The Banach-Steinhaus Theorem in Abstract Duality Pairs

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Abstract

Let $E, F$ be sets and $G$ a Hausdorff, abelian topological group with $b : E \times F \to G$; we refer to $E, F, G$ as an abstract duality pair with respect to $G$ or an abstract triple and denote this by $(E, F : G)$. Let $(E_i, F_i : G)$ be abstract triples for $i = 1, 2$. Let $\mathcal{F}_i$ be a family of subsets of $F_i$ and let $\tau_{\mathcal{F}_i}(E_i) = \tau_i$ be the topology on $E_i$ of uniform convergence on the members of $\mathcal{F}_i$. Let $\Gamma$ be a family of mappings from $E_1$ to $E_2$. We consider conditions which guarantee that $\Gamma$ is $\tau_1 - \tau_2$ equicontinuous. We then apply the results to obtain versions of the Banach-Steinhaus Theorem for both abstract triples and for linear operators between locally convex spaces.
In [CLS] we established versions of the Orlicz-Pettis Theorem for sub-series convergent series in abstract triples or abstract duality pairs based on results which were initiated at New Mexico State University during Professor Li Ronglu’s tenure as a visiting scholar. In this note we present some further results on an equicontinuity version of the Banach-Steinhaus Theorem for abstract triples which were also the result of Professor Li’s visit. After establishing our abstract version of the Banach-Steinhaus Theorem we present several applications to continuous linear operators between locally convex spaces and establish versions of the Banach-Steinhaus Theorem for arbitrary locally convex spaces.

We first recall the definition of abstract triples. Let $E, F$ be sets and $G$ a Hausdorff, abelian topological group with $b : E \times F \to G$; if $x \in E$ and $y \in F$, we often write $b(x, y) = x \cdot y$ for convenience. We refer to $E, F, G$ as an abstract duality pair with respect to $G$ or an abstract triple and denote this by $(E, F : G)$. Note that $(F, E : G)$ is an abstract triple under the map $\overline{b}(y, x) = b(x, y)$. Examples of abstract triples are given in [CLS]; in particular a pair of vector spaces in duality is an example where $G$ is the scalar field.

In what follows $(E_i, F_i : G)$ will denote abstract triples for $i = 1, 2$. Let $\mathcal{F}_i$ be a family of subsets of $F_i$ and let $\tau_{\mathcal{F}_i}(E_i) = \tau_i$ be the topology on $E_i$ of uniform convergence on the members of $\mathcal{F}_i$ so a net $\{x_\alpha\}$ converges to $x \in E_i$ iff $x_\alpha \cdot y \to x \cdot y$ uniformly for $y$ belonging to a member of $\mathcal{F}_i$. Let $\Gamma$ be a family of mappings $T : E_1 \to E_2$. We consider conditions which guarantee that $\Gamma$ is $\tau_1 - \tau_2$ equicontinuous. We then establish several versions of the Banach-Steinhaus Theorem for abstract triples and give applications to continuous linear operators between locally convex spaces.

To motivate the condition which guarantees that $\Gamma$ is $\tau_1 - \tau_2$ equicontinuous, we consider the case of continuous linear operators between locally convex spaces. Let $(E_1, F_1), (E_2, F_2)$ be dual pairs and let $\tau_i$ be the polar topology of uniform convergence on the members of $\mathcal{F}_i$ and let $\Gamma$ be a family of weakly continuous linear operators $T : E_1 \to E_2$. Suppose

\begin{equation}
(*) \text{ for every } B \in \mathcal{F}_2 \text{ there exists } A \in \mathcal{F}_1 \text{ such that } T' B = B T \subseteq A \text{ for every } T \in \Gamma.
\end{equation}

or, taking polars in $E_1$,

\begin{equation}
(**) \ (T' B)_0 = T^{-1} B_0 \supseteq A_0 \text{ for } T \in \Gamma.
\end{equation}

Condition (**) implies that $\Gamma$ is $\tau_1 - \tau_2$ equicontinuous.
We consider abstracting condition (*) to abstract triples. For this regard the elements $y$ of $F_i$ as functions from $E_i \rightarrow G$ defined by $y(x) = x \cdot y$ for $x \in E_i$. We say that the pair $(F_\infty, F_\in)$ satisfies the equicontinuity condition $(E)$ if

$$(E) \text{ for every } B \in F_2 \text{ there exists } A \in F_1 \text{ such that }$$

$$B \Gamma = \{y \circ T : y \in B, T \in \Gamma \} \subset A$$

[Note if $x \in E_1$, $(y \circ T)(x) = y(Tx) = y \cdot Tx$.

