A generalization of Drygas functional equation

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Abstract
We obtain the solutions of the following Drygas functional equation
\[ \sum_{\lambda \in \Phi} f(x + \lambda y + a_{\lambda}) = \kappa f(x) + \sum_{\lambda \in \Phi} f(\lambda y), \quad x, y \in S, \]
where \( S \) is an abelian semigroup, \( G \) is an abelian group, \( f \in G^S \), \( \Phi \)
is a finite automorphism group of \( S \) with order \( \kappa \), and \( a_{\lambda} \in S, \lambda \in \Phi \).

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1. Introduction

Characterizing quasi-inner product spaces, Drygas considers in [9] the functional equation
\[ f(x) + f(y) = f(x - y) + \frac{x^2 + y^2}{2} - f(\frac{x - y}{2}) \]
which can be reduced to the following equation (see [21], Remark 9.2, p. 131)
\[ f(x + y) + f(x - y) = 2f(x) + f(y) + f(-y), \quad x, y \in \mathbb{R} \]
(1.1)
where \( \mathbb{R} \) denotes the set of real numbers.

This equation is known in the literature as Drygas equation and is a generalization of the quadratic functional equation
\[ f(x + y) + f(x - y) = 2f(x) + 2f(y), \quad x, y \in \mathbb{R}. \]
(1.2)
The general solution of Drygas equation was given by Ebanks, Kannappan and Sahoo in [10]. It has the form
\[ f(x) = A(x) + Q(x), \]
where \( A : \mathbb{R} \rightarrow \mathbb{R} \) is an additive function and \( Q : \mathbb{R} \rightarrow \mathbb{R} \) is a quadratic function, see also [17]. A set-valued version of Drygas equation was considered by Smajdor in [23]. The Drygas functional equation on an arbitrary group \( G \) takes the form
\[ f(xy) + f(xy^{-1}) = 2f(x) + f(y) + f(y^{-1}). \]
(1.3)
The solutions of Drygas equation in abelian group are obtained by Stetkær in [24]. Various authors studied the Drygas equation, for example Ebanks et al. [10], Fažiev and Sahoo [11], Jung and Sahoo [17], Łukasik [18], Szabo [26], Yang [27].

There are several functional equations reduced to those of the Drygas functional equation (1.1), i.e. the mixed type additive, quadratic, Jensen and Pexidered equations, we refer, for example, to [1], [2], [4]-[8], [11]-[16], [19].

The present paper is actually a natural extension and complement to the work of Bouikhalene et al [4], Łukasik [18], Sinopoulos [22], Stetkær [25] and many others [2], [3], [10], [20].

We wish through this work to bring and share answers about two aspects: the characterization for solutions of the following Drygas equation
\[ \sum_{\lambda \in \Phi} f(x + \lambda y + a\lambda) = \kappa f(x) + \sum_{\lambda \in \Phi} f(\lambda y), \quad x, y \in S, \]
(1.4)
where $f : S \to G$ is a mapping, $S$ is an abelian semigroup, $G$ is an abelian group, $\Phi$ is a finite automorphism group of $S$, $a_\lambda \in S$, $\lambda \in \Phi$, then to give illustrative examples of such situations.

This equation is an extension form of several equations, for examples,

$$f(x + y) + f(x - y) = 2f(x) + f(y) + f(-y), \quad x, y \in S;$$
$$f(x + y + a) + f(x + y + b) = 2f(x) + 2f(y), \quad x, y \in S;$$
$$f(x + y + a) + f(x - y + b) = 2f(x) + f(y) + f(-y), \quad x, y \in S;$$
$$f(x + y + a) + f(x + \sigma y + b) = 2f(x) + 2f(y), \quad x, y \in S;$$
$$f(x + y) + f(x + \sigma(y)) = 2f(x) + f(y) + f(\sigma(y)), \quad x, y \in S;$$
$$f(x + y) + f(x + \sigma(y)) = 2f(x), \quad x, y \in S;$$

$$\sum_{k=0}^{m-1} f(x + e^{2\pi i k/m} y) = m f(x) + \sum_{k=0}^{m-1} f(e^{2\pi i k/m} y), \quad x, y \in S = G = \mathbb{C}, \quad m \in \mathbb{N}^*, m \geq 2,$$

and

$$\sum_{k=0}^{m-1} f(x + e^{2\pi i k/m} y + a_i) = m f(x), \quad x, y \in S = G = \mathbb{C}, \quad m \in \mathbb{N}^*, m \geq 2,$$

for $a, b, a_1, \ldots, a_{m-1} \in \mathbb{C}$ where $\mathbb{N}^*$ is the set of nonnegative numbers and where $\mathbb{C}$ is the set of all complex numbers.

