

Strongly Bounded Partial Sums

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Abstract

If λ is a scalar sequence space, a series $\sum z_j$ in a topological vector space Z is λ multiplier convergent in Z if the series $\sum_{j=1}^{\infty} t_j z_j$ converges in Z for every $t = \{t_j\} \in \lambda$. If λ satisfies appropriate conditions, a series in a locally convex space X which is λ multiplier convergent in the weak topology is λ multiplier convergent in the original topology of the space (the Orlicz-Pettis Theorem) but may fail to be λ multiplier convergent in the strong topology of the space. However, we show under appropriate conditions on the multiplier space λ that the series will have strongly bounded partial sums.

The Orlicz-Pettis Theorem for locally convex spaces asserts that a series in such a space which is subseries convergent in the weak topology is subseries convergent in the Mackey topology and even in locally convex topologies which are stronger than the Mackey topology ([6], [9]). However, it is the case that a series in a locally convex space which is subseries convergent in the weak topology may fail to be convergent in the strong topology (see Example 1 below). Even though a series which is subseries convergent in the weak topology may fail to be convergent in the strong topology the partial sums of the series are bounded in the strong topology (see Theorem 3 below). If λ is vector space of scalar valued sequences which contains c_{00} , the space of sequences which are eventually 0, a series $\sum z_j$ in a topological vector space (Z, τ) is λ multiplier convergent in Z if the series $\sum_{j=1}^{\infty} t_j z_j$ is τ convergent for every $\{t_j\} \in \lambda$. For example, if m_0 is the sequence space of all scalars with finite range, then a series $\sum z_j$ is m_0 multiplier convergent iff the series is subseries convergent and if $\lambda = l^\infty$, then a series is l^∞ multiplier convergent iff the series is bounded multiplier convergent. In this note we show that if the multiplier space λ satisfies a gliding hump condition and $\sum x_j$ is a series in a locally convex space X which is λ multiplier convergent in the weak topology, then for every $t = \{t_j\} \in \lambda$ the partial sums of the series $\{\sum_{j=1}^n t_j x_j : n \in \mathbf{N}\}$ are strongly bounded.

We begin by fixing the notation and terminology. In what follows X, X' will denote a pair of vector spaces in duality; the weak topology on X from X' will be denoted by $\sigma(X, X')$. The strong topology $\beta(X, X')$ on X is the locally convex topology generated by the semi-norms

$$p_B(x) = \sup\{|x'(x)| : x' \in B\}, x \in X,$$

where B runs over the family of all $\sigma(X', X)$ bounded subsets of X' . See [5], [11] for descriptions of the topologies.

Our results require a gliding hump assumption on the multiplier space λ . If $I \subset \mathbf{N}$, χ_I will denote the characteristic function of I and if $t = \{t_j\}$ is scalar sequence, then $\chi_I t$ will denote the coordinatewise product of t and χ_I . If $m, n \in \mathbf{N}, m \leq n$, the interval $[m, n]$ is $\{j \in \mathbf{N} : m \leq j \leq n\}$; a sequence of intervals $\{I_j\}$ is increasing if $\max I_j < \min I_{j+1}$. A sign is a scalar $s_j = \pm 1$. The sequence space λ has the signed weak gliding hump property (signed WGHP) if whenever $t = \{t_j\} \in \lambda$ and $\{I_j\}$ is an increasing sequence of intervals, there exist a subsequence $\{n_j\}$ and a sequence of signs $\{s_j\}$ such that the coordinate sum of the series $\sum_{j=1}^{\infty} s_j \chi_{I_{n_j}} t \in \lambda$; if all the signs s_j can be chosen equal to 1, λ has the weak gliding hump

property (WGHP). The WGHP was introduced by Noll ([7]) and the signed WGHP was introduced by Stuart ([8]) to treat questions of weak sequential completeness in β duals. The space λ is monotone if $\chi_I t \in \lambda$ whenever $t \in \lambda$ and $I \subset \mathbf{N}$ ([1]). A monotone space obviously has WGHP so spaces such as c_0, l^p ($0 < p \leq \infty$) have WGHP; the non-monotone space cs of convergent series has WGHP and the space bs of bounded series has signed WGHP but not WGHP ([8]). See [3] and [9] for further examples and [2] for descriptions of the sequence spaces.

