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Nonlinear Bessel potentials and generalizations of the Kato class

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Abstract

We study the scale of function spaces M_p introduced by Zamboni. For these spaces, we get a characterization in terms of nonlinear Bessel potentials. This result is based on a known characterization of the Kato class $K_{n,s}$ of order s in terms of Bessel potentials and the space of bounded uniformly continuous functions.

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1. Introduction

The Kato class K_n was introduced and studied by Aizenman and Simon (see [7] and [2]). For $n \geq 3$, it consists of locally integrable functions f on $\mathbf{R}^n$ such that

$$\lim_{r\to 0} \sup_{x\in\mathbf{R^n}} \int_{B(x,r)} \frac{|f(y)|}{|x-y|^{n-2}} dy=0.$$

For $1 < p < n$, the following classes were defined by Zamboni (see [9]): the class $\tilde{M}_p$ of functions f such that

$$\sup_{x\in\mathbf{R}^n}\left\{\int_{B(x,r)} \frac{1}{|x-y|^{n-1}}\left(\int_{B(x,r)} \frac{|f(z)|}{|y-z|^{n-1}}dz\right)^{\frac{1}{p-1}} dy\right\}^{p-1} < \infty,$$

and the class M_p of functions f such that $f \in \tilde{M}_p$ and

$$\lim_{r\to 0} \sup_{x\in\mathbf{R}^n}\left\{\int_{B(x,r)} \frac{1}{|x-y|^{n-1}}\left(\int_{B(x,r)} \frac{|f(z)|}{|y-z|^{n-1}}dz\right)^{\frac{1}{p-1}} dy\right\}^{p-1} = 0.$$

In [3], Davies and Hinz introduced the scale $K_{n,s}$ of the Kato classes of order $s > 0$. It was shown by Gulisashvili (see [4] Theorem 1) that for a locally integrable function f the following conditions are equivalent:

(a) $f \in K_{n,s}$ for $s > 0$;

(b) $J^{-s}|f| \in L_\infty$ and $\lim_{\alpha\to 0^+} \alpha^s \left\|J^{-s}(|f|)_\alpha\right\|_\infty = 0$;

(c) $J^{-s}|f| \in BUC$.

In (a) and (c), the symbol J^{-s} stands for the Bessel potential of order s, BUC denotes the space of bounded uniformly continuous functions on $\mathbf{R}^n$, and $|f|_\alpha(x) = |f(\alpha x)|, x \in \mathbf{R}^n, \alpha > 0$. Previously, this result was obtained for the Kato class K_n and the Kato class of measures $\tilde{K}_n$, in [6] and [5], respectively.

In the present paper, we generalize the theorem formulated above for the classes $\tilde{M}_p$ and M_p, using the nonlinear Bessel potentials (see Theorems 1 and 2 below)

2. Definitions and Notation

In this section, we gather definitions and notation that will be used throughout the paper. We also include several simple lemmas. By $L^1_{loc}(\mathbf{R}^n)$ we will denote the space of functions which are locally integrable on $\mathbf{R}^n$, and by $L^1_{loc,u}$ the space of functions f such that

$$\sup_{x\in\mathbf{R}^n} \int_{B(x,1)} |f(y)|\, dy < \infty.$$

DEFINITION 1. *Let $f \in L^1_{loc}(\mathbf{R}^n)$. For any $1 < p < n$ and $r > 0$, we set*

$\Phi(r) = \sup_{x\in\mathbf{R}^n}\left(\int_{B(x,r)} \frac{1}{|x-y|^{n-1}}\left(\int_{B(x,r)} \frac{|f(z)|dz}{|z-y|^{n-1}}\right)^{\frac{1}{p-1}} dy\right)^{p-1},$

where $B(x,r) = \{y : |x-y| < r\}$.

We say that f *belongs to the space* $\tilde{M}_p(\mathbf{R}^n)$, *if* $\Phi(r) < \infty$ *for all* $r > 0$.

DEFINITION 2. *We say that a function* $f \in M_p(\mathbf{R}^n)$ *if*
$\lim_{r\to 0}\Phi(r) = 0$.

We are now ready to formulate some simple properties of the classes M_p and $\tilde{M}_p$.

LEMMA 1. *(See [9], p. 151) For* $1 < p < n$, *we have*
(i) $M_p(\mathbf{R}^n) \subset \tilde{M}_p(\mathbf{R}^n)$, *and*
(ii) $M_2(\mathbf{R}^n) = K_n$.

From Lemma 1 we conclude that both $M_p(\mathbf{R}^n)$ and $\tilde{M}_p(\mathbf{R}^n)$ are generalizations of K_n.