2. Background results

Let $\mathbb{Z}$ designate the set of integers numbers and $G^S$ the $\mathbb{Z}$-module consisting of all maps from an abelian semigroup $S$ into an abelian group $G$. Let $n \in \mathbb{N}^*$ and $A_n \in G^{S^n}$ be a function, then we say that $A_n$ is $n$-additive provided that

$$A_n(x_1 + y_1, \ldots, x_j + y_j, \ldots, x_n + y_n) = A_n(x_1, \ldots, x_j, \ldots, x_n) + A_n(y_1, \ldots, y_j, \ldots, y_n),$$

for all $x_1, \ldots, x_j, \ldots, x_n, y_1, \ldots, y_j, \ldots, y_n \in E$; we say that $A_n$ is symmetric provided that

$$A_n(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}) = A_n(x_1, x_2, \ldots, x_n)$$

whenever $x_1, x_2, \ldots, x_n \in S$ and $\sigma$ is a permutation of $\{1, 2, \ldots, n\}$. 
If \( k \in \mathbb{N}^* \) and \( A_k \in G^{S^k} \) is symmetric and \( k \)-additive, let \( A^*_k(x) = A_k(x, ..., x) \) for all \( x \in S \). And note that \( A_k^*(rx) = r^k A_k^*(x) \) whenever \( x \in S \) and \( r \in \mathbb{N} \) also we note that

\[
A_k(x + h, ..., x + h) = \sum_{i=0}^{k} C_i^k A_k^{i}(x, ..., x, h, ..., h), \quad x, h \in S.
\]

The function \( A_k^* \) is called a \textit{monomial} function of degree \( k \) associated to \( A_k \).

A function \( p \in G^S \) is called a \textit{generalized polynomial function} of degree \( n \) provided there exist \( A_0 \in S \) and monomial functions \( A_k^* \) (for \( 1 \leq k \leq n \)) such that

\[
p(x) = A_0 + \sum_{i=1}^{n} A_k^*(x),
\]

for all \( x \in S \). We also need to recall the definition of the \textit{linear difference operator} \( \Delta_h, h \in S \) on \( G^S \) by

\[
\Delta_h f(x) = f(x + h) - f(x), \quad h, x \in S.
\]

Notice that these difference operators have important properties such as the linearity property

\[
\Delta_h(\alpha f + \beta g) = \alpha \Delta_h f + \beta \Delta_h g, \quad f, g \in G^S, \quad \alpha, \beta \in \mathbb{Z},
\]

and the commutativity property

\[
\Delta_h f \Delta_{h_1} \cdots \Delta_{h_s} = \Delta_{h_1 h_2 \cdots h_s} = \Delta_{h_{\sigma(1)} h_{\sigma(2)} \cdots h_{\sigma(s)}},
\]

where \( \sigma \) is a permutation of \( \{1, 2, ..., n\} \). There are also other properties such as

\[
\Delta_h^n f(x) = \sum_{i=0}^{n} (-1)^{n-i} C_i^n f(x + ih)
\]

and if \( A_m : S^m \to G \) is a symmetric and \( m \)-additive mapping, then we have

\[
\Delta_{h_1 \cdots h_k} A_m^*(x) = \begin{cases} 
m! A_m(h_1, ..., h_m), & \text{if } k = m \\
0, & \text{if } k > m \end{cases}
\]

We will finish this section with some results which we will need in the sequel. Before that, we need to know that every abelian group \( G \) is said to be \textit{n!-divisible group} when it is divisible uniquely by \( n! \) where \( n \in \mathbb{N}^* \).
Theorem 1. [3],[7],[12],[14],[18],[19]

Let $G$ be an abelian group $n!$-divisible, $n \in \mathbb{N}^*$ and $f \in G^S$, then the following assertions are equivalent.

1. $\Delta^nf(x) = 0$, $x, h \in S$.
2. $\Delta_{h_1...h_n}f(x) = 0$, $x, h_1, ..., h_n \in S$.
3. $f$ is a generalized polynomial function of degree at most $n - 1$.

Lemma 1. [18] Let $G$ be an abelian group $n!$-divisible, $n \in \mathbb{N}^*$, $x_1, x_2, ..., x_n \in G$, then the following properties are fulfilled.

