

The forcing open monophonic number of a graph

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Abstract

For a connected graph G of order $n \geq 2$, and for any minimum open monophonic set S of G , a subset T of S is called a forcing subset for S if S is the unique minimum open monophonic set containing T . A forcing subset for S of minimum cardinality is a minimum forcing subset of S . The forcing open monophonic number of S , denoted by $f_{om}(S)$, is the cardinality of a minimum forcing subset of S . The forcing open monophonic number of G , denoted by $f_{om}(G)$, is $f_{om}(G) = \min(f_{om}(S))$, where the minimum is taken over all minimum open monophonic sets in G . The forcing open monophonic numbers of certain standard graphs are determined. It is proved that for every pair a, b of integers with $0 \leq a \leq b - 4$ and $b \geq 5$, there exists a connected graph G such that $f_{om}(G) = a$ and $om(G) = b$. It is analyzed how the addition of a pendant edge to certain standard graphs affects the forcing open monophonic number.

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Key Words : Monophonic number, open monophonic number, forcing monophonic number, forcing open monophonic number.

1. Introduction

By a graph $G = (V(G), E(G))$ we mean a finite, undirected connected graph without loops or multiple edges. The order and size of G are denoted by n and m , respectively. For basic graph theoretic terminology we refer to Harary [5]. The distance, geodesic and geodetic number of a graph were introduced and studied in [1, 3]. A set S of vertices of a connected graph G is an *open geodetic set* of G if for each vertex v in G , either v is an extreme vertex of G and $v \in S$, or v is an internal vertex of a x - y geodesic for some $x, y \in S$. An open geodetic set of minimum cardinality is a minimum open geodetic set and this cardinality is the *open geodetic number* $og(G)$. The open geodetic number of a graph was introduced and studied in [2, 6, 8]. A chord of a path u_1, u_2, \dots, u_n in G is an edge $u_i u_j$ with $j \geq i + 2$. For two vertices u and v in a connected graph G , a u - v path is called a *monophonic path* if it contains no chords. A *monophonic set* of G is a set $S \subseteq V(G)$ such that every vertex of G is contained in a monophonic path joining some pair of vertices in S . The *monophonic number* $m(G)$ of G is the cardinality of a minimum monophonic set. A subset T of S is called a forcing subset for S if S is the unique minimum monophonic set containing T . A forcing subset for S of minimum cardinality is a minimum forcing subset of S . The forcing monophonic number of S , denoted by $f_m(S)$, is the cardinality of a minimum forcing subset of S . The forcing monophonic number of G , denoted by $f_m(G)$, is $f_m(G) = \min(f_m(S))$, where the minimum is taken over all minimum monophonic sets in G .

A set S of vertices in a connected graph G is an *open monophonic set* if for each vertex v in G , either v is an extreme vertex of G and $v \in S$, or v is an internal vertex of a x - y monophonic path for some $x, y \in S$. An open monophonic set of minimum cardinality is a *minimum open monophonic set* and this cardinality is the *open monophonic number* $om(G)$ of G . The open monophonic number of a graph was introduced and further studied in [9, 10, 11, 12]. The following is the motivation for the introduction of open monophonic concepts and the corresponding forcing concepts. In a communication network, the main problem of protection from intercepting and overhearing is an inevitable one. Hence there is a need for enhancing security in a communication network design. Since graphs lend themselves naturally as models for these real life situations, the security based communication network problems can be studied and settled using the open monophonic and the corresponding forcing concepts. Further, these concepts have interesting applications in Channel Assignment Prob-

lem in Radio Technologies [4].

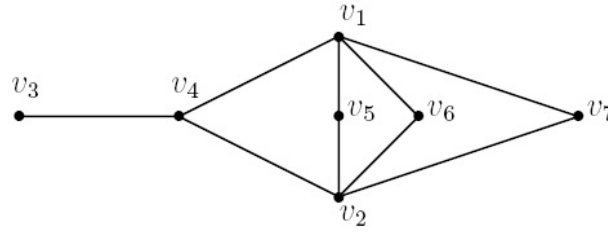
The following theorems are used in the sequel.

Theorem 1.1. [10] Every extreme vertex of a connected graph G belongs to each open monophonic set of G . In particular, if the set S of all extreme vertices of G is an open monophonic set of G , then S is the unique minimum open monophonic set of G .

Theorem 1.2. [12] For a connected graph G of order at least 2, $om(G) = 2$ if and only if there exist two extreme vertices u and v such that every vertex lies on a monophonic path joining u and v .

