Total edge irregularity strength of disjoint union of double wheel graphs

P. Jeyanthi
Govindammal Aditanar College for Women, India
and
A. Sudha
Wavoo Wajeeha Women’s College of Arts & Science, India
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Abstract

An edge irregular total $k$-labeling $f : V \cup E \rightarrow \{1, 2, 3, \ldots, k\}$ of a graph $G = (V, E)$ is a labeling of vertices and edges of $G$ in such a way that for any two different edges $uv$ and $u'v'$ their weights $f(u) + f(uv) + f(v)$ and $f(u') + f(u'v') + f(v')$ are distinct. The total edge irregularity strength tes($G$) is defined as the minimum $k$ for which the graph $G$ has an edge irregular total $k$-labeling. In this paper, we determine the total edge irregularity strength of disjoint union of $p$ isomorphic double wheel graphs and disjoint union of $p$ consecutive non-isomorphic double wheel graphs.

Keywords: Irregularity strength; total edge irregularity strength; edge irregular total labeling, disjoint union of double wheel graphs.

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1. Introduction

The graphs in this paper are simple, finite and undirected. In [2] Baća et al. defined the notion of edge irregular total $k$-labeling of a graph $G$ as a function $\phi : V \cup E \rightarrow \{1, 2, \ldots, k\}$ such that the edge weights $wt_\phi(uv) = \phi(u) + \phi(uv) + \phi(v)$ are distinct for all the edges. That is $wt_\phi(uv) \neq wt_\phi(u'v')$ for every pair of edges $uv, u'v' \in E$. The minimum $k$ for which the graph $G$ has an edge irregular total $k$-labeling is called the total edge irregularity strength of $G$, $tes(G)$. They found a lower bound for the total edge irregularity strength of a graph as

$$
(1.1) \quad tes(G) \geq \max \left\{ \left\lfloor \frac{|E(G)| + 2}{3} \right\rfloor, \left\lceil \frac{(\Delta(G) + 1)}{2} \right\rceil \right\}
$$

where $\Delta(G)$ is the maximum degree of $G$. Ivančo and Jendrol [3] posed the following conjecture.

**Conjecture 1.1** [3] Let $G$ be an arbitrary graph different from $K_5$. Then

$$
(1.2) \quad tes(G) = \max \left\{ \left\lfloor \frac{|E(G)| + 2}{3} \right\rfloor, \left\lceil \frac{(\Delta(G) + 1)}{2} \right\rceil \right\}
$$

Conjecture (1.1) has been verified by several authors for several families of graphs. Motivated by the results in [1,4,5,6] we determine the total edge irregularity strength of the disjoint union of double wheel graphs. A double wheel graph $DW_n$ of size $n$ can be composed of $2C_n + K_1, n \geq 3$, that is it consists of two cycles of size $n$, where all the vertices of the two cycles are connected to a common hub.

2. Main Results

In this section, first we determine the total edge irregularity strength of the disjoint union of $p$ isomorphic double wheel graphs with the vertex set $V(pDW_n) = \{v_j, v_i^j, u_i^j : 1 \leq i \leq n, 1 \leq j \leq p\}$ and the edge set $E(pDW_n) = \{v_jv_i^j, v_ju_i^j, v_i^{j+1}v_i^j, u_i^ju_i^{j+1} : 1 \leq i \leq n, 1 \leq j \leq p\}$ where the subscript $i$ is taken modulo $n$.

**Lemma 2.1.** $tes(2DW_n) = \left\lfloor \frac{8n+2}{3} \right\rfloor, n \geq 3$. 

**Proof.** Since $|E(2DW_n)| = 8n$, $tes(2DW_n) \geq \left\lfloor \frac{8n+2}{3} \right\rfloor$ by (1.1). Let $k = \left\lfloor \frac{8n+2}{3} \right\rfloor$. To prove the reverse inequality we define $f : V \cup E \rightarrow \{1, 2, 3, \ldots, k\}$ as follows:

$f(v_1) = 1$;
$f(v_2) = n$;
$f(v_i^1) = i, 1 \leq i \leq n$;
$f(v_i^2) = k, 1 \leq i \leq n$;
$f(u_i^1) = n, 1 \leq i \leq n$;
$f(u_i^2) = k, 1 \leq i \leq n$;
$f(v_1v_i^1) = 1, 1 \leq i \leq n$;
$f(v_1u_i^1) = 1 + i, 1 \leq i \leq n$;
$f(v_i^1v_{i+1}^1) = 2n + 1 - i, 1 \leq i \leq n$;
$f(u_i^1u_{i+1}^1) = n + 2 + i, 1 \leq i \leq n$;
$f(v_2v_i^2) = 3n + 2 - k + i, 1 \leq i \leq n$;
$f(v_2u_i^2) = 4n + 2 - k + i, 1 \leq i \leq n$;
$f(v_i^2v_{i+1}^2) = 6n + 2 - 2k + i, 1 \leq i \leq n$;
$f(u_i^2u_{i+1}^2) = 7n + 2 - 2k + i, 1 \leq i \leq n$.