1. 
   \begin{align*}
   \sum_{k=1}^{n} (-1)^{n-k} \binom{n}{k} k^i = 0, & i \in \{1, 2, ..., n-1\}, n \neq 1 \\
   \text{and} \quad \sum_{k=1}^{n} (-1)^{n-k} \binom{n}{k} k^n = n!.
   \end{align*}

2. 
   \begin{align*}
   \text{If} \sum_{i=1}^{n} k^i x_i = 0, & k \in \{1, ..., n\}, \text{ then } x_1 = x_2 = ... = x_n = 0.
   \end{align*}

3. Main results

Using the difference operator, we adopt the operatorial approach to characterize the solutions of Drygas equation (1.4) which is not a Jensen equation or a quadratic equation.

In the remainder of this paper, we denote by $S$ an abelian semigroup and by $G$ an abelian $(\kappa+1)!$-divisible group. However, a solution $f$ of Drygas equation (1.4) in the semigroup $S$ can be extended to the monoid $S \cup \{0\}$ (i.e. by adding the zero element to $S$) by setting the value of $f$ to zero. We will then, $f(0) = \frac{1}{2\pi} \sum_{\lambda \in \Phi} f(a_\lambda)$. Without alter the generality of the problem studied and if necessary, we will assume that $S$ admit a zero element.

Lemma 2. Let $\Phi$ be a finite automorphism group of $S$, $\kappa = \text{card} \Phi$, $a_\lambda \in S$ ($\lambda \in \Phi$), $A_0 \in G$ and $A_i \in G^{S^i}$ ($1 \leq i \leq \kappa$) be symmetric and $i$-additive mappings such that

\begin{equation}
   p(x) = A_0 + \sum_{i=1}^{\kappa} A_i^i(x), \quad x \in S
\end{equation}
and

\[ I_p(x, y) = \sum_{\lambda \in \Phi} p(x + \lambda y + a_\lambda) - \kappa p(x) - \sum_{\lambda \in \Phi} p(\lambda y), \ x, y \in S. \] (3.2)

Then we have the following

(a) \[ I_p(0, 0) = \sum_{\lambda \in \Phi} \sum_{i=1}^{\kappa} \mathcal{A}_i(a_\lambda) - \kappa \mathcal{A}_0 \] (3.3)

and

(b) \[ I_p(x, y) = I_p(0, 0) + \sum_{\lambda \in \Phi} \sum_{j=0}^{\kappa-1} \sum_{k=0}^{\kappa-1} \sum_{2 \leq i = \max}^{\kappa} C_i^j C_{i-j}^k \mathcal{A}_i(x, x, a_\lambda, \ldots, a_\lambda, \lambda y, \ldots, \lambda y). \] (3.4)

for all \( x, y \in S \), where \( \max = \max\{j + 1, k + 1, j + k\} \).

**Proof.** By direct calculation, we show that

\[ I_p(0, 0) = \sum_{\lambda \in \Phi} p(a_\lambda) - 2\kappa p(0). \]

Thus, by replacing \( p \) by its expression of \( \mathcal{A}_i, 0 \leq i \leq \kappa \) we obtain (a). For every \( x, y \in S \), we have

\[
I_p(x, y) = \kappa \mathcal{A}_0 + \sum_{\lambda \in \Phi} \left( \sum_{i=1}^{\kappa} \mathcal{A}_i^* (x + \lambda y + a_\lambda) \right) - \sum_{\lambda \in \Phi} \left( p(x) + p(\lambda y) \right) \\
= \sum_{\lambda \in \Phi} \left( \sum_{i=1}^{\kappa} \left( \sum_{j=0}^{i} C_i^j \mathcal{A}_i(x + a_\lambda, \ldots, x + a_\lambda, \lambda y, \ldots, \lambda y) \right) \right) - \sum_{i=1}^{\kappa} \kappa \mathcal{A}_i^* (x) \\
- \sum_{\lambda \in \Phi} \mathcal{A}_i^* (\lambda y) - \kappa \mathcal{A}_0 \\
= \sum_{\lambda \in \Phi} \left( \sum_{i=1}^{\kappa} \left( \sum_{j=0}^{i} C_i^j \sum_{k=0}^{i-j} C_{i-j}^k \mathcal{A}_i(x, a_\lambda, \ldots, a_\lambda, \lambda y, \ldots, \lambda y) \right) \right) \\
\]
A generalization of Drygas functional equation

\[
- \sum_{i=1}^{\kappa} \kappa A_i^*(x) \\
- \sum_{\lambda \in \Phi} A_i^*(\lambda y) - \kappa A_0
\]

