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A generalization of Drygas functional equation

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Abstract

We obtain the solutions of the following Drygas functional equation
$\sum_{\lambda\in\Phi} f(x+\lambda y+a_\lambda) = \kappa f(x) + \sum_{\lambda\in\Phi} f(\lambda y),\ x, y \in S,$
where S is an abelian semigroup, G is an abelian group, $f \in G^S$, Φ is a finite automorphism group of S with order κ, and $a_\lambda \in S$, $\lambda \in \Phi$.

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1. Introduction

Characterizing quasi-inner product spaces, Drygas considers in [9] the functional equation f(x)+f(y)=f(x-y)+$\left\{f\left(\frac{x+y}{2}\right)-f\left(\frac{x-y}{2}\right)\right\}$
which can be reduced to the following equation (see [21], Remark 9.2, p. 131)

$$f(x+y)+f(x-y)=2f(x)+f(y)+f(-y), \quad x,y\in \mathbf{R} \tag{1.1}$$

where $\mathbf{R}$ denotes the set of real numbers.

This equation is known in the literature as *Drygas equation* and is a generalization of the quadratic functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y), \quad x,y\in \mathbf{R}. \tag{1.2}$$

The general solution of Drygas equation was given by Ebanks, Kannappan and Sahoo in [10]. It has the form

$$f(x)=A(x)+Q(x),$$

where $A:\mathbf{R}\longrightarrow\mathbf{R}$ is an additive function and $Q:\mathbf{R}\longrightarrow\mathbf{R}$ is a quadratic function, see also [17]. A set-valued version of Drygas equation was considered by Smajdor in [23]. The Drygas functional equation on an arbitrary group G takes the form

$$f(xy)+f(xy^{-1})=2f(x)+f(y)+f(y^{-1}). \tag{1.3}$$

The solutions of Drygas equation in abelian group are obtained by Stetkær in [24]. Various authors studied the Drygas equation, for example Ebanks et al. [10], Faǐziev and Sahoo [11], Jung and Sahoo [17], Łukasik [18], Szabo [26],Yang [27].

There are several functional equations reduced to those of the Drygas functional equation (1.1), i.e. the mixed type additive, quadratic, Jensen and Pexidered equations, we refer, for example, to [1],[2],[4]-[8], [11]-[16], [19].

The present paper is actually a natural extension and complement to the work of Bouikhalene et al [4], Łukasik [18], Sinopoulos [22], Stetkær [25] and many others [2],[3], [10],[20].

We wish through this work to bring and share answers about two aspects: the characterization for solutions of the following Drygas equation

$$\sum_{\lambda\in\Phi} f(x+\lambda y+a_\lambda)=\kappa f(x)+\sum_{\lambda\in\Phi} f(\lambda y),\ x,y\in S, \tag{1.4}$$

where $f: S \to G$ is a mapping, S is an abelian semigroup, G is an abelian group, Φ is a finite automorphism group of S, $a_\lambda \in S,\ \lambda \in \Phi$, then to give illustrative examples of such situations.

This equation is an extension form of several equations, for examples,

$$f(x+y)+f(x-y)=2f(x)+f(y)+f(-y),\ x,y \in S,$$

$$f(x+y+a)+f(x+y+b)=2f(x)+2f(y),\ x,y \in S,$$

$$f(x+y+a)+f(x-y+b)=2f(x)+f(y)+f(-y),\ x,y \in S,$$

$$f(x+y+a)+f(x+\sigma y+b)=2f(x)+2f(y),\ x,y \in S,$$

$$f(x+y)+f(x+\sigma(y))=2f(x)+f(y)+f(\sigma(y)),\ x,y \in S,$$

$$f(x+y)+f(x+\sigma(y))=2f(x),\ x,y \in S,$$

$$\sum_{k=0}^{m-1} f(x+e^{\frac{2i\pi k}{m}}y)=mf(x)+\sum_{k=0}^{m-1} f(e^{\frac{2i\pi k}{m}}y),\ x,y \in S=G=\mathbf{C},\ m \in \mathbf{N}^*, m \geq 2,$$

and $$\sum_{k=0}^{m-1} f(x+e^{\frac{2i\pi k}{m}}y+a_i)=mf(x),\ x,y \in S=G=\mathbf{C},\ m \in \mathbf{N}^*, m \geq 2,$$

for $a, b, a_1, ..., a_{m-1} \in \mathbf{C}$ where $\mathbf{N}^*$ is the set of nonnegative numbers and where $\mathbf{C}$ is the set of all complex numbers.

