$$

where $s_{12} \neq 0$.

The completion to a matrix with spectrum $\{\lambda_1, \lambda_2\} \subset \Lambda$ is unique and it is equal to

$$\begin{pmatrix} \lambda_1 + \lambda_2 - s_{22} & s_{12} \\ \frac{(\lambda_1 - s_{22})(s_{22} - \lambda_2)}{s_{12}} & s_{22} \end{pmatrix}.$$

∎

Next we present some examples to illustrate scope and limitations of our method in the case of a full prescribed line when considering completion over integers.

Example 5.2. *Consider the partially prescribed matrix*

$$P = \begin{pmatrix} \square & \square & a & 3 \\ \square & \square & \square & 22 \\ \square & \square & \square & 5 \\ \square & \square & \square & 1 \end{pmatrix},$$

where a is an integer. From P we can obtain, using operation Γ_Z and choosing different matrices from the equivalence classes, the following 2×2 partially prescribed matrices:

$$P_1 = \begin{pmatrix} \square & 30 \\ \square & 1 \end{pmatrix},\ P_2 = \begin{pmatrix} \square & 27 \\ \square & 4 \end{pmatrix},\ P_3 = \begin{pmatrix} \square & 25 \\ \square & 6 \end{pmatrix},$$

$$P_4 = \begin{pmatrix} \square & 22 \\ \square & 9 \end{pmatrix},\ P_5 = \begin{pmatrix} \square & 8 \\ \square & 23 \end{pmatrix},\ P_6 = \begin{pmatrix} \square & 5 \\ \square & 26 \end{pmatrix}.$$

Observe that Proposition 5.1 implies that it is not possible to obtain $\begin{pmatrix} \square & 3 \\ \square & 28 \end{pmatrix}$.

Since the sum of the off-diagonal elements in the full prescribed line is not equal to zero, any of the matrices P_i, $i = 1, 2, \ldots, 6$, can be completed to have any real (complex) spectrum over the real (complex) numbers. However the completion, using our method, will only be possible over the integers if there exists two elements λ_1, λ_2 in the prescribed spectrum that satisfy at least one of the following conditions:

1. *$(\lambda_1 - 1)(1 - \lambda_2)$ is divisible by 30,*
2. *$(\lambda_1 - 4)(4 - \lambda_2)$ is divisible by 27,*
3. *$(\lambda_1 - 6)(6 - \lambda_2)$ is divisible by 25,*

4. $(\lambda_1 - 9)(9 - \lambda_2)$ *is divisible by 22,*

5. $(\lambda_1 - 23)(23 - \lambda_2)$ *is divisible by 8,*

6. $(\lambda_1 - 26)(26 - \lambda_2)$ *is divisible by 5.*

Example 5.3. *Consider partially prescribed matrix*

$$P = \begin{pmatrix} \square & \square & a & 3 \\ \square & \square & \square & -2 \\ \square & \square & \square & 5 \\ \square & \square & \square & 1 \end{pmatrix},$$

where a is an integer. Since P can be reduced to $\begin{pmatrix} \square & 1 \\ \square & 6 \end{pmatrix}$ *we can complete P to have any integer spectrum.*

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