The following example shows that a weakly subseries convergent series in a locally convex space may fail to be strongly subseries convergent.

Example 1. Consider the series $\sum e^j$ in l^∞ , where e^j is the sequence with 1 in the j^{th} coordinate and 0 in the other coordinates. The series is $\sigma(l^\infty, l^1)$ subseries convergent but not $\beta(l^\infty, l^1) = \|\cdot\|_\infty$ subseries convergent. Note, however, the partial sums of the series are $\|\cdot\|_\infty$ bounded.

Even though a weak subseries convergent series may fail to be strongly subseries convergent if the multiplier space λ has signed WGHP, the partial sums of a weakly λ multiplier convergent series has partial sums which are strongly bounded. In order to establish this result we require a matrix theorem due to Antosik and Mikusinski which we describe for the convenience of the reader.

Theorem 2. (Antosik-Mikusinski) Let Z be a topological vector space and $x_{ij} \in Z$ for $i, j \in \mathbf{N}$. Suppose (I) $\lim_i x_{ij} = x_j$ exists for every j and (II) for each increasing sequence of positive integers $\{m_j\}$ there exist a subsequence $\{n_j\}$ of $\{m_j\}$ and a sequence of signs $\{s_j\}$ such that $\{\sum_{j=1}^\infty s_j x_{in_j}\}_{i=1}^\infty$ converges. Then $\lim_i x_{ii} = 0$.

More general forms of the theorem can be found in Appendix D of [9]. An infinite matrix $[x_{ij}]$ satisfying conditions (I) and (II) is called a signed \mathcal{K} matrix. The signed version of the Antosik-Mikusinski Matrix Theorem is due to Stuart ([8]).

Theorem 3. Suppose λ has signed WGHP and the series $\sum x_j$ is λ multiplier convergent with respect to $\sigma(X, X')$. Then for every $t = \{t_j\} \in \lambda$ the partial sums $\{\sum_{j=1}^n t_j x_j : n \in \mathbf{N}\}$ are $\beta(X, X')$ bounded.

Proof. Suppose the conclusion fails. Then there exist a $\sigma(X', X)$ bounded set B , an increasing sequence $\{n_k\}$, $\epsilon_k \rightarrow 0$, $\epsilon_k > 0$, and $\delta > 0$ such that

$$\epsilon_k p_B\left(\sum_{j=1}^{n_k} t_j x_j\right) > \delta$$

for every k . For $k = 1$ we have $p_B(\epsilon_1 \sum_{j=1}^{n_1} t_j x_j) > \delta$ so there exists $x'_1 \in B$ such that $\epsilon_1 \left| x'_1 \left(\sum_{j=1}^{n_1} t_j x_j \right) \right| > \delta$. There exists l_1 such that $l \geq l_1$ implies

$$(\#) \quad \epsilon_l p_B\left(\sum_{j=1}^{n_1} t_j x_j\right) < \delta/2.$$

Put $k_1 = 1$ and $k_2 = \max\{l_1, k_1\}$ so $\epsilon_{k_2} p_B(\sum_{j=1}^{n_{k_2}} t_j x_j) > \delta$ and there exists $x'_2 \in B$ such that

$$\epsilon_{k_2} \left| x'_2 \left(\sum_{j=1}^{n_{k_2}} t_j x_j \right) \right| > \delta.$$

Then from $(\#)$,

$$\epsilon_{k_2} \left| x'_2 \left(\sum_{j=n_{k_1}+1}^{n_{k_2}} t_j x_j \right) \right| \geq \epsilon_{k_2} \left| x'_2 \left(\sum_{j=1}^{n_{k_2}} t_j x_j \right) \right| - \epsilon_{k_2} \left| x'_2 \left(\sum_{j=1}^{n_{k_1}} t_j x_j \right) \right| > \delta - \delta/2 = \delta/2.$$

