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Vertex equitable labeling of union of cyclic snake related graphs

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Abstract

Let G be a graph with p vertices and q edges and $A = \{0, 1, 2, \ldots, \lceil \frac{q}{2} \rceil\}$. A vertex labeling $f : V(G) \to A$ induces an edge labeling f^* defined by $f^*(uv) = f(u) + f(v)$ for all edges uv. For $a \in A$, let $v_f(a)$ be the number of vertices v with $f(v) = a$. A graph G is said to be vertex equitable if there exists a vertex labeling f such that for all a and b in $A, |v_f(a) - v_f(b)| \leq 1$ and the induced edge labels are $1, 2, 3, \ldots, q$. In this paper, we prove that key graph $KY(m, n), P(2.QS_n), P(m.QS_n)$, $C(n.QS_m)$, $NQ(m)$ and $K_{1,n} \times P_2$ are vertex equitable graphs.

Keywords : *Vertex equitable labeling, vertex equitable graph, comb graph, key graph, path union graph, quadrilateral snake graph.*

AMS Subject Classification : *05C78.*

1. Introduction

All graphs considered here are simple, finite, connected and undirected. We follow the basic notations and terminology of graph theory as in [1]. A graph labeling is an assignment of integers to the vertices or edges or both, subject to certain conditions. There are several types of labeling and a detailed survey of graph labeling can be found in [2]. The vertex set and the edge set of a graph are denoted by $V(G)$ and $E(G)$ respectively. The concept of vertex equitable labeling was due to Lourdusamy and Seenivasan in [3] and further studied in [4]-[10]. Let G be a graph with p vertices and q edges and $A = \{0, 1, 2, \ldots, \lceil \frac{q}{2} \rceil\}$. A graph G is said to be vertex equitable if there exists a vertex labeling $f : V(G) \to A$ induces an edge labeling f^* defined by $f^*(uv) = f(u) + f(v)$ for all edges uv such that for all a and b in A, $|v_f(a) - v_f(b)| \leq 1$ and the induced edge labels are $1, 2, 3, \ldots, q$, where $v_f(a)$ be the number of vertices v with $f(v) = a$ for $a \in A$. The vertex labeling f is known as vertex equitable labeling. A graph G is said to be a vertex equitable if it admits vertex equitable labeling. In this paper, we extend our study on vertex equitable labeling and prove that key graph $KY(m, n), P(2.QS_n), P(m.QS_n)$, $C(n.QS_m)$, $NQ(m)$ and $K_{1,n} \times P_2$ are vertex equitable graphs. In [3], it is proved that the comb graph $P_n \odot K_1$ is a vertex equitable graph. In the following theorem we give an another vertex equitable labeling for the same graph $P_n \odot K_1$.

Theorem 1.1. *The comb graph $P_n \odot K_1$ is a vertex equitable graph.*

Proof. Let $V(P_n \odot K_1) = \{u_i, v_i : 1 \leq i \leq n\}$ and $E(P_n \odot K_1) = \{u_i v_i : 1 \leq i \leq n\} \cup \{u_i u_{i+1} : 1 \leq i \leq n-1\}$. Here $|V(P_n \odot K_1)| = 2n$ and $E(P_n \odot K_1)| = 2n - 1$. Let $A = \{0, 1, 2, \ldots, \lceil \frac{2n-1}{2} \rceil\}$.

Define a vertex labeling $f : V(P_n \odot K_1) \to A$ as follows:

Case (i). When n is even.
$f(u_{2i-1}) = 2(i-1), f(u_{2i}) = 2i, f(v_{2i-1}) = f(v_{2i}) = 2i - 1$ if $1 \leq i \leq \frac{n}{2}$.

Case (ii). When n is odd.
$f(u_{2i-1}) = 2i - 1, f(v_{2i-1}) = 2(i-1)$ if $1 \leq i \leq \lceil \frac{n}{2} \rceil, f(v_{2i}) = 2i, f(u_{2i}) = 2i - 1$ if $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$. It can be verified that the induced edge labels of $P_n \odot K_1$ are $1, 2, \ldots, 2n-1$ and $|v_f(a) - v_f(b)| \leq 1$ for all $a, b \in A$. Hence $P_n \odot K_1$ is a vertex equitable graph. □

We use the following theorem and definitions in the subsequent section.