2. Forcing open monophonic number of a graph

Definition 2.1. Let G be a connected graph with at least two vertices and S a minimum open monophonic set of G . A subset T of S is called a forcing subset for S if S is the unique minimum open monophonic set containing T . A forcing subset for S of minimum cardinality is a minimum forcing subset of S . The forcing open monophonic number of S , denoted by $f_{om}(S)$, is the cardinality of a minimum forcing subset of S . The forcing open monophonic number of G , denoted by $f_{om}(G)$, is $f_{om}(G) = \min(f_{om}(S))$, where the minimum is taken over all minimum open monophonic sets in G .



G

Figure 2.1.

Example 2.2. For the graph G in Figure 2.1, it is clear that the sets $S_1 = \{v_1, v_2, v_3, v_6\}$ and $S_2 = \{v_1, v_2, v_3, v_7\}$ are the minimum open monophonic

sets of G . Now, $T = \{v_6\}$ is a forcing subset for S_1 and so $f_{om}(S_1) = 1$. Similarly $T' = \{v_7\}$ is a forcing subset for S_2 so that $f_{om}(S_2) = 1$. Hence $f_{om}(G) = 1$.

The next theorem follows immediately from the definition of an open monophonic number and the forcing open monophonic number of a connected graph G .

Theorem 2.3. For every connected graph G , $0 \leq f_{om}(G) \leq om(G) \leq n$.

Remark 2.4. For the complete graph $G = K_n$ ($n \geq 2$), the set of all vertices of K_n is the unique minimum open monophonic set so that $om(G) = n$ and $f_{om}(G) = 0$. Also, all the inequalities in Theorem 2.3 can be strict. For the graph G given in Figure 2.1, $om(G) = 4$, $f_{om}(G) = 1$ and $n = 6$.

Now, we proceed by proving a main result regarding forcing monophonic number of a graph.

Theorem 2.5. Let G be a connected graph. Then

- (i) $f_m(G)$ (or $f_{om}(G)$) = 0 if and only if G has a unique minimum (open) monophonic set.
- (ii) $f_m(G)$ (or $f_{om}(G)$) = 1 if and only if G has at least two minimum (open) monophonic sets, one of which is the unique minimum monophonic set contains one of its elements.
- (iii) $f_m(G)$ (or $f_{om}(G)$) = $m(G)$ (or $om(G)$) if and only if no minimum (open) monophonic set of G is the unique minimum (open) monophonic set containing any of its proper subsets.

Proof. We prove the theorem for forcing monophonic number. The proof for open forcing monophonic number is similar.

(i) Let $f_m(G) = 0$. Then by definition, $f_m(S) = 0$ for some minimum monophonic set S of G so that the empty set ϕ is the minimum forcing subset for S . Hence it follows that S is the unique minimum monophonic set of G . The converse is clear.

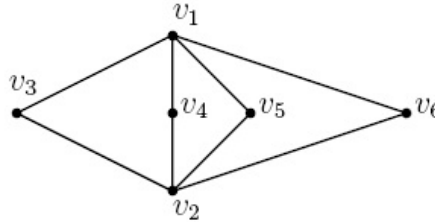
(ii) Let $f_m(G) = 1$. Then by (i), G has at least two minimum monophonic sets. Also, since $f_m(G) = 1$, there is a singleton subset T of a minimum monophonic set S of G such that T is not a subset of any other

minimum monophonic set of G . Thus S is the unique minimum monophonic set containing one of its elements. The converse is clear.

(iii) Let $f_m(G) = m(G)$. Then $f_m(S) = m(G)$ for every monophonic set S in G . Also, $m(G) \geq 2$ and hence $f_m(G) \geq 2$. Then by (i) G has at least two minimum monophonic sets, and so the empty set ϕ is not a minimum forcing subset for any minimum monophonic set of G . Since $f_m(G) = m(G)$, no proper subset of S is a forcing subset of S . Thus no minimum monophonic set of G is the unique minimum monophonic set containing any of its proper subsets. Conversely, assume that no minimum monophonic set of G is the unique minimum monophonic set containing any of its proper subsets. This shows that G contains more than one minimum monophonic set, and no subset of any minimum monophonic set S other than S is a forcing subset for S . Hence it follows that $f_m(G) = m(G)$. \square

Definition 2.6. A vertex v of a connected graph G is said to be an open monophonic vertex of G if v belongs to every minimum open monophonic set of G .

We illustrate this by an example. For the graph G given in Figure 2.2, it is easily verified $S_1 = \{v_1, v_2, v_3, v_6\}$, $S_2 = \{v_1, v_2, v_3, v_5\}$, $S_3 = \{v_1, v_2, v_3, v_4\}$, $S_4 = \{v_1, v_2, v_5, v_6\}$, $S_5 = \{v_1, v_2, v_4, v_6\}$, $S_6 = \{v_1, v_2, v_4, v_5\}$ are the minimum open monophonic sets. Hence the vertices v_1 and v_2 are the open monophonic vertices. It is to be noted that the set $T = \{v_3, v_6\}$ is a forcing subset for S_1 and so $f_{om}(S_1) = 2$. Hence $f_{om}(G) = 2$. Also, every endvertex in a graph is an open monophonic vertex.