Continuing this construction produces an increasing sequence $\{k_p\}$, $\{x'_p\} \subset B$ such that if $[n_{k_{p-1}+1}, n_{k_p}] = I_p$, then

$$(*) \quad \epsilon_{k_p} \left| x'_p \left(\sum_{j \in I_p} t_j x_j \right) \right| > \delta/2.$$

Define an infinite matrix

$$M = [m_{pq}] = [\epsilon_{k_p} x'_p \left(\sum_{j \in I_q} t_j x_j \right)].$$

We claim that M is a signed- \mathcal{K} matrix. First, the columns of M converge to 0 since $\{x'_p\}$ is $\sigma(X', X)$ bounded and $\epsilon_k \rightarrow 0$. Next, given any increasing sequence of integers, by the signed WGHP property there exist a further subsequence $\{r_q\}$ and signs $\{s_q\}$ such that $u = \sum_{q=1}^{\infty} s_q \chi_{I_{r_q}} t \in \lambda$. Then

$$\sum_{q=1}^{\infty} s_q m_{pr_q} = \sum_{q=1}^{\infty} \epsilon_{k_p} x'_p (s_q \sum_{j \in I_{r_q}} t_j x_j) = \epsilon_{k_p} x'_p \left(\sum_{q=1}^{\infty} s_q \sum_{j \in I_{r_q}} t_j x_j \right) = \epsilon_{k_p} x'_p \left(\sum_{q=1}^{\infty} u_q x_q \right),$$

where $\sum_{q=1}^{\infty} u_q x_q$ is the $\sigma(X, X')$ sum of the series. Then $\lim_p \sum_{q=1}^{\infty} s_q m_{pr_q} = 0$ since $\epsilon_p \rightarrow 0$ and $\{x'_p\}$ is $\sigma(X, X')$ bounded. By the Antosik-Mikusinski Matrix Theorem, the diagonal of M converges to 0. But, this contradicts (*) and the result follows.

A similar result under different assumptions was established in [9, 2.6]; the result above requires no topology on the multiplier space (e.g., the spaces d and δ ([4, pages 48 and 68])). In particular, the theorem applies to subseries and bounded multiplier convergent series. It should be pointed out that the series $\sum x_j$ in Theorem 3 is actually λ multiplier convergent in the Mackey topology since it is assumed that λ has signed WGHP (see the Orlicz-Pettis Theorem 4.10 of [9]).

The following example shows that without some assumption on the multiplier space λ the conclusion of Theorem 3 may fail to hold. Let μ be the scalar sequence space of all sequences which are eventually constant. In this case a series $\sum z_j$ in a topological vector space (Z, τ) is μ multiplier convergent with respect to τ iff the series is τ convergent. \square

Example 4. Let c_{00} be the space of all real sequences which are eventually 0 with the sup norm $\|\cdot\|_{\infty}$. The dual of this space is l^1 . The sequence $\{je^j\}$ is $\sigma(l^1, c_{00})$ convergent to 0 since if $t = \{t_j\} \in c_{00}$, then $je^j(t) = 0$ eventually. Set $e^0 = 0$. Then the telescoping series $\sum_{j=1}^{\infty} (je^j - (j-1)e^{j-1})$ is $\sigma(l^1, c_{00})$ convergent to 0 and is, therefore, μ multiplier convergent with respect to $\sigma(l^1, c_{00})$. However, $\beta(l^1, c_{00}) = \|\cdot\|_1$ and if $u = \{1\} \in \mu$, then

$$\left\| \sum_{j=1}^n u_j (je^j - (j-1)e^{j-1}) \right\|_1 = \|ne^n\|_1 = n$$

so the partial sums of the series are not strongly bounded.