We observe that,

$wt(v_1v_i^1) = 2 + i, 1 \leq i \leq n$;
$wt(v_1u_i^1) = n + 2 + i, 1 \leq i \leq n$;
$wt(v_i^1v_{i+1}^1) = 2n + 2 + i, 1 \leq i \leq n$;
$wt(u_i^1u_{i+1}^1) = 3n + 2 + i, 1 \leq i \leq n$;
$wt(v_2v_i^2) = 4n + 2 + i, 1 \leq i \leq n$;
$wt(v_2u_i^2) = 5n + 2 + i, 1 \leq i \leq n$;
$wt(v_i^2v_{i+1}^2) = 6n + 2 + i, 1 \leq i \leq n$;
$wt(u_i^2u_{i+1}^2) = 7n + 2 + i, 1 \leq i \leq n$. \(\square\)

**Theorem 2.2.** Let $n \geq 3$ and $p \geq 3$ be two integers. Then the total edge irregularity strength of disjoint union of $p$ isomorphic double wheel graphs is $\left\lfloor \frac{4pn+2}{3} \right\rfloor$.

**Proof.** Since $|E(pDW_n)| = 4pm$, by (1.1) we have $tes(pW_n) \geq \left\lfloor \frac{4pn+2}{3} \right\rfloor$.

Let $k = \left\lfloor \frac{4pn+2}{3} \right\rfloor$. To prove the reverse inequality, we define the total edge irregular $k$-labeling $f$ for $1 \leq i \leq n$ and $1 \leq j \leq p$ as follows:

$f(v_j) = f(v_i^j) = f(u_i^j) = \min\{(j - 1)2n + 1, k\}$.

For $1 \leq j \leq p$ and $(j - 1)2n + 1 \leq k$,

$f(v_jv_i^j) = i$;

$f(v_ju_i^j) = n + i$;
\[ f(v_i^j v_{i+1}^j) = 2n + i; \]
\[ f(u_i^j u_{i+1}^j) = 3n + i. \]
For \( 1 \leq j \leq p \) and \( (j - 1)2n + 1 > k \),
\[ f(v_j v_i^j) = 4(j - 1)n + 2 - 2k + i; \]
\[ f(v_j^j u_i^j) = 4(j - 1)n + 2 - 2k + n + i; \]
\[ f(v_i^j v_{i+1}^j) = 4(j - 1)n + 2 - 2k + 2n + i; \]
\[ f(u_i^j u_{i+1}^j) = 4(j - 1)n + 2 - 2k + 3n + i. \]

We observe that,
\[ wt(v_j v_i^j) = 4(j - 1)n + 2 + i, 1 \leq i \leq n, 1 \leq j \leq p; \]
\[ wt(v_j u_i^j) = 4(j - 1)n + 2 + n + i, 1 \leq i \leq n, 1 \leq j \leq p; \]
\[ wt(v_i^j v_{i+1}^j) = 4(j - 1)n + 2 + 2n + i, 1 \leq i \leq n, 1 \leq j \leq p; \]
\[ wt(u_i^j u_{i+1}^j) = 4(j - 1)n + 2 + 3n + i, 1 \leq i \leq n, 1 \leq j \leq p . \]

It can be easily verified that all the vertex and edge labels are at most \( k \) and the weights of the edges are pair-wise distinct. Thus the resulting total labeling is the edge irregular total \( k \)-labeling. Figure 1 illustrates the edge irregular total labeling of the disjoint union of 4 copies of double wheel graphs. □
Total edge irregularity strength of disjoint union of double Wheel graphs.