G

Figure 2.2.

Theorem 2.7. Let G be a connected graph and let F be the set of relative complements of the minimum forcing subsets in their respective minimum open monophonic sets in G . Then $\bigcap_{F \in F} F$ is the set of open monophonic vertices of G .

Proof. Let W be the set of all open monophonic vertices of G . If $v \in W$, then v belongs to every minimum open monophonic set S of G . Let $T \subseteq S$ be any minimum forcing subset for any minimum open monophonic set S of G . We claim that $v \notin T$. If $v \in T$, then $T' = T - \{v\}$ is a proper subset of T such that S is the unique minimum open monophonic set containing T' so that T' is a forcing subset for S with $|T'| < |T|$, which is a contradiction because T is a minimum forcing subset for S . Thus $v \notin T$ and so $v \in F$, where F is the relative complement of T in S . Thus $v \in \bigcap_{F \in F} F$ so that $W \subseteq \bigcap_{F \in F} F$. Now, if $v \in \bigcap_{F \in F} F$, then v belongs to the relative complement of T in S for every T and every S such that $T \subseteq S$, where T is a minimum forcing subset for S . Since F is the relative complement of T in S , we have $F \subseteq S$ and thus $v \in S$ for every S . Hence v is an open monophonic vertex of G so that $v \in W$. It follows that $\bigcap_{F \in F} F \subseteq W$, and so $W = \bigcap_{F \in F} F$. \square

Theorem 2.8. Let G be a connected graph and W be the set of all open monophonic vertices of G . Then $f_{om}(G) \leq om(G) - |W|$.

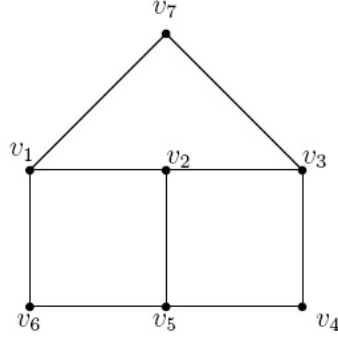
Proof. Let S be any minimum open monophonic set of G . Then $om(G) = |S|$. Since W is the set of all open monophonic vertices of G , it follows that $W \subseteq S$ and S is the unique minimum open monophonic set containing $S - W$. Hence $f_{om}(G) \leq |S - W| = |S| - |W| = om(G) - |W|$. \square

The following result is an easy consequence of Theorems 1.1 and 2.8.

Corollary 2.9. If G is a connected graph with k extreme vertices, then $f_{om}(G) \leq om(G) - k$.

We observe that the bound in Theorem 2.8 is sharp. For the graph G given in Figure 2.2, $om(G) = 4$, $f_{om}(G) = 2$ and $W = \{v_1, v_3\}$ is the set of open monophonic vertices of G . Hence $f_{om}(G) = om(G) - |W|$. Also, the inequality in Theorem 2.11 can be strict. For the graph G given in Figure 2.3, it is clear that $S_1 = \{v_1, v_3, v_5\}$ and $S_2 = \{v_4, v_6, v_7\}$ are the only two

minimum open monophonic sets of G so that $om(G) = 3$, $f_{om}(G) = 1$ and $W = \phi$. Hence $f_{om}(G) < om(G) - |W|$.



G

Figure 2.3.

Theorem 2.10. For any cycle $G = C_n (n \geq 4)$,

$$f_{om}(G) = \begin{cases} 3 & \text{if } n \geq 7 \\ 1 & \text{if } n = 6 \\ 4 & \text{if } n = 5 \\ 0 & \text{if } n = 4. \end{cases}$$

Proof. Let $G = C_n (n \geq 4)$ be the cycle $C_n : v_1, v_2, v_3, \dots, v_n, v_1$.

For $n = 4$, $S = \{v_1, v_2, v_3, v_4\}$ is the unique open monophonic set of G so that by Theorem 2.5(i), $f_{om}(G) = 0$.

For $n = 5$, the sets $S_1 = \{v_1, v_2, v_3, v_4\}$, $S_2 = \{v_1, v_2, v_3, v_5\}$, $S_3 = \{v_1, v_2, v_4, v_5\}$, $S_4 = \{v_2, v_3, v_4, v_5\}$, $S_5 = \{v_1, v_3, v_4, v_5\}$ are the minimum open monophonic sets of G . It is clear that no proper subset of any of the sets $S_i (i = 1, 2, 3, 4, 5)$ is a forcing subset so that $f_{om}(G) = 4$.