**Theorem 1.** If $(F_1, F_2)$ satisfies condition $(E)$, the $\Gamma$ is $\tau_1 - \tau_2$ equicontinuous.

**Proof.** Suppose the net $\{x_\delta\}$ in $E_1$ converges to $x \in E_1$ with respect to $\tau_1$ so $x_\delta \cdot y \rightarrow x \cdot y$ uniformly when $y$ belongs to a member of $F_1$. Let $B \in F_2, z \in B$ and let $A$ be as in condition $(E)$. Then $z \circ T \in A$ for every $T \in \Gamma, z \in B$ so $z \cdot T x_\delta \rightarrow z \cdot Tx$ uniformly for $T \in \Gamma, z \in B$ by the definition of convergence in $\tau_1$. Therefore, $T x_\delta \rightarrow T x$ in $\tau_2$ uniformly for $T \in \Gamma$. $\square$

The case of a single operator satisfying condition $(E)$ is of interest.

**Corollary 2.** Suppose $T : E_1 \rightarrow E_2$ is such that for every $B \in F_2$ there exists $A \in F_1$ such that $BT \subset A$. Then $T$ is $\tau_1 - \tau_2$ continuous.

**Corollary 3.** (Banach-Steinhaus) Suppose $\{T_\alpha\}$ is a net of maps from $E_1$ to $E_2$ such that $\tau_2 - \lim_\alpha T_\alpha x = Tx$ exists for every $x \in E_1$. If $\Gamma = \{T_\alpha\}$ satisfies condition $(E)$, then $T$ is $\tau_1 - \tau_2$ continuous.

**Proof.** Suppose the net $\{x_\delta\}$ is $\tau_1$ convergent to $x \in E_1$. Then by hypothesis $\tau_2 - \lim_\alpha T_\alpha x_\delta = T x_\delta$ for each $\delta$. Also, by Theorem 1, $\tau_2 - \lim_\delta \lim_\alpha T_\alpha x_\delta = \lim_\alpha \lim_\delta T_\alpha x_\delta = \lim_\alpha \lim_\delta T_\alpha x_\delta = \lim_\alpha \lim_\delta T_\alpha x_\delta = \lim_\alpha \lim_\delta T_\alpha x_\delta = \lim_\alpha T_\alpha x$ (DSI.7.6) and $T$ is $\tau_1 - \tau_2$ continuous. $\square$

We next consider conditions for which $(E)$ holds and establish versions of the Banach-Steinhaus Theorem for topological vector spaces.

In what follows $G$ will be a Hausdorff topological vector space.

We first give a motivation for the conditions which appear in a version of the Banach-Steinhaus Theorem for abstract triples.
Suppose \((E, F)\) is a dual pair and \(\tau_F\) is a polar topology on \(E\) of uniform convergence on the members of \(F\). Recall a subset \(C \subset E\) is \(\tau_F\) bounded iff \(BC = \{(y, x) : y \in B, x \in C\}\) is bounded for every \(B \in F\). We abstract this condition to abstract triples.

**Definition 4.** A subset \(C \subset E_2\) is \(F_2\) bounded if \(C \cdot B = \{x \cdot y : y \in B, x \in C\}\) is bounded in \(G\) for every \(B \in F_2\).

We give an equicontinuity version of the Banach-Steinhaus Theorem for abstract triples.

**Theorem 5.** Suppose \(\Gamma\) is pointwise \(F_2\) bounded on \(E_1\) (i.e., for every \(x \in E_1\) the set \(\Gamma x\) is \(F_2\) bounded in \(E_2\)). Let \(B\) be the family of subsets of \(F_1\) which are pointwise bounded on \(E_1\). Then the pair \((B, F_2)\) satisfies condition \((E)\). Hence, \(\Gamma\) is \(\tau_B - \tau_2\) equicontinuous.

**Proof.** Let \(B \in F_2\). We claim \(B \Gamma \in B\). Let \(x \in E_1\). Since \(\Gamma x\) is \(F_2\) bounded, \(B(\Gamma x)\) is bounded in \(G\) so \(B \Gamma \in B\). Therefore, \((B, F_2)\) satisfies condition \((E)\) and the result follows from Theorem 1. \(\square\)

From Corollary 3 we have another version of the Banach-Steinhaus Theorem.

**Corollary 6.** Let \(\{T_\alpha\}\) be a net of maps from \(E_1 \to E_2\) which is pointwise \(F_2\) bounded on \(E_1\). If \(\tau_2 - \lim_\alpha T_\alpha x = Tx\) exists for every \(x \in E_1\), then \(T\) is \(\tau_B - \tau_2\) continuous.

We also have the more familiar form of the Banach-Steinhaus Theorem for sequences.