2. Background results

Let $\mathbf{Z}$ designate the set of integers numbers and G^S the $\mathbf{Z}$-module consisting of all maps from an abelian semigroup S into an abelian group G. Let $n \in \mathbf{N}^*$ and $\mathcal{A}_n \in G^{S^n}$ be a function, then we say that $\mathcal{A}_n$ is *n-additive* provided that

$$\mathcal{A}_n(x_1+y_1, ..., x_j+y_j, ..., x_n+y_n) = \mathcal{A}_n(x_1, ..., x_j, ..., x_n)+\mathcal{A}_n(y_1, ..., y_j, ..., y_n),$$

for all $x_1, ..., x_j, ..., x_1,\ y_1, ..., y_j, ..., y_n \in E$; we say that $\mathcal{A}_n$ is *symmetric* provided that

$$\mathcal{A}_n(x_{\sigma(1)}, x_{\sigma(2)}, ..., x_{\sigma(n)}) = \mathcal{A}_n(x_1, x_2, ..., x_n)$$

whenever $x_1, x_2, ..., x_n \in S$ and σ is a permutation of $\{1, 2, ..., n\}$.

If $k \in \mathbf{N}^*$ and $\mathcal{A}_k \in G^{S^k}$ is symmetric and k-additive, let $\mathcal{A}_k^*(x) = \mathcal{A}_k(\underbrace{x,...,x}_{k})$ for all $x \in S$. And note that $\mathcal{A}_k^*(rx) = r^k \mathcal{A}_k^*(x)$ whenever $x \in S$ and $r \in \mathbf{N}$ also we note that

$$\mathcal{A}_k(x+h,...,x+h) = \sum_{i=0}^{k} C_k^i \mathcal{A}_k(\underbrace{x,...,x}_{i}, \underbrace{h,...,h}_{k-i}),\ x,h \in S.$$

The function $\mathcal{A}_k^*$ is called a *monomial* function of degree k associated to $\mathcal{A}_k$.

A function $p \in G^S$ is called a *generalized polynomial function* of degree n provided there exist $\mathcal{A}_0 \in S$ and monomial functions $\mathcal{A}_k^*$ (for $1 \leq k \leq n$) such that

$$p(x) = \mathcal{A}_0 + \sum_{i=1}^{n} \mathcal{A}_k^*(x),$$

for all $x \in S$. We also need to recall the definition of the *linear difference operator* $\Delta_h,\ h \in S$ on G^S by

$$\Delta_h f(x) = f(x+h) - f(x),\ h,\ x \in S.$$

Notice that these difference operators have important properties such as the linearity property

$$\Delta_h(\alpha f + \beta g) = \alpha \Delta_h(f) + \beta \Delta_h(g),\ f,g \in G^S,\ \alpha,\beta \in \mathbf{Z},$$

and the commutativity property

$$\triangle_{h_1}\triangle_{h_2}...\triangle_{h_s} = \triangle_{h_1 h_2 ... h_s} = \triangle_{h_{\sigma(1)} h_{\sigma(2)} ... h_{\sigma(s)}},$$

where σ is a permutation of $\{1, 2, ..., n\}$. There are also other properties such as

$$\Delta_h^n f(x) = \sum_{i=0}^{n} (-1)^{n-i} C_n^i f(x+ih)$$

and if $\mathcal{A}_m : S^m \to G$ is a symmetric and m-additive mapping, then we have

$$\triangle_{h_1...h_k} \mathcal{A}_m^*(x) = \begin{cases} m!\mathcal{A}_m(h_1,...,h_m), & \text{if } k = m \\ 0, & \text{if } k > m \end{cases}.$$

We will finish this section with some results which we will need in the sequel. Before that, we need to know that every abelian group G is said to be $n!$-*divisible group* when it is divisible uniquely by $n!$ where $n \in \mathbf{N}^*$.

Theorem 1. *[3],[7],[12],[14],[18],[19]*

Let G be an abelian group $n!$-divisible, $n \in \mathbf{N}^*$ and $f \in G^S$, then the following assertions are equivalent.

1. $\triangle_h^n f(x) = 0,\ x, h \in S$.
2. $\triangle_{h_1 \dots h_n} f(x) = 0,\ x, h_1, ..., h_n \in S$.
3. f *is a generalized polynomial function of degree at most* $n-1$.

Lemma 1. *[18] Let G be an abelian group $n!$-divisible, $n \in \mathbf{N}^*$, $x_1, x_2, ..., x_n \in G$, then the following properties are fulfilled.*

1.
$$\sum_{k=1}^{n}(-1)^{n-k}C_n^k k^i = 0,\ i \in \{1, 2, ..., n-1\}, n \neq 1 \tag{2.1}$$
and
$$\sum_{k=1}^{n}(-1)^{n-k}C_n^k k^n = n!. \tag{2.2}$$

2.
$$\text{If } \sum_{i=1}^{n} k^i x_i = 0,\ k \in \{1, ..., n\}, \quad \text{then } x_1 = x_2 = ... = x_n = 0. \tag{2.3}$$

3. Main results

Using the difference operator, we adopt the operatorial approach to characterize the solutions of Drygas equation (1.4) which is not a Jensen equation or a quadratic equation.