For $n = 6$, the sets $S_1 = \{v_1, v_3, v_5\}$ and $S_2 = \{v_2, v_4, v_6\}$ are the only two minimum open monophonic sets of G . Since $\{v_1\}$ is a forcing subset for S_1 , it follows that $f_{om}(G) = 1$.

Let $G = C_n (n \geq 7)$. Then any set containing three independent vertices of G is a minimum open monophonic set so that $om(G) = 3$. Since $n \geq 7$, it is clear that no 2-element subset of any minimum open monophonic set S is a forcing subset for S . Hence $f_{om}(G) = 3$. \square

Theorem 2.11. For any wheel $W_n = K_1 + C_{n-1}$ ($n \geq 4$),

$$f_{om}(W_n) = \begin{cases} 3 & \text{if } n \geq 8 \\ 1 & \text{if } n = 7 \\ 4 & \text{if } n = 6 \\ 0 & \text{if } n = 4, 5. \end{cases}$$

Proof. Let $C_{n-1} : v_1, v_2, \dots, v_{n-1}v_1$. Let $K_1 = \{x\}$. Then $W_n = C_{n-1} + K_1$

For $n = 4$, W_4 is the complete graph K_4 so that $f_{om}(G) = 0$.

For $n = 5$, the set $S = \{v_1, v_2, v_3, v_4\}$ is the unique open monophonic set of G , then by Theorem 2.5(i), $f_{om}(G) = 0$.

For $n = 6$, it is clear that no 3-element subset of vertices of W_6 is an open monophonic set of W_6 . The sets $S_1 = \{v_1, v_2, v_3, v_4\}$, $S_2 = \{v_1, v_2, v_3, v_5\}$, $S_3 = \{v_1, v_2, v_4, v_5\}$, $S_4 = \{v_2, v_3, v_4, v_5\}$ and $S_5 = \{v_1, v_3, v_4, v_5\}$ are the minimum open monophonic sets of G . It is easily verified that no subset of S_i ($1 \leq i \leq 5$) containing at most 3 elements is a forcing subset for S_i . Hence it follows that $f_{om}(W_6) = 4$.

For $n = 7$, the sets $S_1 = \{v_1, v_3, v_5\}$ and $S_2 = \{v_2, v_4, v_6\}$ are the only two minimum open monophonic sets of W_7 . Since $\{v_1\}$ is a forcing subset for S_1 , it follows that $f_{om}(W_7) = 1$.

Let $W_n = K_1 + C_{n-1}$ ($n \geq 8$). Then any set containing three independent vertices of W_n is a minimum open monophonic set so that $om(W_n) = 3$. Since $n \geq 8$, it is clear that no 2-element subset of any minimum open monophonic set S is a forcing subset for S . Hence $f_{om}(G) = 3$. \square

Theorem 2.12. For the complete bipartite graph $G = K_{r,s}$ ($2 \leq r \leq s$),

$$f_{om}(G) = \begin{cases} 0 & \text{if } r = s = 2 \\ 2 & \text{if } 2 = r < s \\ 4 & \text{if } 3 \leq r \leq s \end{cases}.$$

Proof. **Case 1.** $r = s = 2$. Then G is the cycle C_4 and by Theorem 2.10, $f_{om}(G) = 0$.

Case 2. $2 = r < s$. Let U and W be the partite sets of $K_{r,s}$ with $|U| = r$ and $|S| = s$. It is clear that the minimum open monophonic sets S of G are obtained by choosing the two elements of U and any two elements from W . It is easily verified that no single vertex subset of any of these sets is a forcing subset for S . Also, it is clear that any 2-element subset of W is a forcing subset for one of these minimum open monophonic sets S of G . Hence $f_{om}(G) = 2$.

Case 3. $3 \leq r \leq s$. Then any minimum open monophonic set S is got by taking any two elements from U and any two elements from W . It is easily verified that no 3-element subset of any minimum open monophonic sets S of G is a forcing set for S . Hence it follows that $f_{om}(G) = 4$. \square

It was already seen that $f_{om}(G) = 0$ if G is a complete graph or a tree or the cycle C_4 or the wheels W_4 and W_5 . Hence we leave the following problem as an open question.

Problem 2.13. Characterize graphs G for which (i) $f_{om}(G) = 0$ (ii) $f_{om}(G) = om(G)$.

Theorem 2.14. If G is a connected graph with $om(G) = 2$, then $f_{om}(G) = 0$.

Proof. Let $S = \{u, v\}$ be any minimum open monophonic set of G . Then by Theorem 1.2, u and v are the only extreme vertices such that every vertex of G lies on a monophonic path joining u and v . It follows from Theorem 1.1 that S is the unique minimum open monophonic set of G . Hence by Theorem 2.5(i), $f_{om}(G) = 0$. \square

Theorem 2.15. Given any positive integer with $n \geq 2$, there exists a connected graph G of order $2n$ such that $om(G) = n$ and $f_{om}(G) = 0$.