\[ tes(4DW_6) = 33 \]
Now we determine the total edge irregularity strength of the disjoint union of \( p \) consecutive \( (n_{i+1} = n_i + 1, i \geq 1) \) non-isomorphic double wheel graphs with the vertex set \( V \left( \bigcup_{j=1}^{p} DW_{n_j} \right) \) and the edge set \( E \left( \bigcup_{j=1}^{p} DW_{n_j} \right) \)

where \( V \left( \bigcup_{j=1}^{p} DW_{n_j} \right) = \left\{ v_j, v_i^j, u_i^j : 1 \leq i \leq n_j, 1 \leq j \leq p \right\} \) and \( E \left( \bigcup_{j=1}^{p} DW_{n_j} \right) = \left\{ v_i v_{i+1}^j, v_i u_{i+1}^j, u_i^j u_{i+1}^j : 1 \leq i \leq n_j, 1 \leq j \leq p \right\} \) where the subscript \( i \) is taken modulo \( n \).

**Lemma 2.3.** Let \( n_1 \geq 3 \) be an integer and \( n_2 = n_1 + 1 \). Then \( tes(DW_{n_1} \cup DW_{n_2}) = \left\lfloor \frac{8n_1 + 6}{3} \right\rfloor \).

**Proof.** Let \( k = \left\lfloor \frac{8n_1 + 6}{3} \right\rfloor \). Then by (1.1), \( tes(DW_{n_1} \cup W_n) \geq k \). Now to prove the reverse inequality, we define an edge irregular \( k \)-labeling of \( f \) as follows:

\[
\begin{align*}
  f(v_1) &= 1; \\
  f(v_i^1) &= i, 1 \leq i \leq n_1; \\
  f(v_i^2) &= f(u_i^2) = k, 1 \leq i \leq n_2; \\
  f(u_i^1) &= n_1, 1 \leq i \leq n_1; \\
  f(v_2) &= n_1 + 1; \\
  f(v_1 v_i^1) &= 1, 1 \leq i \leq n_1; \\
  f(v_1 u_i^1) &= 1 + i, 1 \leq i \leq n_1; \\
  f(v_2 v_i^1) &= 3n_1 + 1 - k + i, 1 \leq i \leq n_2; \\
  f(v_i^1 u_{i+1}) &= 2n_1 + 1 - i, 1 \leq i \leq n_1; \\
  f(u_i^1 u_{i+1}) &= n_1 + 2 + i, 1 \leq i \leq n_1; \\
  f(v_2 u_i^1) &= 4n_1 + 2 - k + i, 1 \leq i \leq n_2; \\
  f(v_i^2 u_{i+1}) &= 6n_1 + 4 - 2k + i, 1 \leq i \leq n_2; \\
  f(u_i^2 u_{i+1}) &= 7n_1 + 5 - 2k + i, 1 \leq i \leq n_2.
\end{align*}
\]

We observe that,

for \( 1 \leq i \leq n_j, j = 1, 2 \)

\[
\begin{align*}
  wt(v_1 v_i^1) &= 2 + i; \\
  wt(v_1 u_i^1) &= n_1 + 2 + i; \\
  wt(v_1 v_i^1 u_{i+1}) &= 2n_1 + 2 + i; \\
  wt(u_i^1 u_{i+1}) &= 3n_1 + 2 + i; \\
  wt(v_2 v_i^1) &= 4n_1 + 2 + i; \\
  wt(v_2 u_i^1) &= 5n_1 + 3 + i; \\
  wt(v_i^2 v_i^2) &= 6n_1 + 4 + i;
\end{align*}
\]
Lemma 2.4. Let $n_1, n_2, n_3$ be integers and $n_1 \geq 3$. Then \( \text{tes}(DW_{n_1} \cup DW_{n_2} \cup DW_{n_3}) = \left\lfloor \frac{2(6n_1+7)}{3} \right\rfloor \).

Proof. Since \( |E(DW_{n_1} \cup DW_{n_2} \cup DW_{n_3})| = 12(n_1 + 1) \) by (1.1), \( \text{tes}(DW_{n_1} \cup DW_{n_2} \cup DW_{n_3}) \geq \left\lfloor \frac{12(n_1+1)+2}{3} \right\rfloor = \left\lfloor \frac{2(6n_1+7)}{3} \right\rfloor \). Let \( k = \left\lceil \frac{2(6n_1+7)}{3} \right\rceil \).