**Corollary 7.** Let \(T_k : E_1 \to E_2\) and suppose \(\tau_2 - \lim_k T_kx = Tx\) exists for each \(x \in E_1\). Then \(T\) is \(\tau_B - \tau_2\) continuous.

**Proof.** For each \(x \in E_1\), \(\{T_kx\}\) is \(F_2\) bounded so the corollary above applies. \(\square\)

We can also give a generalization of Theorem 1. Let \(A_1\) be a family of subsets of \(E_1\) and let \(B_1\) be a family of subsets of \(F_1\) which is uniformly bounded on members of \(A_1\) (i.e., \(B_1\) is \(A_1\) bounded).

**Theorem 8.** Suppose \(\Gamma A\) is \(F_2\) bounded for every \(A \in A_1\). Then \(\Gamma\) is \(\tau_{B_1} - \tau_2\) equicontinuous.
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Proof. As in the proof of Theorem 5 the pair $(B_1, F_2)$ satisfies condition $(E)$. □

In the case of Theorem 5, the family $A_1$ consists of singletons.

We now give applications of the results above for abstract triples to continuous linear operators between locally convex spaces and obtain versions of the Banach-Steinhaus Theorem for arbitrary locally convex spaces.

In what follows let $(E_1, F_1), (E_2, F_2)$ be dual pairs with polar topologies $\tau_i$ on $E_i$ of uniform convergence on members of $F_i$. Let $\Gamma$ be a family of $\tau_1 - \tau_2$ continuous linear operators. Let $\beta(E_i, F_i)$ be the strong topology on $E_i$ from the duality. From Theorem 5 we obtain an equicontinuity version of the Banach-Steinhaus Theorem.

**Theorem 9.** If $\Gamma$ is pointwise bounded on $E_1$, then $\Gamma$ is $\beta(E_1, F_1) - \tau_2$ equicontinuous.

Proof. In Theorem 5 the family $B$ is the family of $\sigma(F_1, E_1)$ bounded sets so $\tau_B = \beta(E_1, F_1)$ and the result follows from Theorem 5. □

Note that the family $\Gamma$ may fail to be equicontinuous with respect to the original topology of $E_1$ but that the result above holds for arbitrary locally convex spaces with no assumptions on the domain space $E_1$. If $E_1$ is a barrelled space, then the original topology of $E_i$ is $\beta(E_1, F_1)$ so Theorem 7 gives one of the usual forms of the Banach-Steinhaus Theorem or the Uniform Boundedness Principle (see [Sw1] 24.11,[Wi1] 9.3.4). This result was established in [Sw2]. As noted in [LC] there are non-barrelled spaces which carry the strong topology so Theorem 9 gives a proper extension of the usual form of the Banach-Steinhaus Theorem for barrelled spaces.

From Corollary 7 and Theorem 9 we also obtain the sequential version of the Banach-Steinhaus Theorem.

**Theorem 10.** Let $T_k : E_1 \to E_2$ be a sequence of $\tau_1 - \tau_2$ continuous linear operators such that $\tau_2 - \lim_k T_k x = Tx$ exists for every $x \in E_1$. Then $T$ is $\beta(E_1, F_1) - \tau_2$ continuous and $\{T_k\}$ is $\beta(E_1, F_1) - \tau_2$ equicontinuous.

Again note that $T$ may fail to be continuous with respect to the original topology of $E_1$. This result was established in [LC].

We can obtain an improvement of Theorem 9 for Banach-Mackey spaces. Recall a locally convex space $E_1$ is a Banach-Mackey space if the bounded subsets of $E_1$ are strongly bounded ([Wi1]10.4.3). For example, any sequentially complete locally convex space is a Banach-Mackey space ([Wi1] 10.4.8). We denote the topology on $E_1$ of uniform convergence on the $\beta(F_1, E_1)$ bounded subsets of $F_1$ by $\beta^*(E_1, F_1)$ (see [Sw1]20,[Wi1]10.1).
Theorem 11. Suppose $E_1$ is a Banach-Mackey space. If $\Gamma$ is pointwise bounded on $E_1$, then $\Gamma$ is $\beta^*(E_1, F_1) - \tau_2$ equicontinuous.

Proof. By the Banach-Mackey property the family $B$ of Theorem 5 is the family of all $\beta(F_1, E_1)$ bounded subsets of $F_1$ so $\tau_B = \beta^*(E_1, F_1)$ and the result follows from Theorem 5.

Note $\beta^*(E_1, F_1) \subset \beta(E_1, F_1)$ so Theorem 11 improves the conclusion of Theorem 9 for Banach-Mackey spaces. We can also obtain an improvement of Theorem 10 for Banach-Mackey spaces. □

Theorem 12. Suppose $E_1$ is a Banach-Mackey space. Let $T_k : E_1 \to E_2$ be a sequence of $\tau_1 - \tau_2$ continuous linear operators such that $\lim_k T_k x = Tx$ exists for every $x \in E_1$. Then $T$ is $\beta^*(E_1, F_1) - \tau_2$ continuous and $\{T_k\}$ is $\beta^*(E_1, F_1) - \tau_2$ equicontinuous.