Proof. Let P_n be a path on n vertices. Let G be the graph obtained by adding a pendant edge at each vertex of P_n . Then it is clear that $om(G) = n$ and $f_{om}(G) = 0$. \square

The following graphs given in Figure 2.4 and Figure 2.5 are such that $om(G) = 2$. Hence by Theorem 2.14, $f_{om}(G) = 0$ for these two graphs G .

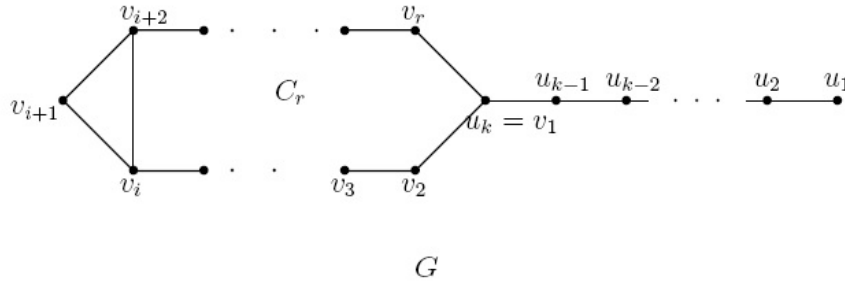


Figure 2.4

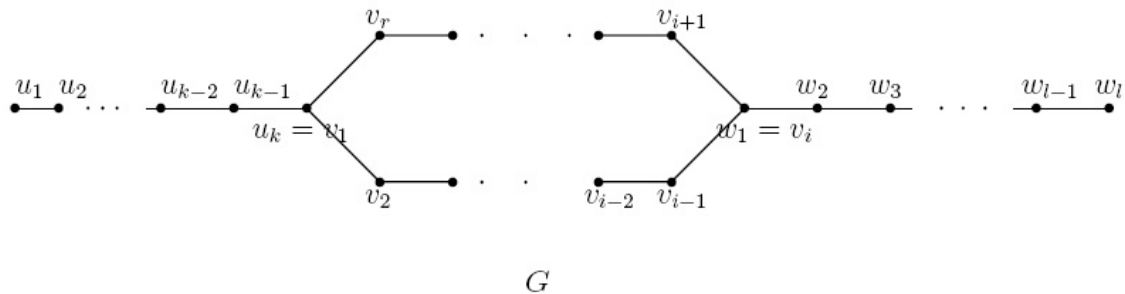


Figure 2.5.

Theorem 2.16. For every pair a, b of integers with $0 \leq a \leq b-4$ and $b \geq 5$, there exists a connected graph G such that $f_{om}(G) = a$ and $om(G) = b$.

Case 1. $a = 0$. Let $G = K_b$. Then, by Theorems 2.5(i) and 1.1, we have $f_{om}(G) = 0$ and $om(G) = b$.

Case 2. $a = 1$.

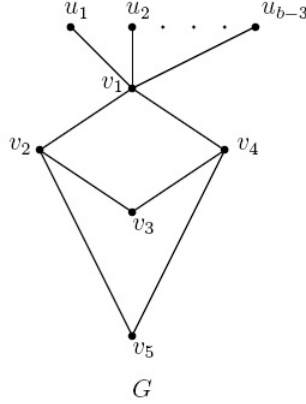


Figure 2.4.

Let G be the graph in Figure 2.4, obtained from $C_4 : v_1, v_2, v_3, v_4, v_1$ by first adding $b - 3$ new vertices $u_1, u_2, u_3, \dots, u_{b-3}$ to C_4 and joining each u_i ($1 \leq i \leq b - 3$) to the vertex v_1 of C_4 ; and also adding the new vertex v_5 and joining the edges v_5v_2 and v_5v_4 . Let $S = \{u_1, u_2, \dots, u_{b-3}\}$ be the set of all extreme vertices of G . By Theorem 1.1, every open monophonic set of G contains S . Then G contains exactly two minimum open monophonic sets namely $S_1 = S \cup \{v_2, v_3, v_4\}$ and $S_2 = S \cup \{v_2, v_4, v_5\}$. Thus $om(G) = |S| + 3 = b$. Since S_1 is the unique minimum open monophonic set containing v_3 , it follows from Theorem 2.5(ii) that $f_{om}(G) = 1 = a$.