If the multiplier space λ satisfies a stronger gliding hump assumption and the space X satisfies an additional condition, the conclusion of Theorem 3 can be strengthened.

The space λ is a K-space if λ has a Hausdorff vector topology under which the coordinate maps $t = \{t_j\} \rightarrow t_j$ are continuous from λ into the scalar field. Let \mathcal{F} be a family of subsets of λ . The K-space λ has the signed \mathcal{F} gliding hump property (signed \mathcal{F} GHP) if for every $F \in \mathcal{F}$ whenever $\{t^k\} \subset F$ is bounded and $\{I_k\}$ is an increasing sequence of intervals, there exist subsequence $\{n_k\}$ and a sequence of signs $\{s_k\}$ such that the coordinate sum of the series $\sum_{k=1}^{\infty} s_k \chi_{I_{n_k}} t^k$ belongs to λ ; if all the signs can be chosen equal to 1, then λ has the \mathcal{F} gliding hump property

(\mathcal{F} GHP). For example, if \mathcal{F} consists of all the finite subsets of λ , signed \mathcal{F} GHP (\mathcal{F} GHP) is just the signed WGHP (WGHP) and if \mathcal{F} consists of all the bounded subsets of λ , signed \mathcal{F} GHP (\mathcal{F} GHP) corresponds to the signed strong gliding hump property (strong gliding hump property). The spaces l^∞ and m_0 have the strong gliding hump property while the space bs has the signed strong gliding hump property but not the strong gliding hump property. For other spaces with the \mathcal{F} GHP see [10]. For further examples of spaces with the strong gliding hump property, see Appendix B of [9] and [3].

For the result we require the following observation. Recall a locally convex space X is a Banach-Mackey space if bounded subsets of X are strongly bounded ([11,10.4.3]). For example, any barrelled space is a Banach-Mackey space since it carries the strong topology while every sequentially complete locally convex space is a Banach-Mackey space ([11,10.4.8]).

Lemma 5. *Assume X is a Banach-Mackey space and λ is a K -space. If $A \subset \lambda$ is bounded and $B \subset X'$ is $\sigma(X', X)$ bounded, then for every $x \in X$, $k \in \mathbf{N}$,*

$$p_B(\{t_k x : t = \{t_j\} \in A\}) < \infty.$$

Proof. Since λ is a K -space, the set $\{t_k x : t = \{t_j\} \in A\}$ is bounded in X and, therefore, strongly bounded by the Banach-Mackey assumption. \square

Theorem 6. *Suppose X is a Banach-Mackey space, λ has signed \mathcal{F} GHP and the series $\sum x_j$ is λ multiplier convergent with respect to $\sigma(X, X')$. Then for every bounded subset $A \in \mathcal{F}$ the partial sums $\{\sum_{j=1}^n t_j x_j : t \in A, n \in \mathbf{N}\}$ are $\beta(X, X')$ bounded.*

Proof. Suppose the conclusion fails. Then there exist a $\sigma(X', X)$ bounded set B , $\{t^k\} \subset A$, an increasing sequence $\{n_k\}$, $\epsilon_k \rightarrow 0$, $\epsilon_k > 0$, and $\delta > 0$ such that

$$\epsilon_k p_B(\sum_{j=1}^{n_k} t_j^k x_j) > \delta$$

for every k . For $k = 1$ we have $p_B(\epsilon_1 \sum_{j=1}^{n_1} t_j^1 x_j) > \delta$ so there exists $x'_1 \in B$ such that $\epsilon_1 \left| x'_1(\sum_{j=1}^{n_1} t_j^1 x_j) \right| > \delta$. By the lemma there exists l_1 such that $l \geq l_1$ implies

$$(\#) \quad \epsilon_l p_B(\sum_{j=1}^{n_1} t_j^l x_j) < \delta/2.$$

Put $k_1 = 1$ and $k_2 = \max\{l_1, k_1\}$ so $\epsilon_{k_2} p_B(\sum_{j=1}^{n_{k_2}} t_j^{k_2} x_j) > \delta$ and there exists $x'_2 \in B$ such that

$$\epsilon_{k_2} \left| x'_2 \left(\sum_{j=1}^{n_{k_2}} t_j^{k_2} x_j \right) \right| > \delta.$$