= \[I_p(0, 0) + \sum_{\lambda \in \Phi} \sum_{j=0}^{\kappa-1} \sum_{k=0}^{\kappa-1} \sum_{0 \leq i = \max\{j+1,k+1,j+k\}}^{\kappa} C_i^j C_k^{i-j}
A_i(x, \ldots, x, a_{\lambda}, \ldots, a_{\lambda}, \lambda y, \ldots, \lambda y),
\]

from where (b) follows. \(\square\)

**Lemma 3.** Let \(\Phi\) be a finite automorphism group of \(S\), \(\kappa = \text{card} \Phi\), \(a_{\lambda} \in S\) \((\lambda \in \Phi)\), \(A_0 \in G\) and \(A_i \in G^{S^i}\) \((1 \leq i \leq \kappa)\) be symmetric and \(i\)-additive mappings such that

\[
p(x) = A_0 + \sum_{i=1}^{\kappa} A_i^*(x), \quad x \in S
\]

and

\[
I_p(x, y) = \sum_{\lambda \in \Phi} p(x + \lambda y + a_{\lambda}) - \kappa p(x) - \sum_{\lambda \in \Phi} p(\lambda y), \quad x, y \in S.
\]

Then the following are equivalent.

1. \(I_p(x, y) = 0, \quad x, y \in S\). \(\quad (3.7)\)

2. \(p\) is a solution of Eq. (1.4).

3. \(A_0 \in G\) and the mappings \(A_i, \quad 1 \leq i \leq \kappa\), satisfy the following two equalities,

a)

\[
\sum_{\lambda \in \Phi} \sum_{i=1}^{\kappa} A_i^*(a_{\lambda}) = \kappa A_0
\]

and
\[ \sum_{i=\text{max}}^{\kappa} C_i^j C_{i-j}^k \sum_{\lambda \in \Phi} A_i(x, x, ..., a, \lambda, a, \lambda, \lambda, ..., \lambda) = 0, \quad x, y \in S, \quad (3.9) \]

\[ 0 \leq k \leq \kappa - 1, \quad 0 \leq j \leq \kappa - 1, \quad 2 \leq i = \text{max} = \text{max}\{k+1, j+1, k+j+1\} \leq \kappa. \]

**Proof.** Note first that by Lemma 2 the condition (2) is satisfied if and only if the condition (1) is satisfied. Suppose that (1) is satisfied, then by Lemma 2 we obtain (3)(a) and we have:

\[ I_p(x, y) = \sum_{\lambda \in \Phi} \sum_{j=0}^{\kappa-1} \sum_{k=0}^{\kappa-1} C_i^j C_{i-j}^k A_i(x, x, ..., a, \lambda, a, \lambda, \lambda, ..., \lambda), \quad x, y \in S, \quad (3.10) \]

for all \( x, y \in S \). To prove (3) we define, for every \( 0 \leq j \leq \kappa - 1, \quad 0 \leq k \leq \kappa - 1 \) the mappings \( g_j, h_{(k,j)} : S \times S :\rightarrow G \) by

\[ g_j(x, y) = \sum_{\lambda \in \Phi} \sum_{k=0}^{\kappa-1} C_i^j C_{i-j}^k A_i(x, x, ..., a, \lambda, a, \lambda, \lambda, ..., \lambda), \quad x, y \in S, \]

\[ h_{(k,j)}(x, y) = \sum_{\lambda \in \Phi} \sum_{j=0}^{\kappa-1} C_i^j C_{i-j}^k A_i(x, x, ..., a, \lambda, a, \lambda, \lambda, ..., \lambda), \quad x, y \in S. \]

Note that,

\[ I_p(x, y) = \sum_{j=0}^{\kappa-1} g_j(x, y), \quad \sum_{k=0}^{\kappa-1} h_{(k,j)}(x, y) = g_j(x, y) \]

and

\[ g_0(x, y) = h_{(0,j)}(x, y) = h_{(k,0)}(x, y) = 0, \quad \text{for all } x, y \in S. \]

However, as

\[ g_j(x, ny) = n^j g_j(x, y), \quad n \in \mathbb{N}^+, \quad x, y \in S, \quad 0 \leq j \leq \kappa - 1, \]

we have

\[ \sum_{j=0}^{\kappa-1} n^j g_j(x, y) = \sum_{j=0}^{\kappa-1} g_j(x, ny) = 0, \quad n \in \mathbb{N}^+, \quad x, y \in S. \]
By Lemma 1, we get
\[ g_j(x, y) = 0, \quad x, y \in S, \quad 0 \leq j \leq \kappa - 1. \]