Case 3. $a \geq 2$. Let $F_i : u_i, v_i, w_i$ be a path of order 3 ($1 \leq i \leq a$). Let $P_3 : u, v, w$ be a path of order 3. Let H be the graph obtained from the graphs P_3 and F_i ($1 \leq i \leq a$) by first adding the $2a$ edges uu_i and ww_i for $1 \leq i \leq a$; and also adding $b - a - 2$ new vertices $y_1, y_2, \dots, y_{b-a-3}, y_{b-a-2}$ and joining y_i ($1 \leq i \leq b - a - 3$) with w and y_{b-a-2} with u . Let G be the graph in Figure 2.5 obtained from H by adding the new vertex x and joining each u_i and w_i ($1 \leq i \leq a$) with x , and also adding the new vertices x_1 and x_2 and joining the edges xx_1 and vx_2 . We prove that $om(G) = b$ and $f_{om}(G) = a$. Let $S = \{x_1, x_2, y_1, y_2, \dots, y_{b-a-3}, y_{b-a-2}\}$ be the set of all extreme vertices of G . By Theorem 1.1, every open monophonic set of G contains S . Since the vertices v_1, v_2, \dots, v_a do not lie on a monophonic path joining any pair of vertices of S , S is not an open monophonic set of G . Let $S_i = \{u_i, w_i\}$ ($1 \leq i \leq a$). We observe that every open monophonic set of G must contain at least one vertex from each X_i ($1 \leq i \leq a$). Thus

$om(G) \geq b - a + a = b$. Now, since the set $X = S \cup \{u_1, u_2, \dots, u_a\}$ is an open monophonic set of G , it follows that $om(G) \leq |X| = b$. Thus $om(G) = b$. Next, we show that $f_{om}(G) = a$. Since every open monophonic set of G contains S , it follows from Theorem 2.7, that $f_{om}(G) \leq om(G) - |S| = b - (b - a) = a$. Since $om(G) = b$ and every open monophonic set of G contains S , it is easily seen that every minimum open monophonic set Y is of the form $S \cup \{z_1, z_2, \dots, z_n\}$, where $z_i \in S_i (1 \leq i \leq a)$.

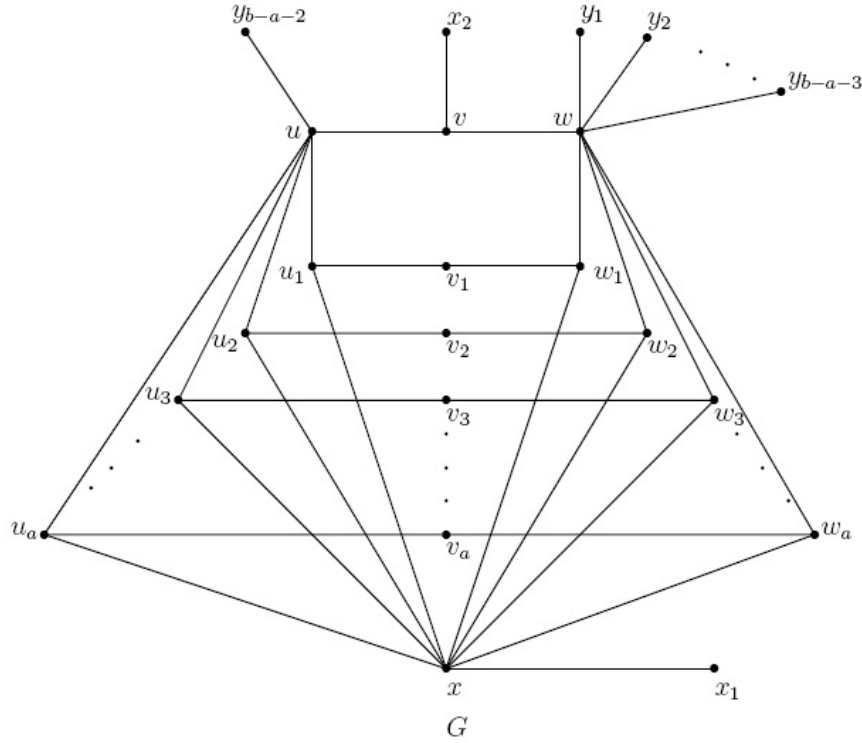


Figure 2.4.

Now, let T be a proper subset of Y with $|T| < a$. Then there is a vertex $z_j (1 \leq j \leq a)$ such that $z_j \notin T$. Let t_j be a vertex of S_j distinct from z_j . Then $Z = (Y - \{z_j\}) \cup \{t_j\}$ is an open monophonic set properly containing T . Thus Y is not the unique minimum open monophonic set containing T . Thus T is not a forcing subset of Y . This is true for all minimum open monophonic sets of G and so it follows that $f_{om}(G) = a$. \square

3. Addition of a pendant edge and forcing open monophonic number

A main problem in graph theory that commonly arises is how the value of a parameter is affected by making a small change in the graph. In this section, we study how forcing open monophonic number of a graph is affected by the addition of a pendant edge. In this section we analyse how the forcing open monophonic numbers of certain standard graphs are affected by the addition of a pendant edge.