Theorem 1.2. *[3] The cycle C_n is a vertex equitable graph if and only if $n \equiv 0$ or $3(mod\ 4)$.*

Theorem 1.3. *[7] The kC_4-snake is a vertex equitable graph.*

Theorem 1.4. *[4] Let $G_1(p_1, 2n+1)$ and $G_2(p_2, q_2)$ be any two vertex equitable graphs with equitable labeling f and g respectively. Let u and v be the vertices of G_1 and G_2 respectively such that $f(u) = n+1$ and $g(v) = 0$. Then the graph G obtained by joining u and v by an edge is a vertex equitable graph.*

Theorem 1.5. *[9] Let $G_1(p_1, q), G_2(p_2, q), \ldots, G_m(p_m, q)$ be the vertex equitable graphs with q is odd and u_i, v_i be the vertices of $G_i (1 \le i \le m)$ labeled by 0 and $\lceil \frac{q}{2} \rceil$. Then the graph G obtained by joining v_1 with u_2 and v_2 with u_3 and v_3 with u_4 and so on until we join v_{m-1} with u_m by an edge is also a vertex equitable graph.*

Definition 1.6. *Let $NQ(m)$ be the n^{th} quadrilateral snake obtained from the path $u_1, u_2, \ldots, u_m$ by joining u_i, u_{i+1} with $2n$ new vertices v_j^i and $w_j^i, 1 \le i \le m-1, 1 \le j \le n$.*

Definition 1.7. *The key graph is a graph obtained from K_2 by appending one vertex of C_m to one end point and comb graph $P_n \odot K_1$ to the other end of K_2. It is denoted as $KY(m, n)$.*

Definition 1.8. *[11] Let $G_1, G_2, \ldots, G_n, n \ge 2$ be n graphs and u_i be a vertex of G_i for $1 \le i \le n$. The graph obtained by adding an edge between u_i and u_{i+1} for $1 \le i \le n-1$ is called a path union of $G_1, G_2, \ldots, G_n$ and is denoted by $P(G_1, G_2, \ldots, G_n)$. When all the n graphs are isomorphic to a graph G, it is denoted by $P(n.G)$.*

Definition 1.9. *Let $G_1, G_2, \ldots, G_n$, be n graphs and u_i be a vertex of G_i for $1 \le i \le n$. The graph obtained by adding an edge between u_i and $u_{i+1} (1 \le i \le n-1), u_n$ and u_1 is called a cycle union of $G_1, G_2, \ldots, G_n$ and is denoted by $C(G_1, G_2, \ldots, G_n)$. When all the n graphs are isomorphic to a graph G, it is denoted by $C(n.G)$.*

2. Main Results

Theorem 2.1. *The key graph $KY(m, n)$ is a vertex equitable graph if $m \equiv 0$ or $3(mod\ 4)$.*

Proof.
Case(i). $m \equiv 3(mod\ 4)$.

Let $G_1 = C_m, G_2 = P_n \odot K_1$. Since G_1 has m edges and G_2 has $2n-1$ edges, By Theorem 1.1 and Theorem 1.2, $P_n \odot K_1, C_m$ are vertex equitable graphs. Hence, by Theorem 1.4, $KY(m,n)$ is a vertex equitable graph.

Case(ii). $m \equiv 0(mod\ 4)$.

Let $G_1 = P_n \odot K_1, G_2 = C_m$. Since G_1 has $2n-1$ edges and G_2 has m edges, By Theorem 1.1 and Theorem 1.2 $P_n \odot K_1$ and C_n are vertex equitable graphs. Hence by Theorem 1.4, $KY(m,n)$ is a vertex equitable graph. □

An example for the vertex equitable labeling of $KY(7,5)$ is shown in Figure 1.