In the remainder of this paper, we denote by S an abelian semigroup and by G an abelian $(\kappa+1)!$-divisible group. However, a solution f of Drygas equation (1.4) in the semigroup S can be extended to the monoid $S \cup \{0\}$ (i.e. by adding the zero element to S) by setting the value of f to zero. We will then, $f(0) = \frac{1}{2\kappa}\sum_{\lambda \in \Phi} f(a_\lambda)$. Without alter the generality of the problem studied and if necessary, we will assume that S admit a zero element.

Lemma 2. *Let Φ be a finite automorphism group of S, $\kappa = card\Phi$, $a_\lambda \in S$ $(\lambda \in \Phi)$, $\mathcal{A}_0 \in G$ and $\mathcal{A}_i \in G^{S^i}$ $(1 \leq i \leq \kappa)$ be symmetric and i-additive mappings such that*

$$p(x) = \mathcal{A}_0 + \sum_{i=1}^{\kappa} \mathcal{A}_i^*(x), \quad x \in S \tag{3.1}$$

and

$$I_p(x,y) = \sum_{\lambda\in\Phi} p(x+\lambda y+a_\lambda) - \kappa p(x) - \sum_{\lambda\in\Phi} p(\lambda y),\ x,y\in S. \tag{3.2}$$

Then we have the following

(a) $$I_p(0,0) = \sum_{\lambda\in\Phi}\sum_{i=1}^{\kappa} \mathcal{A}_i(a_\lambda) - \kappa\mathcal{A}_0$$

(3.3)
and

(b) $I_p(x,y) =$
$I_p(0,0) + \sum_{\lambda\in\Phi}\sum_{j=0}^{\kappa-1}\sum_{k=0}^{\kappa-1}\sum_{2\le i=max}^{\kappa} C_i^j C_{i-j}^k \mathcal{A}_i(\underbrace{x,...,x}_{k}, a_\lambda,...,a_\lambda, \underbrace{\lambda y,...,\lambda y}_{j}),$

(3.4)

for all $x,y\in S$, where $max = max\{j+1, k+1, j+k\}$.

Proof. By direct calculation, we show that

$$I_p(0,0) = \sum_{\lambda\in\Phi} p(a_\lambda) - 2\kappa p(0).$$

Thus, by replacing p by its expression of $\mathcal{A}_i, 0\le i\le\kappa$ we obtain (a). For every $x,y\in S$, we have

$$\begin{aligned}
& I_p(x,y) \\
= \ & \kappa\mathcal{A}_0 + \sum_{\lambda\in\Phi}\left(\sum_{i=1}^{\kappa}\left(\mathcal{A}_i^*(x+\lambda y+a_\lambda)\right)\right) - \sum_{\lambda\in\Phi}\left(p(x)+p(\lambda y)\right) \\
= \ & \sum_{\lambda\in\Phi}\left(\sum_{i=1}^{\kappa}\left(\sum_{j=0}^{i} C_i^j\mathcal{A}_i(x+a_\lambda,...,x+a_\lambda,\underbrace{\lambda y,...,\lambda y}_{j})\right)\right) - \sum_{i=1}^{\kappa}\kappa\mathcal{A}_i^*(x) \\
- \ & \sum_{\lambda\in\Phi}\mathcal{A}_i^*(\lambda y) - \kappa\mathcal{A}_0 \\
= \ & \sum_{\lambda\in\Phi}\left(\sum_{i=1}^{\kappa}\left(\sum_{j=0}^{i} C_i^j \sum_{k=0}^{i-j} C_{i-j}^k \mathcal{A}_i(\underbrace{x,...,x}_{k}, a_\lambda,...,a_\lambda, \underbrace{\lambda y,...,\lambda y}_{j})\right)\right)
\end{aligned}$$

$$
\begin{aligned}
& -\sum_{i=1}^{\kappa} \kappa \mathcal{A}_i^*(x) \\
- \quad & \sum_{\lambda\in\Phi} \mathcal{A}_i^*(\lambda y) - \kappa \mathcal{A}_0 \\
= \quad & I_p(0,0) + \sum_{\lambda\in\Phi} \sum_{j=0}^{\kappa-1} \sum_{k=0}^{\kappa-1} \sum_{2\le i=max\{j+1,k+1,j+k\}}^{\kappa} C_i^j C_{i-j}^k \\
& \mathcal{A}_i(\underbrace{x,...,x}_{k}, a_\lambda, ..., a_\lambda, \underbrace{\lambda_y, ..., \lambda y}_{j}),
\end{aligned}
$$