Theorem 3.1. For the graph $G = C_n + e (n \geq 3)$, $om(G) = \begin{cases} 3 & \text{if } n \geq 5 \\ 4 & \text{if } n = 4 \\ 3 & \text{if } n = 3. \end{cases}$

and

$$f_{om}(G) = \begin{cases} 2 & \text{if } n \geq 6 \\ 0 & \text{if } n = 3, 4, 5. \end{cases}$$

Proof. For the graph $G = C_3 + e$ where $e = v_1x$, the set $S = \{x, v_2, v_3\}$ is the unique minimum open monophonic set of G and so $om(G) = 3$, and by Theorem 2.5 (i), $f_{om}(G) = 0$.

For the graph $G = C_4 + e$, where $e = v_1x$, the set $S = \{x, v_2, v_3, v_4\}$ is the unique minimum open monophonic set of G and so $om(G) = 4$, and by Theorem 2.5(i), $f_{om}(G) = 0$.

For the graph $G = C_5 + e$ where $e = v_1x$, the set $S = \{x, v_3, v_4\}$ is the unique minimum open monophonic set of G and so $om(G) = 3$, and by Theorem 2.5(i), $f_{om}(G) = 0$.

Let $C_n : v_1, v_2, v_3, \dots, v_n, v_1$ be a cycle and let $G = C_n + e (n \geq 6)$ with $e = v_1x$. Let v_j and v_k be the vertices adjacent to v_i . Then it is clear that the $(n-3)C_2$ sets $S = \{x, u, v\}$ with u and v distinct from v_j and v_k are the only minimum open monophonic sets of G . It follows that $om(G) = 3$. Since no singleton subset of S is a forcing subset for S and since $\{u, v\}$ is the unique minimum forcing subset for S , it follows that $f_{om}(G) = 2$. \square

Theorem 3.2. Let $W_n = C_{n-1} + K_1 (n \geq 5)$, where $C_{n-1} : v_1, v_2, \dots, v_{n-1}v_1$ and $K_1 = \{v_n\}$. Then for the graph $G = W_n + e$ where W_n is a wheel on n vertices and e is the pendant edge added at a vertex on C_{n-1} ,

$$om(G) = \begin{cases} 3 & \text{if } n \geq 6 \\ 4 & \text{if } n = 5 \end{cases} \quad \text{and} \quad f_{om}(G) = \begin{cases} 2 & \text{if } n \geq 7 \\ 0 & \text{if } n = 5, 6. \end{cases}$$

Proof. For the graph $G = W_5 + e$, where $e = v_1x$, the set $S = \{x, v_2, v_3, v_4\}$ is the unique minimum open monophonic set of G so that $om(G) = 4$. Hence by Theorem 2.5(i), $f_{om}(G) = 0$.

For the graph $G = W_6 + e$, where $e = v_1x$, the set $S = \{x, v_3, v_4\}$ is the unique minimum open monophonic set of G so that $om(G) = 3$. Hence by Theorem 2.5(i), $f_{om}(G) = 0$.

For the graph $G = W_7 + e$, where $e = v_1x$, the sets $S_1 = \{x, v_3, v_4\}$, $S_2 = \{x, v_3, v_5\}$ and $S_3 = \{x, v_4, v_5\}$ are the minimum open monophonic sets of G , it is follows that $om(G) = 3$. It is clear that the set $T_1 = \{v_3, v_4\}$ is a minimum forcing subset of S_1 so that $f_{om}(G) = 2$.

Let $G = C_n + e$ ($n \geq 8$) with $e = v_i x$. Let v_j and v_k be the vertices adjacent to v_i . Then it is clear that the $(n-3)C_2$ sets $S = \{x, u, v\}$ with u and v distinct from v_i, v_j and v_k are the only minimum open monophonic sets of G so that $om(G) = 3$. Since no singleton subset of S is a forcing subset for S and since $\{u, v\}$ is the unique minimum forcing subset for S , it follows that $f_{om}(G) = 2$. \square

Theorem 3.3. Let $W_n = C_{n-1} + K_1$ ($n \geq 5$), where $C_{n-1} : v_1, v_2, \dots, v_{n-1}v_1$ and $K_1 = \{v_n\}$. Then for the graph $G = W_n + e$ where W_n is a wheel on n vertices and e is the pendant edge added at a vertex on v_n ,

$$om(G) = \begin{cases} 4 & \text{if } n \geq 7 \\ 5 & \text{if } n = 5, 6. \end{cases}$$

and

$$f_{om}(G) = \begin{cases} 3 & \text{if } n \geq 8 \\ 1 & \text{if } n = 7 \\ 4 & \text{if } n = 6 \\ 0 & \text{if } n = 5. \end{cases}$$

Proof. Let $W_n = C_{n-1} + K_1$ where $C_{n-1} : v_1, v_2, \dots, v_{n-1}v_1$ and $K_1 = \{v_n\}$.