We deduced from the definition of \( h_{(k,j)} \) that
\[ h_{(k,j)}(nx, y) = n^k h_{(k,j)}(x, y), \quad n \in \mathbb{N}^*, \quad x, y \in S, \quad 0 \leq k \leq \kappa - j, \quad 0 \leq j \leq \kappa - 1, \]
and we have
\[ \sum_{k=0}^{\kappa-1} n^k h_{(k,j)}(x, y) = \sum_{k=0}^{\kappa-1} h_{(k,j)}(nx, y) = g_j(nx, y) = 0, \quad n \in \mathbb{N}^*, \quad x, y \in S, \quad 0 \leq j \leq \kappa - 1. \]

By the same manner as above we obtain
\[ h_{(k,j)}(x, y) = 0, \quad j \in \{0, ..., \kappa - 1\}, \quad k \in \{0, ..., \kappa - 1\}. \]

Thus, Lemma 1 gives the expected result, (3)(b). The converse of this implication is immediate. This completes the proof.

**Lemma 4.** Let \( \Phi \) be a finite automorphism group of \( S \), \( \kappa = \text{card} \Phi \), \( a_\lambda \in S \) \((\lambda \in \Phi)\), and \( f \in G^S \) such that
\[ \sum_{\lambda \in \Phi} f(x + \lambda y + a_\lambda) = \kappa f(x) + \sum_{\lambda \in \Phi} f(\lambda y), \quad x, y \in S. \tag{3.11} \]

Then, for every \( x, y \in S \), \( \Delta^\kappa_y f(x) \) is independent of \( x \) and we have
\[ \Delta^{\kappa+1}_y f(x) = 0, \quad x, y \in S. \tag{3.12} \]

**Proof.** The proof used here goes along the same lines as the one in [18]. We will denote by \( \Phi_{i,j} \subset \Phi \), \( i \in \{0, ..., \kappa\} \), \( j \in \{1, ..., C^i_\kappa\} \) the \( C^i_\kappa \) pairwise different sets such that \( \text{card} \Phi_{i,j} = \kappa - i \) and by \( g \in G^S \), the application defined by
\[ g(y) = -\sum_{i=0}^{\kappa} (-1)^{\kappa-i} \sum_{j=1}^{C^i_\kappa} f \left( \sum_{\lambda \in \Phi_{i,j}} \lambda y \right), \quad y \in S. \]

Let \( \lambda \in \Phi \), \( i \in \{0, ..., \kappa\} \) and \( j \in \{1, ..., C^i_\kappa\} \), then the set \( \lambda \Phi_{i,j} \subset \Phi \) has \( \kappa - i \) elements. So, there is \( k \in \{1, ..., C^i_\kappa\} \) satisfies the following two equalities.
\[ \lambda \Phi_{ij} = \Phi_{i,k} \text{ and } \lambda^{-1} \Phi_{i,k} = \Phi_{i,j}. \]

It follows,

\[ \sum_{j=1}^{C_k^i} f \left( \sum_{\mu \in \Phi_{i,j}} \lambda \mu y \right) = \sum_{j=1}^{C_k^i} f \left( \sum_{\mu \in \Phi_{i,j}} \mu y \right), \quad x \in S. \]

For given \( x, y \), we set

\[ u_i = x + iy, \quad v_{ij} = \sum_{\mu \in \Phi_{i,j}} \mu y, \quad i \in \{0, \ldots, \kappa\}, \quad j \in \{1, \ldots, C_k^i\}. \]

Otherwise, let \( \lambda \in \Phi, \quad i \in \{0, \ldots, \kappa\} \text{ and } j \in \{1, \ldots, C_k^i\} \), then we have the following two cases:

**Case 1.** Let \( \lambda^{-1} \in \Phi_{i,j} \), then \( i \neq \kappa \) and, \( \Phi_{i,j} = \Phi_{i+1,j} \cup \{\lambda^{-1}\} \).

It follows that

\[ u_i + \lambda v_{ij} = x + iy + \sum_{\mu \in \Phi_{i,j}} \lambda \mu y \]
\[ = x + (i + 1)y + \sum_{\mu \in \Phi_{i+1,k}} \lambda \mu y \]
\[ = u_{i+1} + \lambda v_{i+1,k}, \]

for a suitable \( k \) in \( \{1, \ldots, C_k^{i+1}\} \).