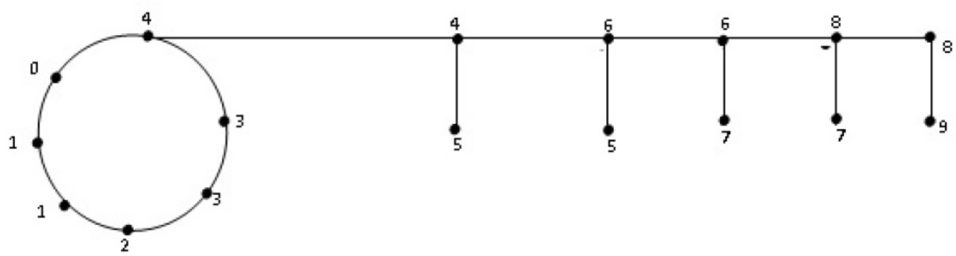

Figure 1

Theorem 2.2. *The path union graph $P(2.QS_n)$ is a vertex equitable graph.*

Proof. Let $V(P(2.QS_n)) = \{u_i, v_{ij}, w_{ij} : 1 \leq i \leq 2, 1 \leq j \leq n\}$ and $E(P(2.QS_n)) = \{u_1u_2, u_iv_{i1}, u_iw_{i1} : 1 \leq i \leq 2\} \cup \{u_{ij}v_{ij}, u_{ij}w_{ij} : 1 \leq i \leq 2, 1 \leq j \leq n\} \cup \{u_{ij}v_{ij+1}, u_{ij}w_{ij+1} : 1 \leq i \leq 2, 1 \leq j \leq n-1\}$. Here $|V(P(2.QS_n))| = 6n+2$ and $|E(P(2.QS_n))| = 8n+1$. Let $A = \{0, 1, 2, \ldots, \lceil \frac{8n+1}{2} \rceil\}$.

Define a vertex labeling $f : V((P(2.QS_n))) \to A$ as follows:

$f(u_1) = 0, f(u_2) = \lceil \frac{8n+1}{2} \rceil$.

For $1 \leq j \leq n$, $f(u_{1j}) = f(w_{1j}) = 2j$, $f(v_{1j}) = 2j-1$, $f(v_{2j}) = f(v_{2j}) = f(u_{2j}) = \lceil \frac{8n+1}{2} \rceil - 2j$, $f(w_{2j}) = \lceil \frac{8n+1}{2} \rceil - 2j + 1$.

It can be verifed that the induced edge labels of $P(2.QS_n)$ are $1, 2, \ldots, 8n+1$ and $|v_f(a) - v_f(b)| \leq 1$ for all $a, b \in A$. Hence $P(2.QS_n)$ is a vertex equitable graph. $\square$

Theorem 2.3. *The path union graph $P(m.QS_n)$ is a vertex equitable graph if $m > 2$.*

Proof. Here $V|P(m.QS_n)| = m(3n+1)$ and $E|P(m.QS_n)| = 4mn + m - 1$.

Case(i). m is even.

Let $G_i = P(2.QS_n)$ for $1 \leq i \leq \frac{m}{2}$. By Theorem 2.2, $P(2.QS_n)$ is a vertex equitable graph. Since each G_i has $8n+1$ edges, by Theorem 1.5, $P(m.QS_n)$ admits vertex equitable labeling if m is even.

Case(ii). m is odd and take $m = 2k+1$.

By Case (i) $P(2k.QS_n)$ is a vertex equitable graph. By Theorem 1.3, nC_4 snake is a vertex equitable graph. Let $G_1 = P(2k.QS_n)$ and $G_2 = nC4$. Since G_1 has $8mn + 2m - 1$ edges, by Theorem 1.4, $P(2m+1.QS_n)$ admits vertex equitable labeling. $\square$

An example for the vertex equitable labeling of the graph obtained by the path union of 4 copies of $3C_4$-snake is shown in Figure 2.