For the graph $G = W_5 + e$, where $e = v_5x$, the set $S = \{x, v_1, v_2, v_3, v_4\}$ is the unique minimum open monophonic set of G so that $om(G) = 5$. By Theorem 2.5(i), $f_{om}(G) = 0$.

For the graph $G = W_6 + e$, where $e = v_6x$, the sets $S_1 = \{x, v_1, v_2, v_3, v_4\}$, $S_2 = \{x, v_2, v_3, v_4, v_5\}$, $S_3 = \{x, v_1, v_2, v_3, v_5\}$, $S_4 = \{x, v_1, v_3, v_4, v_5\}$ and $S_5 = \{x, v_1, v_2, v_4, v_5\}$ are the unique minimum open monophonic sets of G so that $om(G) = 5$. Since $T_1 = \{v_1, v_2, v_3, v_4\}$ is a minimum forcing subset of S_1 , it follows that $f_{om}(G) = 4$.

For the graph $G = W_7 + e$, where $e = v_7x$, the sets $S_1 = \{x, v_1, v_3, v_5\}$, $S_2 = \{x, v_2, v_4, v_6\}$ are the only two minimum open monophonic sets of

G , so that $om(G) = 4$. Since $\{v_1\}$ is a minimum forcing subset for S_1 , it follows that $f_{om}(G) = 1$.

Let $G = W_n + e$ ($n \geq 8$) with $e = v_n x$. Let v_i, v_j, v_k be three non-adjacent vertices on C_{n-1} . Then it is clear that the sets $S = \{x, v_i, v_j, v_k\}$ are the only minimum open monophonic sets of G so that $om(G) = 4$. Since $n \geq 8$, it is clear that no 2-element subset of any minimum open monophonic set S is a forcing subset for S . Since $\{v_i, v_j, v_k\}$ is a forcing subset for S , it follows that $f_{om}(G) = 3$. \square

Theorem 3.4. For the graph $G = K_n + e$ ($n \geq 2$), $om(G) = n$ and $f_{om}(G) = 0$.

Proof. For the graph $G = K_n + e$, where $e = v_1 x$ ($n \geq 2$), the set $S = \{x, v_2, \dots, v_n\}$ is the unique open monophonic set of G so that $om(G) = n$. By Theorem 2.5(i), $f_{om}(G) = 0$. \square

We observe that for any non-trivial tree T , it is clear that $T + e$ is also a tree so that $f_{om}(T + e) = 0$.

Theorem 3.5. Let $U = \{u_1, u_2, \dots, u_r\}$ and $W = \{w_1, w_2, \dots, w_s\}$ be the partite sets of $K_{r,s}$. Then for the graph $G = K_{r,s} + e$ ($2 \leq r \leq s$), $om(G) = 4$,

$$f_{om}(G) = \begin{cases} 0 & \text{if } r = s = 2 \\ 1 & \text{if } 3 \leq r \leq s \end{cases}$$

and

$$\text{for } 2 = r < s, f_{om}(G) = \begin{cases} 2 & \text{if } e = u_1 x \\ 1 & \text{if } e = w_1 x \end{cases}$$

Proof. For $r = s = 2$, we have $K_{r,s} = C_4$ and by Theorem 3.1, $om(K_{r,s} + e) = 4$ and $f_{om}(K_{r,s} + e) = 0$.

For $2 = r < s$, let $e = u_1 x$. Then any set S formed by taking x, u_2 along with any two elements w_i, w_j of W forms a minimum open monophonic set of G so that $om(G) = 4$. It is clear that the subset $T = \{w_i, w_j\}$ is a forcing subset of S and so $f_{om}(G) = 2$. Similarly, if $e = u_2 x$, then again $om(G) = 4$ and $f_{om}(G) = 2$. Now, for $e = w_1 x$, the sets $S_i = \{u_1, u_2, x, w_i\}$ ($w_i \neq w_1$) is the minimum open monophonic sets of G so that $om(G) = 4$. Then $\{w_i\}$ is a forcing subset of S_i so that $f_{om}(G) = 1$. Similarly, if $e = w_i x$, where $i \neq 1$, then again $om(G) = 4$ and $f_{om}(G) = 1$.

For $3 \leq r \leq s$, let $e = u_1 x$. Then a set S_i formed by taking any two elements of W together with x and u_i such that $i \neq 1$ forms a minimum

open monophonic set. Hence $om(G) = 4$. Also $\{u_i\}$ is the forcing subset of S_i so that $f_{om}(G) = 1$. Similarly, if $e = u_ix (i \neq 1)$ or $e = w_ix$, then again $om(G) = 4$ and $f_{om}(G) = 1$.

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