**Case 2.** Let \( \lambda^{-1} \in \Phi_{i,j} \), then \( i \neq 0 \) and, \( \Phi_{i-1,j} = \Phi_{i,j} \cup \{\lambda^{-1}\} \). We can write,

\[ u_i + \lambda v_{ij} = x + iy + \sum_{\mu \in \Phi_{i,j}} \lambda \mu y \]
\[ = x + (i - 1)y + \sum_{\mu \in \Phi_{i-1,k}} \lambda \mu y \]
\[ = u_{i-1} + \lambda v_{i-1,k}, \]

for a suitable \( k \) in \( \{1, \ldots, C_k^{i+1}\} \). Taking into account (3.13) and the calculation results of the previous two cases, we have:
\[ \kappa \Delta_y^\kappa f(x) - \kappa g(y) \]

\[ = \kappa \sum_{i=0}^\kappa (-1)^{\kappa-i} C_i^\kappa f(x + iy) + \kappa \sum_{i=0}^{\kappa-1} \sum_{j=1}^{C_i^\kappa} f \left( \sum_{\mu \in \Phi_{i,j}} \mu y \right) \]

\[ = \kappa \sum_{i=0}^\kappa (-1)^{\kappa-i} C_i^\kappa f(x + iy + \lambda y) + \sum_{i=0}^{\kappa-1} \sum_{j=1}^{C_i^\kappa} \sum_{\mu \in \Phi_{i,j}} f \left( \sum_{\lambda \in \Phi} \lambda \mu y \right) \]

\[ = \sum_{i=0}^\kappa (-1)^{\kappa-i} \sum_{j=1}^{C_i^\kappa} \left( \kappa f(u_i) + \sum_{\lambda \in \Phi} f(\lambda v_{ij}) \right) \]

\[ = \sum_{i=0}^\kappa (-1)^{\kappa-i} \sum_{j=1}^{C_i^\kappa} f(u_i + \lambda v_{ij} + a\lambda) \]

\[ = 0, \ x, y \in S. \]

This shows that for every \( x, y \in S \), \( \Delta_y^\kappa f(x) \) is independent of \( x \) and

\[ \Delta_y^\kappa f(x + y) - \Delta_y^\kappa f(x) = 0, \ x, y \in S, \]

and more accurately

\[ \Delta_y^{\kappa+1} f(x) = 0, \ x, y \in S, \]

from which the desired result follows. □

**Remark 1.** Under the assumptions of Lemma 4, if in addition we assume that

\[ \sum_{\lambda \in \Phi} f(\lambda y) = 0, \ y \in S, \]

then

\[ \Delta_y^\kappa f(x) = 0, \ x, y \in S. \]

**Theorem 2.** Let \( f \in G^S \), \( \Phi \) a finite automorphism group of \( S \), \( \kappa = \text{card} \Phi \) and \( a\lambda \in S \ (\lambda \in \Phi) \). Then the function \( f : S \rightarrow G \) is a solution of equation

\[ \sum_{\lambda \in \Phi} f(x + \lambda y + a\lambda) = \kappa f(x) + \sum_{\lambda \in \Phi} f(\lambda y), \ x, y \in S, \]

if and only if \( f \) has the following form

\[ f(x) = A_0 + \sum_{i=1}^\kappa A_i^\kappa(x), \ x \in S, \]
where $A_0 \in G$ and $A_k : S^k \to G$, $k \in \{1, 2, ..., \kappa\}$ are symmetric and $k$-additive functions satisfying the two conditions:

i) $\sum_{i=\max}^\kappa C_i^j C_{i-j}^k \sum_{\lambda \in \Phi} A_i(\underbrace{x, x, ..., x}_{k}, a_\lambda, ..., a_\lambda, \lambda y, \lambda y, ..., \lambda y) = 0$, $x, y \in S$, $0 \leq k \leq \kappa - 1$, $0 \leq j \leq \kappa - 1$, $2 \leq \max = \max\{j + 1, k + 1, k + j\} \leq i \leq \kappa$ and

ii) $\sum_{\lambda \in \Phi} \sum_{i=1}^\kappa A_i^\lambda(a_\lambda) = \kappa A_0$.

**Proof.** The necessary condition is obtained by Lemma 4, Theorem 1 and Lemma 3. By Lemma 3 we get the sufficient condition which completes the proof of Theorem. \qed

**Remark 2.** Under the assumptions of Theorem 2, if in addition we assume that

$$\sum_{\lambda \in \Phi} f(\lambda y) = 0, y \in S,$$

then the result (with some modifications on the control of indices $i, j$ and $k$) can be obtained by requiring the assumption ”$G$ is $\kappa$-divisible” instead of ”$G$ is $(\kappa + 1)$-divisible”.