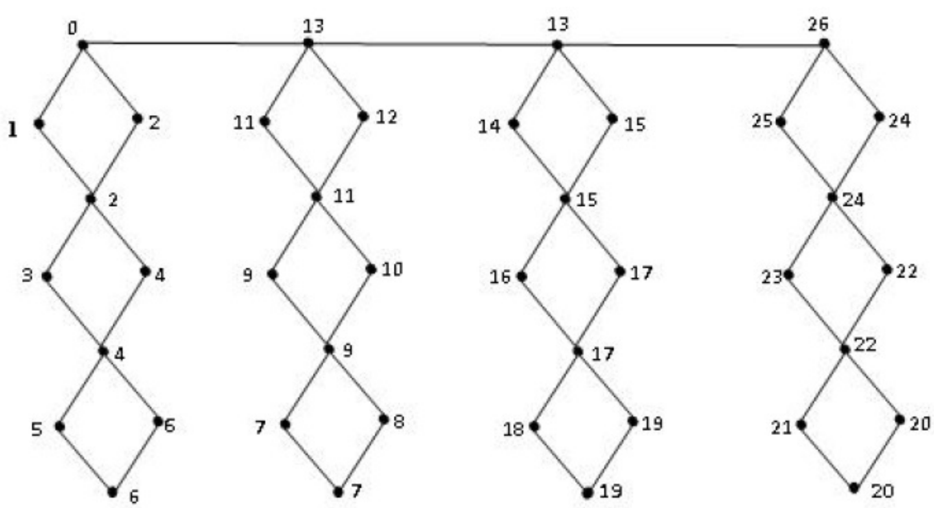

Figure 2

Theorem 2.4. *The graph obtained by the cycle union of n copies of mC_4-snake, $C(n.QS_m)$ is a vertex equitable graph if $n \equiv 0, 3(mod\ 4)$.*

Proof. Let $V(C(n.QS_m)) = \{u_i, u_{ij}, v_{ij}, w_{ij} : 1 \leq i \leq n, 1 \leq j \leq m\}$ and $E(C(n.QS_m)) = \{u_iu_{i+1} : 1 \leq i \leq n-1\} \cup \{u_nu_1\} \cup \{u_iv_{i1}, uw_{i1} : 1 \leq i \leq n\} \cup \{u_{ij}v_{ij}, u_{ij}w_{ij} : 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{u_{ij}v_{ij+1}, u_{ij}w_{ij+1} : 1 \leq i \leq n, 1 \leq j \leq m-1\}$. Here $|V(C(n.QS_m))| = 3mn + n$ and $|E(C(n.QS_m))| = 4mn + n$. Let $A = \{0, 1, 2, \ldots, \lceil \frac{4mn+n}{2} \rceil\}$.

Define a vertex labeling $f : V(C(n.QS_m)) \to A$ as follows:

Case (i). $n \equiv 0(mod\ 4)$.

$f(u_{2i}) = (4m+1)i$ if $1 \leq i \leq \frac{n}{2}$,

$$f(u_{2i-1}) = \begin{cases} (4m+1)(i-1) & \text{if } 1 \leq i \leq \frac{n}{4} \\ (4m+1)(i-1)+1 & \text{if } \frac{n}{4}+1 \leq i \leq \frac{n}{2}, \end{cases}$$

For $1 \leq j \leq m$,

$$f(v_{(2i-1)j}) = \begin{cases} (4m+1)(i-1)+2j & \text{if } 1 \leq i \leq \frac{n}{4} \\ (4m+1)(i-1)+(2j-1) & \text{if } \frac{n}{4}+1 \leq i \leq \frac{n}{2}, \end{cases}$$

$f(v_{(2i)j}) = (4m+1)i - 1 - 2(j-1)$ if $1 \leq i \leq \frac{n}{2}$,

$$f(w_{(2i-1)j}) = \begin{cases} (4m+1)(i-1)+2j-1 & \text{if } 1 \leq i \leq \frac{n}{4} \\ (4m+1)(i-1)+2j & \text{if } \frac{n}{4}+1 \leq i \leq \frac{n}{2} \end{cases}$$

$$f(w_{(2i)j}) = \begin{cases} (4m+1)i-2j & \text{if } 1 \leq i \leq \frac{n}{4} \\ (4m+1)i-2(j-1) & \text{if } \frac{n}{4}+1 \leq i \leq \frac{n}{2}, \end{cases}$$

$$f(u_{(2i-1)j}) = \begin{cases} (4m+1)(i-1)+2j & \text{if } 1 \leq i \leq \frac{n}{4} \\ (4m+1)(i-1)+1+2j & \text{if } \frac{n}{4}+1 \leq i \leq \frac{n}{2}, \end{cases}$$

$f(u_{(2i)j}) = (4m+1)i - 2j$ if $1 \leq i \leq \frac{n}{2}$.

Case (ii). $n \equiv 3(mod\ 4)$.