Now to prove the reverse inequality, we define an edge irregular \( k \)-labeling \( f \) as follows:

- \( f(v_1^1) = 1 \);
- \( f(u_1^1) = f(u_1^2) = 1, 1 \leq i \leq n_1 \);
- \( f(v_2) = f(v_2^2) = f(u_2^2) = 2n + 1, 1 \leq i \leq n_2 \);
- \( f(v_3) = 3n_1 + 3 \);
- \( f(u_3^3) = f(v_3^3) = k, 1 \leq i \leq n_3 \);
- \( f(v_1v_1^1) = i, 1 \leq i \leq n_1 \);
- \( f(v_1u_1^1) = n_1 + i, 1 \leq i \leq n_1 \);
- \( f(v_2v_2^2) = i, 1 \leq i \leq n_2 \);
- \( f(v_2u_2^2) = n_1 + 1 + i, 1 \leq i \leq n_2 \);
- \( f(v_3v_3^3) = 5n_1 + 3 - k + i, 1 \leq i \leq n_3 \);
- \( f(v_3u_3^3) = 6n_1 + 5 - k + i, 1 \leq i \leq n_3 \);
- \( f(v_1v_1^1) = 2n_1 + i, 1 \leq i \leq n_1 \);
- \( f(u_1v_1^1) = 3n_1 + i, 1 \leq i \leq n_1 \);
- \( f(v_2v_2^2) = 2n_1 + 2 + i, 1 \leq i \leq n_2 \);
- \( f(u_2v_2^2) = 3n_1 + 3 + i, 1 \leq i \leq n_2 \);
- \( f(v_3v_3^3) = 10n_1 + 10 - 2k + i, 1 \leq i \leq n_3 \);
- \( f(u_3v_3^3) = 11n_1 + 12 - 2k + i, 1 \leq i \leq n_3 \).

We observe that, for \( 1 \leq i \leq n_j, j = 1, 2, 3 \)

- \( wt(v_1v_1^1) = 2 + i \);
- \( wt(v_1u_1^1) = n_1 + 2 + i \);
- \( wt(v_2v_2^2) = 4n_1 + 2 + i \);
- \( wt(v_2u_2^2) = 5n_1 + 3 + i \);
- \( wt(v_3v_3^3) = 8n_1 + 6 + i \).
\[ wt(v_3u_3^3) = 9n_1 + 8 + i; \]
\[ wt(v_1^1v_1^1) = 2n_1 + 2 + i; \]
\[ wt(u_1^1u_1^1) = 3n_1 + 2 + i; \]
\[ wt(v_2^2v_2^2) = 6n_1 + 4 + i; \]
\[ wt(u_2^2u_2^2) = 7n_1 + 5 + i; \]
\[ wt(v_3^3v_3^3) = 10n_1 + 10 + i; \]
\[ wt(u_3^3u_3^3) = 11n_1 + 12 + i. \]

It can be easily verified that all the vertex and edge labels are at most \( k \) and the weights of the edges are pair-wise distinct. Thus the resulting total labeling is the edge irregular total \( k \)-labeling. \( \square \)

**Theorem 2.5.** The total edge irregularity strength of the disjoint union of \( p \) (\( p \geq 4 \)) consecutive non-isomorphic double wheel graphs is \( \left\lceil \frac{2p(2n_1+p-1)+2}{3} \right\rceil \).

**Proof.** Since \( |E\left( \bigcup_{j=1}^{p} DW_{n_j} \right) | = 2p(2n_1+p-1) \) by (1.1) tesi \( \bigcup_{j=1}^{p} DW_{n_j} \geq \left\lceil \frac{2p(2n_1+p-1)+2}{3} \right\rceil \). To prove the reverse inequality we define the total edge irregular \( k \)-labeling \( f \) for \( 1 \leq i \leq n_j \) and \( 1 \leq j \leq p \) as follows:
\[ f(v_1) = f(v_1^1) = f(u_1^1) = 1, 1 \leq i \leq n_1; \]
\[ f(v_1v_1^1) = i, 1 \leq i \leq n_1; \]
\[ f(v_1^1u_1^1) = n_1 + i, 1 \leq i \leq n_1; \]
\[ f(v_1^1v_1^1) = 2n_1 + i, 1 \leq i \leq n_1; \]
\[ f(u_1^1u_1^1) = 3n_1 + i, 1 \leq i \leq n_1; \]
\[ f(v_j) = f(v_j^i) = f(u_j^i) = \min\left\{ 2\left( \sum_{s=1}^{j-1} n_s \right) + 1, k \right\}, \ 1 \leq i \leq n_j \text{ and } 2 \leq j \leq p. \]