**4. Consequences**

The following corollaries are immediate consequences of Theorem 2. On this occasion, we obtain the following three corollaries 1, 2 and 3 which have been proved by Sinopoulos [22], Stetkær [25], Lukasik [18], Bouikhalene and Elqorachi [4] respectively.

**Corollary 1.** [22][25] Let $\sigma : S \to S$ be an involution of $S$ and $G$ be an abelian group divisible by 2. Then the function $f : S \to G$ is a solution of equation

$$(4.1) \quad f(x + y) + f(x + \sigma(y)) = 2f(x) + f(y) + f(\sigma(y)), \quad x, y \in S$$

if and only if $f$ has the following form

$$(4.2) \quad f(x) = A_1(x) + A_2^\sigma(x), \quad x \in S,$$

where $A_1 : S \to G$ is an arbitrary additive function and $A_2 : S \times S \to G$ is an arbitrary symmetric biadditive function with $A_2(x, y) + A_2(x, \sigma(y)) = 0, x, y \in S$. 


Corollary 2. [18] Let $S$ be an abelian semigroup, $G$ be an abelian group divisible by $\kappa!$, $\Phi$ be a finite automorphism group of $S$ with order $\kappa$. Then the function $f : S \to G$ is a solution of equation

\[ \sum_{\lambda \in \Phi} f(x + \lambda y) = \kappa f(x) + \sum_{\lambda \in \Phi} f(\lambda y), \quad x, y \in S, \]

if and only if $f$ has the following form

\[ f(x) = \sum_{i=1}^{\kappa} A_i^i(x), \quad x \in S, \]

where $A_k : S^k \to G, \; k \in \{1, 2, \ldots, \kappa\}$ are arbitrary symmetric and $k$-additive functions which satisfy the following condition:

\[ \sum_{\lambda \in \Phi} A_i(x, \ldots, x, \underbrace{\lambda y, \lambda y, \ldots, \lambda y}_j) = 0, \quad x, y \in S, \; 1 \leq j \leq i - 1, \; 2 \leq i \leq \kappa. \]

Proof. In this case, with the notations of Theorem 2, as $\{a_\lambda, \; \lambda \in \Phi\} = \{0\}, \; k + j = i$. Furthermore, we can write that

\[ 0 = \sum_{i=\text{max}(k+j,k+1)}^{\kappa} C_i^k C_{i-j}^j \sum_{\lambda \in \Phi} A_i k(x, \ldots, x, a_\lambda, \ldots, a_\lambda, j \lambda y, \ldots, \lambda y) \]

\[ = \sum_{i=k+j}^{\kappa} C_i^k \sum_{\lambda \in \Phi} A_i k(x, \ldots, x, j \lambda y, \ldots, \lambda y) \]

\[ = \sum_{j=1}^{i-1} C_i^j \sum_{\lambda \in \Phi} A_i(x, \ldots, x, j \lambda y, \ldots, \lambda y), \quad x, y \in S, \; 2 \leq i \leq \kappa. \]

For $1 \leq j \leq i - 1, \; 2 \leq i \leq \kappa$, we define the mappings $q(j,i) : S \times S \to G$ by

\[ q(j,i)(x, y) = C_i^j \sum_{\lambda \in \Phi} A_i(x, \ldots, x, j \lambda y, \ldots, \lambda y) x, y \in S. \]

So, we have

\[ q(j,i)(x, ny) = n^j q(j,i)(x, y), \quad x, y \in S, \; n \in \mathbb{N}^*, \; 0 \leq j \leq i - 1, \; 2 \leq i \leq \kappa \]

and

\[ \sum_{j=1}^{\kappa} n^j q(j,i)(x, y) = \sum_{j=1}^{\kappa} q(j,i)(x, ny) = 0, \quad x, y \in S, \; 2 \leq i \leq \kappa. \]

According to Lemma 1 we get the sought result. \(\square\)
Corollary 3. [4] Let $S$ be an abelian group, $G$ be a Banach space and $a \in S$. Then, the general solution $f : S \to G$ of the functional equation

$$f(x + y + a) = f(x) + f(y), \ x, y \in S,$$

(4.5) is

$$f(x) = A_1(a) + A_1(x), x \in S.$$  

(4.6)

where $A_1 : S \to G$ is an arbitrary additive function.