We can also obtain a corollary of Theorem 8.

Corollary 13. Let $A_1$ be the family of all $\sigma(E_1, F_1)$ bounded subsets of $E_1$ and $B_1$ be the family of all $\beta(F_1, E_1)$ bounded subsets of $F_1$. If $\Gamma$ is uniformly bounded on members of $A_1$, then $\Gamma$ is $\beta^*(E_1, F_1) - \tau_2$ equicontinuous.

Proof. $\tau_{B_1} = \beta^*(E_1, F_1)$ so the result follows from Theorem 8. □

Corollary 14 about a single mapping also has an interesting application to linear operators.

Corollary 14. Suppose $T : E_1 \to E_2$ is a bounded linear operator. Then $T$ is $\beta^*(E_1, F_1) - \tau_2$ continuous.

Note that $T$ may not be continuous with respect to the original topology of $E_1$. Consider the identity operator on an infinite dimensional normed space when the domain has the weak topology and the range the norm topology.

The result in Corollary 2 also has an application to a Hellinger-Toeplitz result for linear operators. Let $X, Y$ be locally convex spaces with duals $X', Y'$. A property $\mathcal{P}$ of subsets $B$ of a dual space $Y'$ is said to be linearly invariant if for every continuous linear operator $T : X \to Y$ there exists $A \subset X'$ with property $\mathcal{P}$ such that $BT = T'B \subset A$. For example, the family of subsets with finite cardinal, the weak* compact sets, the weak* convex compact sets, the weak* bounded sets, etc.
If $\mathcal{P}$ is a linearly invariant property, let $P(X, X')$ be the locally convex topology of uniform convergence on the members of $X'$ with property $\mathcal{P}$. From Corollary 2 we have a Hellinger-Toeplitz result in the spirit of Wilansky ([Wi1]11.2.6).

**Corollary 15.** If $T : X \to Y$ is a continuous linear operator, then $T$ is $P(X, X') - P(Y, Y')$ continuous.

In particular, $T$ is continuous with respect to the Mackey topologies and strong topologies ([Wi1]11.2.6).

Finally we indicate an application concerning automatic continuity of matrix transformations between sequence spaces. Let $\lambda_1, \lambda_2$ be scalar sequence spaces containing $c_0$, the space of sequences with finite range and if $a = \{a_j\} \in \lambda_1^\beta$, the $\beta$-dual of $\lambda_1$, $t = \{t_j\} \in \lambda_1$, we write $a \cdot t = \sum_{j=1}^{\infty} a_j t_j$. Assume that $\lambda_i$ has a locally convex polar topology $\tau_i$ from the duality pair $\lambda_i, \lambda_i^\beta$ and that $A = [a_{ij}]$ is an infinite matrix which maps $\lambda_1$ into $\lambda_2$.

Under assumptions on the sequence spaces, we use Theorem 10 to show that $A$ is continuous with respect to appropriate topologies. First, we assume that the $\beta$-dual of $\lambda_1$ is contained in the topological dual $\lambda_1^0$ and then we assume that $\lambda_2$ is an AK-space under its topology (i.e., the canonical unit vectors $\{e^i\}$ form a Schauder basis for $\lambda_2$ ([Wi2] 4.2.13,[Sw3] B.2). Now let $a^i$ be the $i^{th}$ row of the matrix $A$ so $a^i \in \lambda_1^\beta \subset \lambda_1^0$ and define $A_k : \lambda_1 \to \lambda_2$ by $A_k t = \sum_{i=1}^{k} (a^i \cdot t) e^i$. Then $A_k$ is $\tau_1 - \tau_2$ continuous and $\tau_2 - \lim_k A_k t = \sum_{i=1}^{\infty} (a^i \cdot t) e^i = At$ by the AK assumption. By the Banach-Steinhaus Theorem 10, $\{A_k\}$ is $\beta(\lambda_1, \lambda_1) - \tau_2$ equicontinuous and $A$ is $\beta(\lambda_1, \lambda_2) - \tau_2$ continuous, an automatic continuity result. In particular, if $\lambda_1 = \lambda_2 = l^2$, then this result implies that any matrix mapping $l^2$ into itself is continuous; this is the classic theorem of Hellinger and Toeplitz ([K2] 34.7). Further automatic continuity theorems for matrix mappings can be found in [K2] 34.7 and [Sw4] 12.6.

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