$f(u_{2i}) = (4m+1)(i-1) + (2m+1)$ if $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$,

$$f(u_{2i-1}) = \begin{cases} (4m+1)(i-1)+2m & \text{if } 1 \leq i \leq \lceil \frac{n}{4} \rceil \\ (4m+1)(i-1)+(2m+1) & \text{if } \lceil \frac{n}{4} \rceil + 1 \leq i \leq \lceil \frac{n}{2} \rceil, \end{cases}$$

$For 1 \leq j \leq m$,

$f(u_{(2i)j}) = (4m+1)(i-1) + (2m+1) + 2j$ if $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$,

$$f(u_{(2i-1)j}) = \begin{cases} (4m+1)(i-1)+2m-2j & \text{if } 1 \leq i \leq \lceil \frac{n}{4} \rceil \\ (4m+1)(i-1)+2m-2j+1 & \text{if } \lceil \frac{n}{4} \rceil + 1 \leq i \leq \lceil \frac{n}{2} \rceil, \end{cases}$$

$$f(v_{(2i-1)j}) = \begin{cases} (4m+1)(i-1)+2m-2(j-1) & \text{if } 1 \leq i \leq \lceil \frac{n}{4} \rceil \\ (4m+1)(i-1)+2m+1-2j & \text{if } \lceil \frac{n}{4} \rceil + 1 \leq i \leq \lceil \frac{n}{2} \rceil, \end{cases}$$

$$f(v_{(2i)j}) = \begin{cases} (4m+1)(i-1)+2m+2j-1 & \text{if } 1 \leq i \leq \lceil \frac{n}{4} \rceil \\ (4m+1)(i-1)+2m+2j & \text{if } \lceil \frac{n}{4} \rceil + 1 \leq i \leq \lfloor \frac{n}{2} \rfloor, \end{cases}$$

$$f(w_{2i-1)j}) = \begin{cases} (4m+1)(i-1)+2m-2j+1 & \text{if } 1 \leq i \leq \lceil \frac{n}{4} \rceil \\ (4m+1)(i-1)+2m-2(j-1) & \text{if } \lceil \frac{n}{4} \rceil + 1 \leq i \leq \lceil \frac{n}{2} \rceil, \end{cases}$$

$$f(w_{(2i)j}) = \begin{cases} (4m+1)(i-1)+2m+2j & \text{if } 1 \leq i \leq \lceil \frac{n}{4} \rceil \\ (4m+1)(i-1)+2m+2j+1 & \text{if } \lceil \frac{n}{4} \rceil + 1 \leq i \leq \lfloor \frac{n}{2} \rfloor. \end{cases}$$

It can be verified tha the induced edge labels of $C(n.QS_m)$ are $1, 2, \ldots, 4mn+n$ and $|v_f(a) - v_f(b)| \leq 1$ for all $a, b \in A$. Hence $C(n.QS_m)$ is a vertex equitable graph. $\square$

An example for the vertex equitable labeling of the graph obtained by the cycle union of 7 copies of $2C_4$-snake is shown in Figure 3.

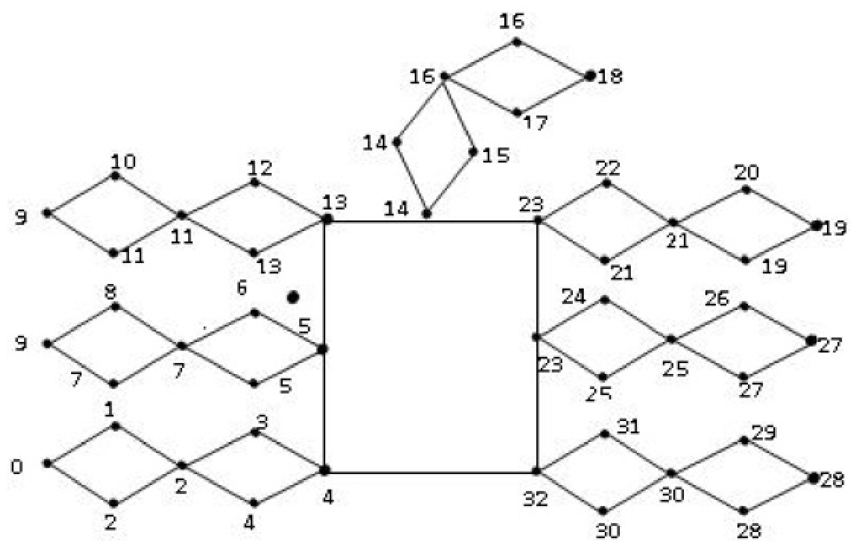

Figure 3

Theorem 2.5. *The n^{th} quadrilateral snake $NQ(m)$ is a vertex equitable graph if $n \geq 2$ is even.*