In the following corollaries we prove new others special cases of the equation 1.4 that is, according to our knowledge, not in the literature.

Corollary 4. Let $S$ be an abelian semigroup, $G$ be an abelian group divisible by 2 and $a, b \in S$. Then, the general solution $f : S \to G$ of the functional equation

$$f(x + y + a) + f(x - y + b) = 2f(x) + f(y) + f(-y), \ x, y \in S,$$

(4.7) is

$$f(x) = \frac{1}{2} \left( A_1(a + b) \right) + A_1(x) + A_2^*(x), x \in S.$$  

(4.8)

where $A_1 : S \to G$ is an arbitrary additive function and $A_2 : S \times S \to G$ is an arbitrary symmetric biadditive function with $A_2(x, a) = A_2(x, b) = 0, x \in S$.

Corollary 5. Let $S$ be an abelian semigroup, $\sigma$ be an involution of $S$, $G$ be an abelian group divisible by 2 and $a, b \in S$. Then, the general solution $f : S \to G$ of the functional equation

$$f(x + y + a) + f(x + \sigma(y) + b) = 2f(x) + f(y) + f(\sigma(y)), \ x, y \in S,$$

(4.9) is

$$f(x) = \frac{1}{2} \left( A_1(a + b) \right) + A_1(x) + A_2^*(x), x \in S.$$  

(4.10)

where $A_1 : S \to G$ is an arbitrary additive function and $A_2 : S \times S \to G$ is an arbitrary symmetric biadditive function with

$A_2(x, a) = A_2(x, b) = 0, x \in S$ and $A_2(x, y) + A_2(x, \sigma(y)) = 0, x, y \in S.$
Proof. Keeping in mind the notation of Theorem 2, we apply it where \( \kappa = 2 \). Then there are an element \( A_0 \in G \) and symmetric \( i \)-additives mappings \( A_i \in G^S, \ i \in \{1, 2\} \) satisfy

1. \( f(x) = A_0 + A_1(x) + A_2^*(x), \ x \in S \)

on the other side, they satisfy the following conditions of Theorem 2:

2. i) \( k = 0, j = 1, A_2(a, y) + A_2(b, \sigma(y)) = 0, \ y \in S \),
   
   ii) \( k = 1, j = 0, A_2(y, a) + A_2(y, b) = 0, \ y \in S \),
   
   iii) \( k = 1, j = 1, A_2(x, y) + A_2(x, \sigma(y)) = 0, \ x, y \in S \).

Thus, \( A_2(y, a) = A_2(y, b) = 0, \ y \in S; \ A_2(x, y) + A_2(x, \sigma(y)) = 0, \ x, y \in S \) and \( 2A_0 = A_1(a + b) \). \( \square \)

Corollary 6. Let \( j \) be a primitive cube root of unity and \( a \) be complex number. Then, the general continuous solution \( f : C \to C \) of the functional equation

\[
  f(x+y+j\alpha)+f(x+jy+j^2\alpha)+f(x+j^2y+a) = 3f(x)+f(y)+f(jy)+f(j^2y), \ x, y \in C,
\]

is of the form

\[
  f(x) = \alpha_1 x + \beta_1 \bar{x} + \alpha_2 x^2 + \beta_2 \bar{x}^2,
\]

where \( \alpha_1, \beta_1, \alpha_2, \beta_2 \in C \).

Proof. According the Theorem 2, there are \( \alpha_0 \in C \), and symmetric \( i \)-additive mappings \( A_i : C^i \to C, \ i \in \{1, 2, 3\} \) such that

\[
  f(z) = \alpha_0 + A_1(z) + A_2^*(z) + A_3^*(z), \ z \in C.
\]

Taking into account that \( j \) is a primitive cube root of unity, we have \( 1 + j + j^2 = 0 \). In addition, the continuity of \( f \) show that \( A_1, A_2 \) et \( A_3 \) can be written as the following

\[
  A_1(z) = \alpha_1 z + \beta_1 \bar{z}, \ \alpha_1, \alpha_2 \in C,
\]

\[
  A_2^*(z) = \alpha_2 z^2 + \beta_2 z^2 + \beta_3 |z|^2, \ \alpha_2, \beta_2, \beta_3 \in C,
\]

\[
  A_3^*(z) = \gamma_1 z^3 + \gamma_2 \bar{z}^3, \ \gamma_1, \gamma_2 \in C.
\]

So the conditions of Theorem 2 do not satisfy where \( \gamma_1 = \gamma_2 = \beta_3 = 0 \) which finish the proof. \( \square \)
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