from where (b) follows. □

Lemma 3. *Let Φ be a finite automorphism group of S, $\kappa = card\Phi$, $a_\lambda \in S$ $(\lambda \in \Phi)$, $\mathcal{A}_0 \in G$ and $\mathcal{A}_i \in G^{S^i}$ $(1 \le i \le \kappa)$ be symmetric and i-additive mappings such that*

$$
p(x) = \mathcal{A}_0 + \sum_{i=1}^{\kappa} \mathcal{A}_i^*(x), \ \ x \in S \tag{3.5}
$$

and

$$
I_p(x,y) = \sum_{\lambda\in\Phi} p(x+\lambda y + a_\lambda) - \kappa p(x) - \sum_{\lambda\in\Phi} p(\lambda y), \ x, y \in S. \tag{3.6}
$$

Then the following are equivalent.

1.
$$
I_p(x,y) = 0, \ x, y \in S. \tag{3.7}
$$

2. *p is a solution of Eq. (1.4).*

3. *$\mathcal{A}_0 \in G$ and the mappings $\mathcal{A}_i$, $1 \le i \le \kappa$, satisfy the following two equalities,*

   a)
$$
\sum_{\lambda\in\Phi} \sum_{i=1}^{\kappa} \mathcal{A}_i^*(a_\lambda) = \kappa \mathcal{A}_0 \tag{3.8}
$$

   and

b)

$$\sum_{i=max}^{\kappa} C_i^j C_{i-j}^k \sum_{\lambda\in\Phi} \mathcal{A}_i(\underbrace{x,x,...,x}_{k}, a_\lambda,...,a_\lambda, \underbrace{\lambda y, \lambda y, ..., \lambda y}_{j}) = 0,\ x,y\in S,$$

(3.9)

$0\le k\le \kappa-1,\ 0\le j\le \kappa-1,\ 2\le i=max=max\{k+1,j+1,j+k\}\le\kappa.$

Proof. Note first that by Lemma 2 the condition (2) is satisfied if and only if the condition (1) is satisfied. Suppose that (1) is satisfied, then by Lemma 2 we obtain (3)(a) and we have:

$$I_p(x,y)=\sum_{\lambda\in\Phi}\sum_{j=0}^{\kappa-1}\sum_{k=0}^{\kappa-1}\sum_{2\le i=max(j+1,k+1,j+k)}^{\kappa} C_i^j C_{i-j}^k$$

$$\mathcal{A}_i(\underbrace{x,...,x}_{k}, a_\lambda,...,a_\lambda, \underbrace{\lambda y,...,\lambda y}_{j})=0, \tag{3.10}$$

for all $x,y\in S$. To prove (3) we define, for every $0\le j\le\kappa-1,\ 0\le k\le\kappa-1$ the mappings $g_j,\ h_{(k,j)} : S\times S :\to G$ by

$$g_j(x,y)=\sum_{\lambda\in\Phi}\sum_{k=0}^{\kappa-1}\sum_{i=j+k}^{\kappa} C_i^j C_{i-j}^k \mathcal{A}_i(\underbrace{x,...,x}_{k}, a_\lambda,...,a_\lambda, \underbrace{\lambda y,...,\lambda y}_{j}),\ x,y\in S,$$

$$h_{(k,j)}(x,y)=\sum_{\lambda\in\Phi}\sum_{1\le i=j+k}^{\kappa} C_i^j C_{i-j}^k \mathcal{A}_i(\underbrace{x,...,x}_{k}, a_\lambda,...,a_\lambda, \underbrace{\lambda y,...,\lambda y}_{j}),\quad x,y\in S.$$

Note that,

$$I_p(x,y)=\sum_{j=0}^{\kappa-1} g_j(x,y),\ \sum_{k=0}^{\kappa-1} h_{(k,j)}(x,y)=g_j(x,y)$$

and

$g_0(x,y)=h_{(0,j)}(x,y)=h_{(k,0)}(x,y)=0$, for all $x,y\in S$. However, as

$$g_j(x,ny)=n^j g_j(x,y),\quad n\in\mathbf{N}^*,\ x,y\in S,\ 0\le j\le\kappa-1,$$

we have

$$\sum_{j=0}^{\kappa-1} n^j g_j(x,y)=\sum_{j=0}^{\kappa-1} g_j(x,ny)=0,\ n\in\mathbf{N}^*,\ x,y\in S.$$