Then from (#),

$$\epsilon_{k_2} \left| x'_2 \left(\sum_{j=n_{k_1}+1}^{n_{k_2}} t_j^{k_2} x_j \right) \right| \geq \epsilon_{k_2} \left| x'_2 \left(\sum_{j=1}^{n_{k_2}} t_j^{k_2} x_j \right) \right| - \epsilon_{k_2} \left| x'_2 \left(\sum_{j=1}^{n_{k_1}} t_j^{k_2} x_j \right) \right| > \delta - \delta/2 = \delta/2.$$

Continuing this construction produces an increasing sequence $\{k_p\}$, $\{x'_p\} \subset B$, $\{t^{k_p}\} \subset A$ such that if $[n_{k_{p-1}+1}, n_{k_p}] = I_p$, then

$$(*) \quad \epsilon_{k_p} \left| x'_p \left(\sum_{j \in I_p} t_j^{k_p} x_j \right) \right| > \delta/2.$$

Define an infinite matrix

$$M = [m_{pq}] = [\epsilon_{k_p} x'_p \left(\sum_{j \in I_q} t_j^{k_q} x_j \right)].$$

We claim that M is a signed- \mathcal{K} matrix. First, the columns of M converge to 0 since $\{x'_p\}$ is $\sigma(X', X)$ bounded and $\epsilon_k \rightarrow 0$. Next, given any increasing sequence of integers, by the signed \mathcal{F} GHP property there exist a further subsequence $\{r_q\}$ and signs $\{s_q\}$ such that $u = \sum_{q=1}^{\infty} s_q \chi_{I_{r_q}} t^{k_{r_q}} \in \lambda$. Then

$$\begin{aligned} \sum_{q=1}^{\infty} s_q m_{pr_q} &= \sum_{q=1}^{\infty} \epsilon_{k_p} x'_p \left(s_q \sum_{j \in I_{r_q}} t_j^{k_{r_q}} x_j \right) \\ &= \epsilon_{k_p} x'_p \left(\sum_{q=1}^{\infty} s_q \sum_{j \in I_{r_q}} t_j^{k_{r_q}} x_j \right) = \epsilon_{k_p} x'_p \left(\sum_{q=1}^{\infty} u_q x_q \right), \end{aligned}$$

where $\sum_{q=1}^{\infty} u_q x_q$ is the $\sigma(X, X')$ sum of the series. Then $\sum_{q=1}^{\infty} s_q m_{pr_q} \rightarrow 0$ since $\epsilon_p \rightarrow 0$ and $\{x'_p\}$ is $\sigma(X', X)$. By the Antosik-Mikusinski Matrix Theorem, the diagonal of M converges to 0. But, this contradicts (*) and the result follows. \square

A similar result under different assumptions is given in 2.6 of [9]. In particular, since the spaces l^∞ and m_0 have the strong gliding hump property, the theorem covers the cases where the series are subseries or bounded multiplier convergent.

As the following example shows the Banach-Mackey assumption in Theorem 6 cannot be dropped.

Example 7. Let $\lambda = c_{00}$ with the sup-norm and X, X' be any dual pair without the Banach-Mackey property. Let $\{x_j\} \subset X$ be weakly bounded but not strong bounded. Then the series $\sum x_j$ is c_{00} multiplier convergent in any locally convex topology, but $\{e^k\}$ is $\|\cdot\|_\infty$ bounded and $\sum_{j=1}^k e_j^k x_j = x_k$ so the partial sums of the series $\sum_{j=1}^\infty e_j^k x_j$ are not uniformly strong bounded. Note c_{00} is monotone and, therefore, has WGHP.

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