Proof. Let $V(NQ(m)) = \{u_i/1 \leq i \leq m\} \cup \{v_j^i/1 \leq i \leq m-1, 1 \leq j \leq n\} \cup \{w_j^i/2 \leq i \leq m, 1 \leq j \leq n\}, E(NQ(m)) = \{u_iu_{i+1}/1 \leq i \leq n-1\} \cup \{u_iv_j^i/1 \leq i \leq m-1, 1 \leq j \leq n\} \cup \{u_iw_j^i/2 \leq i \leq m, 1 \leq j \leq n\}$. Clearly $NQ(m)$ has $2(m-1)n+m$ vertices and $3(m-1)n+m-1$ edges. Let $A = \{0, 1, 2, \ldots, \lceil \frac{3n(m-1)+m-1}{2} \rceil\}$.

Define a vertex labeling $f : V(NQ(m)) \to A$ as follows:

For $1 \leq i \leq m$, $f(u_i) = \lceil \frac{(3n+1)(i-1)}{2} \rceil$,

For $1 \leq i \leq \lfloor \frac{m}{2} \rfloor, 1 \leq j \leq n$, $f(v_j^{2i-1}) = (3n+1)(i-1)+j$,

$f(v_j^{2i}) = (3n+1)(i-1) + \lceil \frac{(3n+1)}{2} \rceil + (j-1)$.

For $1 \leq i \leq \lfloor \frac{m}{2} \rfloor, 1 \leq j \leq \frac{n}{2}$, $f(w_j^{2i-1}) = (3n+1)(i-1) + \lceil \frac{(3n+1)}{2} \rceil - 2j$,

$f(w_{\frac{n}{2}+j}^{2i-1}) = (3n+1)(i-1) + \lceil \frac{(3n+1)}{2} \rceil - (2j-1)$.

For $1 \leq i \leq \lfloor \frac{m}{2} \rfloor, 1 \leq j \leq \frac{n}{2}$, $f(w_j^{2i}) = (3n+1)i - (2j-1)$,

$f(w_{\frac{n}{2}+j}^{2i}) = (3n+1)i - (2j-2)$.

It can be verified that the induced edge labels of $NQ(m)$ are $1, 2, \ldots, 3(m-1)n+m-1$ and $|v_f(a) - v_f(b)| \leq 1$ for all $a, b \in A$. Hence $NQ(m)$ is a vertex equitable graph. $\square$

An example for the vertex equitable labeling of $4Q(4)$ is shown in Figure 4.

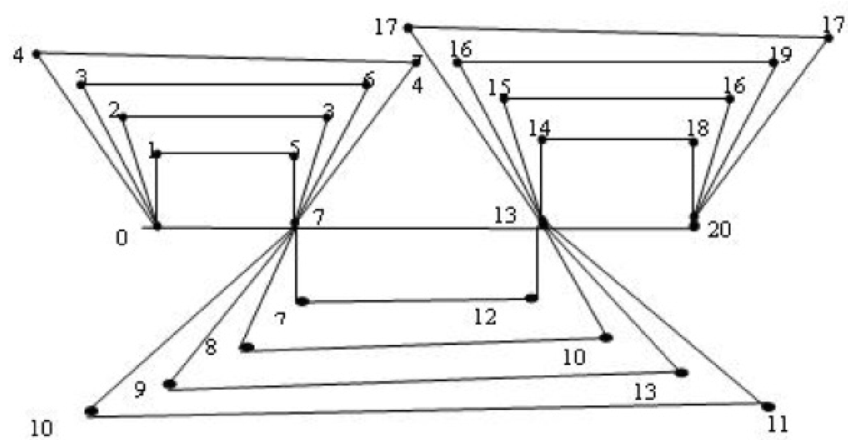

Figure 4

Corollary 2.6. *The book graph $K_{1,n} \times P_2$ is a vertex equitable graph.*

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