By Lemma 1, we get

$$g_j(x,y)=0,\quad x,y\in S,\ 0\le j\le \kappa-1.$$

We deduced from the definition of $h_{(k,j)}$ that

$$h_{(k,j)}(nx,y)=n^k h_{(k,j)}(x,y),\ \ n\in \mathbf{N}^*,\ x,y\in S,\ 0\le k\le \kappa-j,\ 0\le j\le \kappa-1,$$

and we have

$$\sum_{k=0}^{\kappa-1} n^k h_{(k,j)}(x,y)=\sum_{k=0}^{\kappa-1} h_{(k,j)}(nx,y)=g_j(nx,y)=0,\ n\in \mathbf{N}^*,\ x,y\in S,$$
$$0\le j\le \kappa-1.$$

By the same manner as above we obtain

$$h_{(k,j)}(x,y)=0,\ j\in\{0,...,\kappa-1\},\ k\in\{0,...,\kappa-1\}.$$

Thus, Lemma 1 gives the expected result, $(3)(b)$. The converse of this implication is immediate. This completes the proof. □

Lemma 4. *Let Φ be a finite automorphism group of S, $\kappa=card\Phi$, $a_\lambda\in S$ ($\lambda\in\Phi$), and $f\in G^S$ such that*

$$\sum_{\lambda\in\Phi} f(x+\lambda y+a_\lambda)=\kappa f(x)+\sum_{\lambda\in\Phi} f(\lambda y),\quad x,y\in S. \tag{3.11}$$

Then, for every $x,y\in S$, $\Delta_y^\kappa f(x)$ is independent of x and we have

$$\Delta_y^{\kappa+1} f(x)=0,\ \ x,y\in S. \tag{3.12}$$

Proof. The proof used here goes along the same lines as the one in [18]. We will denote by $\Phi_{i,j}\subset\Phi,\ i\in\{0,...,\kappa\},\ j\in\{1,...,C_\kappa^i\}$ the C_κ^i pairwise different sets such that $card\Phi_{i,j}=\kappa-i$ and by $g\in G^S$, the application defined by

$$g(y)=-\sum_{i=0}^{\kappa}(-1)^{\kappa-i}\sum_{j=1}^{C_\kappa^i} f\left(\sum_{\lambda\in\mathcal{G}_{i,j}}\lambda y\right),\ \ y\in S.$$

Let $\lambda\in\Phi,\ i\in\{0,...,\kappa\}$ and $j\in\{1,...,C_\kappa^i\}$, then the set $\lambda\Phi_{ij}\subset\Phi$ has $\kappa-i$ elements. So, there is $k\in\{1,...,C_\kappa^i\}$ satisfies the following two equalities

$$\lambda\Phi_{ij} = \Phi_{i,k} \text{ and } \lambda^{-1}\Phi_{i,k} = \Phi_{i,j}.$$

It follows,

$$\sum_{j=1}^{C_\kappa^i} f\left(\sum_{\mu\in\Phi_{i,j}} \lambda\mu y\right) = \sum_{j=1}^{C_\kappa^i} f\left(\sum_{\mu\in\Phi_{i,j}} \mu y\right), \quad x \in S. \tag{3.13}$$

For given x, y, we set

$$u_i = x + iy,\ v_{ij} = \sum_{\mu\in\Phi_{i,j}} \mu y,\ \ i \in \{0, ..., \kappa\},\ j \in \{1, ..., C_\kappa^i\}.$$

Otherwise, let $\lambda \in \Phi$, $i \in \{0, ..., \kappa\}$ and $j \in \{1, ..., C_\kappa^i\}$, then we have the following two cases:

Case 1. Let $\lambda^{-1} \in \Phi_{i,j}$, then $i \neq \kappa$ and, $\Phi_{i,j} = \Phi_{i+1,j} \cup \{\lambda^{-1}\}$.

It follows that

$$\begin{aligned} u_i + \lambda v_{ij} &= x + iy + \sum_{\mu\in\Phi_{i,j}} \lambda\mu y \\ &= x + (i+1)y + \sum_{\mu\in\Phi_{i+1,k}} \lambda\mu y \\ &= u_{i+1} + \lambda v_{i+1,k}, \end{aligned}$$

for a suitable k in $\{1, ..., C_\kappa^{i+1}\}$.

Case 2. Let $\lambda^{-1} \in \Phi_{i,j}$, then $i \neq 0$ and, $\Phi_{i-1,j} = \Phi_{i,j} \cup \{\lambda^{-1}\}$. We can write,

$$\begin{aligned} u_i + \lambda v_{i,j} &= x + iy + \sum_{\mu\in\Phi_{i,j}} \lambda\mu y \\ &= x + (i-1)y + \sum_{\mu\in\Phi_{i-1,k}} \lambda\mu y \\ &= u_{i-1} + \lambda v_{i-1,k}, \end{aligned}$$

for a suitable k in $\{1, ..., C_\kappa^{i+1}\}$. Taking into account (3.13) and the calculation results of the previous two cases, we have:

$$
\begin{aligned}
&\kappa\Delta_y^{\kappa} f(x)-\kappa g(y)\\
=\;& \kappa\sum_{i=0}^{\kappa}(-1)^{\kappa-i}C_{\kappa}^{i} f(x+iy)+\kappa\sum_{i=0}^{\kappa-1}(-1)^{\kappa-i}\sum_{j=1}^{C_{\kappa}^{i}} f\left(\sum_{\mu\in\Phi_{i,j}}\mu y\right)\\
=\;& \kappa\sum_{i=0}^{\kappa}(-1)^{\kappa-i}C_{\kappa}^{i} f(x+iy+)+\sum_{i=0}^{\kappa-1}(-1)^{\kappa-i}\sum_{j=1}^{C_{\kappa}^{i}}\sum_{\lambda\in\Phi} f\left(\sum_{\mu\in\Phi_{i,j}}\lambda\mu y\right)\\
=\;& \sum_{i=0}^{\kappa}(-1)^{\kappa-i}\sum_{j=1}^{C_{\kappa}^{i}}\left(\kappa f(u_i)+\sum_{\lambda\in\Phi} f(\lambda v_{ij})\right)\\
=\;& \sum_{i=0}^{\kappa}(-1)^{\kappa-i}\sum_{j=1}^{C_{\kappa}^{i}}\sum_{\lambda\in\Phi} f(u_i+\lambda v_{ij}+a_{\lambda})\\
=\;& 0,\; x,y\in S.
\end{aligned}
$$

This shows that for every $x,y\in S$, $\Delta_y^{\kappa} f(x)$ is independent of x and

$$\Delta_y^{\kappa} f(x+y)-\Delta_y^{\kappa} f(x)=0,\; x,y\in S,$$

and more accurately

$$\Delta_y^{\kappa+1} f(x)=0,\; x,y\in S,$$

from which the desired result follows. □

Remark 1. *Under the assumptions of Lemma 4, if in addition we assume that*

$$\sum_{\lambda\in\Phi} f(\lambda y)=0,\; y\in S,$$

then

$$\Delta_y^{\kappa} f(x)=0,\;\; x,y\in S.$$

Theorem 2. Let $f\in G^S$, Φ a finite automorphism group of S, $\kappa=card\Phi$ and $a_\lambda\in S$ $(\lambda\in\Phi)$. Then the function $f:S\to G$ is a solution of equation

$$\sum_{\lambda\in\Phi} f(x+\lambda y+a_{\lambda})=\kappa f(x)+\sum_{\lambda\in\Phi} f(\lambda y),\; x,y\in S, \tag{3.14}$$

if and only if f has the following form

$$f(x)=\mathcal{A}_0+\sum_{i=1}^{\kappa}\mathcal{A}_i^{*}(x),\; x\in S, \tag{3.15}$$

where $\mathcal{A}_0 \in G$ and $\mathcal{A}_k : S^k \to G,\ k \in \{1, 2, ..., \kappa\}$ are symmetric and k-additive functions satisfying the two conditions:

i) $\sum_{i=max}^{\kappa} C_i^j C_{i-j}^k \sum_{\lambda\in\Phi} \mathcal{A}_i(\underbrace{x, x, ..., x}_{k}, a_\lambda, ..., a_\lambda, \underbrace{\lambda y, \lambda y, ..., \lambda y}_{j}) = 0,\ x, y \in S$,
$0 \le k \le \kappa - 1,\ 0 \le j \le \kappa - 1,\ 2 \le max = max\{j + 1, k + 1, k + j\} \le i \le \kappa$ and

ii) $\sum_{\lambda\in\Phi} \sum_{i=1}^{\kappa} \mathcal{A}_i^*(a_\lambda) = \kappa\mathcal{A}_0$.

Proof. The necessary condition is obtained by Lemma 4, Theorem 1 and Lemma 3. By Lemma 3 we get the sufficient condition which completes the proof of Theorem. □

Remark 2. *Under the assumptions of Theorem 2, if in addition we assume that*

$$\sum_{\lambda\in\Phi} f(\lambda y) = 0,\ y \in S,$$

then the result (with some modifications on the control of indices i, j and k) can be obtained by requiring the assumption "G is κ!-divisible" instead of "G is $(\kappa + 1)$!-divisible".

4. Consequences

The following corollaries are immediate consequences of Theorem 2. On this occasion, we obtain the following three corollaries 1, 2 and 3 which have been proved by Sinopoulos [22], Stetkær [25], Łukasik [18], Bouikhalene and Elqorachi [4] respectively.