For \( 1 \leq i \leq n_j \) and \( 2 \leq i \leq p, 2 \left[ \sum_{s=1}^{j-1} n_s \right] + 1 \leq k, \)
\[ f(v_j^i) = i; \]
\[ f(v_ju_j^i) = n_j + i; \]
\[ f(v_j^i) = 2n_j + i; \]
\[ f(u_j^i) = 3n_j + i. \]

We observe that,
\[ wt(v_jv_j^i) = 2 \left[ 2 \left( \sum_{s=1}^{j-1} n_s \right) + 1 \right] + i; \]
\[ wt(v_ju_j^i) = 2 \left[ 2 \left( \sum_{s=1}^{j-1} n_s \right) + 1 \right] + n_j + i; \]
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\[ wt(v^i_jv^i_{j+1}) = 2 \left[ 2 \left( \sum_{s=1}^{j-1} n_s \right) + 1 \right] + 2n_j + i; \]

\[ wt(u^i_ju^i_{j+1}) = 2 \left[ 2 \left( \sum_{s=1}^{j-1} n_s \right) + 1 \right] + 3n_j + i. \]

For \( 1 \leq i \leq n_j \) and \( 2 \leq i \leq p \), \( 2 \left( \sum_{s=1}^{j-1} n_s \right) + 1 > k, \)

\[ f(v_jv^j_i) = 4n_{j-1} + 2 \left[ 2 \left( \sum_{s=1}^{j-2} n_s \right) + 1 \right] - 2k + i; \]

\[ f(v_ju^j_i) = 4n_{j-i} + 2 \left[ 2 \left( \sum_{s=1}^{j-2} n_s \right) + 1 \right] + n_j - 2k + i; \]

\[ f(v^{j}_i v^{j}_{i+1}) = 4n_{j-i} + 2 \left[ 2 \left( \sum_{s=1}^{j-2} n_s \right) + 1 \right] + 2n_j - 2k + i; \]

\[ f(u^j_i u^{j}_{i+1}) = 4n_{j-i} + 2 \left[ 2 \left( \sum_{s=1}^{j-2} n_s \right) + 1 \right] + 3n_j - 2k + i. \]

We observe that, \( wt(v^j_i v^j_{i+1}) = 4n_{j-1} + 2 \left[ 2 \left( \sum_{s=1}^{j-2} n_s \right) + 1 \right] + i; \)

\[ wt(v_ju^j_i) = 4n_{j-i} + 2 \left[ 2 \left( \sum_{s=1}^{j-2} n_s \right) + 1 \right] + n_j + i; \]

\[ wt(v^j_i v^{j}_{i+1}) = 4n_{j-i} + 2 \left[ 2 \left( \sum_{s=1}^{j-2} n_s \right) + 1 \right] + 2n_j + i; \]

\[ wt(u^j_i u^{j}_{i+1}) = 4n_{j-i} + 2 \left[ 2 \left( \sum_{s=1}^{j-2} n_s \right) + 1 \right] + 3n_j + i. \]

It can be easily verified that all the vertex and edge labels are at most \( k \) and the weights of the edges are pair-wise distinct. Thus the resulting total labeling is the edge irregular \( k \)-labeling. Figure 2 illustrates the edge irregular total labelings of the disjoint union of 4 Consecutive non-isomorphic double wheel graphs \( DW_3 \cup DW_4 \cup DW_5 \cup DW_6 \). \( \square \)
Figure 2

Total edge irregularity strength of disjoint union of 4 non-isomorphic double Wheel graphs.

$$tes(DW_3 \cup DW_4 \cup DW_5 \cup DW_6) = 25$$
3. Conclusion

In this paper we determine the total edge irregularity strength of the disjoint union of \( p \) isomorphic double wheel graphs and disjoint union of \( p \) consecutive non-isomorphic double wheel graphs. We conclude this paper by stating the following open problem.

**Open problem:**
For \( m \geq 2 \), find the exact value of the total edge irregularity strength of a disjoint union of \( m \) arbitrary double wheel graphs.

References


**P. Jeyanthi**
Research Centre,
Department of Mathematics
Govindammal Aditanar College for Women
Tiruchendur-628 215, Tamil Nadu,
India
e-mail: jeyajeyanthi@rediffmail.com
and

A. Sudha
Department of Mathematics
Wavoo Wajeeha Women’s College of Arts & Science,
Kayalpatnam - 628 204, Tamil Nadu, India
e-mail: sudhathanalakshmi@gmail.com