Corollary 1. *[22][25] Let $\sigma : S \to S$ be an involution of S and G be an abelian group divisible by 2. Then the function $f : S \to G$ is a solution of equation*

$$f(x + y) + f\Big(x + \sigma(y)\Big) = 2f(x) + f(y) + f\Big(\sigma(y)\Big),\ \ x, y \in S \tag{4.1}$$

if and only if f has the following form

$$f(x) = \mathcal{A}_1(x) + \mathcal{A}_2^*(x),\ x \in S, \tag{4.2}$$

where $\mathcal{A}_1 : S \to G$ is an arbitrary additive function and $\mathcal{A}_2 : S \times S \to G$ is an arbitrary symmetric biadditive function with $\mathcal{A}_2(x, y) + \mathcal{A}_2\Big(x, \sigma(y)\Big) = 0,\ x, y \in S$.

Corollary 2. *[18] Let S be an abelian semigroup, G be an abelian group divisible by $\kappa!$, Φ be a finite automorphism group of S with order κ. Then the function $f : S \to G$ is a solution of equation*

$$\sum_{\lambda\in\Phi} f(x+\lambda y) = \kappa f(x) + \sum_{\lambda\in\Phi} f(\lambda y),\ x,y\in S, \tag{4.3}$$

if and only if f has the following form

$$f(x) = \sum_{i=1}^{\kappa} \mathcal{A}_i^*(x),\ x\in S, \tag{4.4}$$

where $\mathcal{A}_k : S^k \to G,\ k\in\{1,2,...,\kappa\}$ are arbiyrary symmetric and k-additive functions which satisfy the following condition:

$\sum_{\lambda\in\Phi} \mathcal{A}_i(x,x,...,x,\underbrace{\lambda y,\lambda y,...,\lambda y}_{j}) = 0,\ x,y\in S,\quad 1\le j\le i-1,\ 2\le i\le\kappa.$

Proof. In this case, with the notations of Theorem 2, as $\{a_\lambda,\ \lambda\in\Phi\} = \{0\}$, $k+j=i$.

Furthermore, we can write that

$$\begin{aligned} 0 &= \sum_{i=max(k+j,k+1)} C_i^k C_j^{i-j} \sum_{\lambda\in\Phi} \mathcal{A}_i k(\underbrace{x,...,x}, a_\lambda,...,a_\lambda, j\underbrace{\lambda y,...,\lambda y}) \\ &= \sum_{i=k+j}^{\kappa} C_i^k \sum_{\lambda\in\Phi} \mathcal{A}_i k(\underbrace{x,...,x}, j\underbrace{\lambda y,...,\lambda y}) \\ &= \sum_{j=1}^{i-1} C_i^k \sum_{\lambda\in\Phi} \mathcal{A}_i(x,...,x,j\underbrace{\lambda y,...,\lambda y}),\ x,y\in S,\ 2\le i\le\kappa. \end{aligned}$$

For $1\le j\le i-1,\ 2\le i\le\kappa$, we define the mappings $q_{(j,i)} : S\times S\to G$ by

$$q_{(j,i)}(x,y) = C_i^j \sum_{\lambda\in\Phi} \mathcal{A}_i(x,...,x,j\underbrace{\lambda y,\lambda y,...,\lambda y})\ x,y\in S.$$

So, we have

$$q_{(j,i)}(x,ny) = n^j q_{(j,i)}(x,y),\ x,y\in S,\ n\in\mathbf{N}^*,\ 0\le j\le i-1,\ 2\le i\le\kappa$$

and

$$\sum_{j=1}^{\kappa} n^j q_{(j,i)}(x,y) = \sum_{j=1}^{\kappa} q_{(j,i)}(x,ny) = 0,\ x,y\in S,\ 2\le i\le\kappa.$$

According to Lemma1 we get the sought result. $\square$

Corollary 3. *[4] Let S be an abelian group, G be a Banach space and $a \in S$. Then, the general solution $f : S \to G$ of the functional equation*

$$f(x+y+a) = f(x) + f(y), \quad x, y \in S, \tag{4.5}$$

is

$$f(x) = \mathcal{A}_1(a) + \mathcal{A}_1(x), x \in S. \tag{4.6}$$

where $\mathcal{A}_1 : S \to G$ is an arbitrary additive function.

In the following corollaries we prove new others special cases of the equation 1.4 that is, according to our knowledge, not in the literature.

Corollary 4. *Let S be an abelian semigroup, G be an abelian group divisible by 2 and $a, b \in S$. Then, the general solution $f : S \to G$ of the functional equation*

$$f(x+y+a) + f(x-y+b) = 2f(x) + f(y) + f(-y), \quad x, y \in S, \tag{4.7}$$

is

$$f(x) = \frac{1}{2}\Big(\mathcal{A}_1(a+b)\Big) + \mathcal{A}_1(x) + \mathcal{A}_2^*(x), x \in S \tag{4.8}$$

where $\mathcal{A}_1 : S \to G$ is an arbitrary additive function and $\mathcal{A}_2 : S \times S \to G$ is an arbitrary symmetric biadditive function with $\mathcal{A}_2(x,a) = \mathcal{A}_2(x,b) = 0,\ x \in S$.

Corollary 5. *Let S be an abelian semigroup, σ be an involution of S, G be an abelian group divisible by 2 and $a, b \in S$. Then, the general solution $f : S \to G$ of the functional equation*

$$f(x+y+a) + f\Big(x+\sigma(y)+b\Big) = 2f(x) + f(y) + f\Big(\sigma(y)\Big), \quad x, y \in S, \tag{4.9}$$

is

$$f(x) = \frac{1}{2}\Big(\mathcal{A}_1(a+b)\Big) + \mathcal{A}_1(x) + \mathcal{A}_2^*(x), x \in S, \tag{4.10}$$

where $\mathcal{A}_1 : S \to G$ is an arbitrary additive function and $\mathcal{A}_2 : S \times S \to G$ is an arbitrary symmetric biadditive function with

$$\mathcal{A}_2(x,a) = \mathcal{A}_2(x,b) = 0,\ x \in S \ \text{ and } \ \mathcal{A}_2(x,y) + \mathcal{A}_2\Big(x,\sigma(y)\Big) = 0,\ x, y \in S.$$

Proof. Keeping in mind the notation of Theorem 2, we apply it where $\kappa = 2$. Then there are an element $\mathcal{A}_0 \in G$ and symmetric i-additives mappings $\mathcal{A}_i \in G^{S^i}, \ i \in \{1,2\}$ satisfy

1. $f(x) = \mathcal{A}_0 + \mathcal{A}_1(x) + \mathcal{A}_2^*(x), \ x \in S$
   on the other side, they satisfy the following conditions of Theorem 2
   :

2. i) $k = 0, j = 1, \ \mathcal{A}_2(a, y) + \mathcal{A}_2(b, \sigma(y)) = 0, \ y \in S,$
   ii) $k = 1, j = 0, \ \mathcal{A}_2(y, a) + \mathcal{A}_2(y, b) = 0, \ y \in S,$
   iii) $k = 1, j = 1, \ \mathcal{A}_2(x, y) + \mathcal{A}_2(x, \sigma(y)) = 0, \ x, y \in S.$

Thus, $\mathcal{A}_2(y, a) = \mathcal{A}_2(y, b) = 0, \ y \in S; \ \mathcal{A}_2(x, y) + \mathcal{A}_2(x, \sigma(y)) = 0, \ x, y \in S$ and $2\mathcal{A}_0 = \mathcal{A}_1(a + b)$. □

Corollary 6. *Let j be a primitive cube root of unity and a be complex number. Then, the general continuous solution $f : \mathbf{C} \to \mathbf{C}$ of the functional equation*

$$f(x+y+ja)+f(x+jy+j^2a)+f(x+j^2y+a) = 3f(x)+f(y)+f(jy)+f(j^2y), \quad x, y \in \mathbf{C}, \tag{4.11}$$

is of the form

$$f(x) = \alpha_1 x + \beta_1 \overline{x} + \alpha_2 x^2 + \beta_2 \overline{x}^2, \tag{4.12}$$

where $\alpha_1, \beta_1, \alpha_2, \beta_2 \in \mathbf{C}$.

Proof. According the Theorem 2, there are $\alpha_0 \in \mathbf{C}$, and symmetric i-additive mappings $\mathcal{A}_i : \mathbf{C}^i \to \mathbf{C}, \ i \in \{1, 2, 3\}$ such that

$$f(z) = \alpha_0 + \mathcal{A}_1(z) + \mathcal{A}_2^*(z) + \mathcal{A}_3^*(z), \ z \in \mathbf{C}.$$

Taking into account that j is a primitive cube root of unity, we have $1 + j + j^2 = 0$. In addition, the continuity of f show that $\mathcal{A}_1, \mathcal{A}_2$ et $\mathcal{A}_3$ can be written as the following

$\mathcal{A}_1(z) = \alpha_1 z + \beta_1 \overline{z}, \ \alpha_1, \alpha_2 \in \mathbf{C},$

$\mathcal{A}_2^*(z) = \alpha_2 z^2 + \beta_2 \overline{z}^2 + \beta_3 |z|^2, \ \alpha_2, \beta_2, \beta_3 \in \mathbf{C},$

$\mathcal{A}_3^*(z) = \gamma_1 z^3 + \gamma_2 \overline{z}^3, \ \gamma_1, \gamma_2 \in \mathbf{C}.$

So the conditions of Theorem 2 do not satisfy where $\gamma_1 = \gamma_2 = \beta_3 = 0$ which finish the proof. □

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