REMARK 1. *The following example shows that* K_n *is properly contained in* $M_p(\mathbf{R}^n)$ *for* $p > 2$. *It is known that the function* $f(x) = |x|^{-2}$ *is not in the Kato class* K_n. *However,* $f \in M_p$. *Indeed,*

$$(2.1)\lim_{r\to 0}\sup_x\left\{\int_{B(x,r)} \frac{1}{|x-y|^{n-2}}\left(\int_{B(x,r)} \frac{dz}{|z|^2\,|z-y|^{n-1}}\right)^{\frac{1}{p-1}} dy\right\}^{p-1} = 0.$$

This can be shown by splitting the domain of integration in the interior integral into the following three parts $B(x,r)\bigcap\{|z| < \frac{1}{2}|y|\}$, $B(x,r)\bigcap\{\frac{1}{2}|y| \le |z| \le \frac{3}{2}|y|\}$ *and* $B(x,r)\bigcap\{|z| > \frac{3}{2}|y|\}$. *After routine calculations we see that*

$\int_{B(x,r)} \frac{dz}{|z|^2|z-y|^{n-1}}$

is majorized by $C\,|y|^{-1}$. *Finally we have*

$C\sup_x\left\{\int_{B(x,r)} \frac{dy}{|y|^{\frac{1}{p-1}}|x-y|^{n-1}}\right\}^{p-1} \to 0$ *as* $r \to 0$,

this shows that (2.1) holds. Thus, $f \in \bigcap_{p>2} M_p$.

REMARK 2. *(i) For* $0 < r < 1$, *it is not hard to check that for* $1 < p \le 2$ *the expression*

$$\|f\|_{\tilde{M}_p(\mathbf{R}^n)} = \sup_{x\in\mathbf{R}^n}\left(\int_{B(x,1)} \frac{1}{|x-y|^{n-1}}\left(\int_{B(x,1)} \frac{|f(z)|}{|z-y|^{n-1}}dz\right)^{\frac{1}{p-1}}dy\right)^{p-1}$$

(2.2)

defines a norm on $\tilde{M}_p(\mathbf{R}^n)$.
(ii) For $p > 2$, *the expression (2.2) satisfies the following inequality.*

$$\|f+g\|_{\tilde{M}_p(\mathbf{R}^n)} \le 2^{p-2}\Big(\|f\|_{\tilde{M}_p(\mathbf{R}^n)} + \|g\|_{\tilde{M}_p(\mathbf{R}^n)}\Big), \tag{2.3}$$

for all f *and* g *in* $\tilde{M}_p(\mathbf{R}^n)$. *If* U *is a neighborhood of* 0, *from (2.3) we have* $2^{p-1}U + 2^{p-1}U \subset U$,
then $\tilde{M}_p(\mathbf{R}^n\ )$ *is a topological vector space.*

LEMMA 2. $\tilde{M}_p(\mathbf{R}^n) \subset L^1_{loc,u}(\mathbf{R}^n)$ *for* $1 < p < n$.

Proof. Let $f \in \tilde{M}_p(\mathbf{R}^n)$, and fix $r_0 > 0$. Then there exists a positive constant C such that $\Phi(r_0) \le C$. It follows that

$$\sup_{x\in\mathbf{R}^n}\left(\int_{B(x,r_0)} \frac{1}{|x-y|^{n-1}}\left(\int_{B(x,r_0)} \frac{f(z)}{|x-y|^{n-1}}dz\right)^{\frac{1}{p-1}}dy\right)^{p-1}$$
$$\ge \sup_{x\in\mathbf{R}^n}\left(\int_{B(x,r_0)} \frac{dy}{r_0^{n-1}}\left(\int_{B(x,r_0)} \frac{f(z)}{(2r_0)^{n-1}}dz\right)^{\frac{1}{p-1}}\right)^{p-1}$$
$$\ge \sup_{x\in\mathbf{R}^n}\left(\frac{1}{2r_0}\right)^{n-1}\left(\frac{m(B(x,r_0))}{r_0^{n-1}}\right)^{p-1}\int_{B(x,r_0)} f(z)\,dz.$$

Therefore

$$\sup_{x\in\mathbf{R}^n}\int_{B(x,r_0)} f(z)\,dz < BC,$$

where

B= $(2\mathrm{r}_0)^{n-1}(r_0 m(B(0,1)))^{p-1}$.

Finally, let $B(x,1) \subseteq \bigcup_{k=1}^n B(x_k, r_0)$, then

$$\sup_{x\in\mathbf{R}^n}\int_{B(x,1)} f(z)\,dz \le \sum_{k=1}^n \sup_{x\in\mathbf{R}^n}\int_{B(x_k,r_0)} f(z)\,dz,$$

so

$\sup_{x\in\mathbf{R}^n}\int_{B(x,1)} f(z)\,dz < \infty$

therefore

$\tilde{M}_p(\mathbf{R}^n) \subset L^1_{loc,u}(\mathbf{R}^n)$. $\square$

LEMMA 3. *For* $1 < p < n$, $\tilde{M}_p(\mathcal{R}^n)$ *is a complete space.*

Proof. Let $\{f_n\}_{n\in\mathcal{N}}$ be a Cauchy sequence in $\overline{B}(0,r) = \{f \in \tilde{M}_p(\mathcal{R}^n) : f_{\tilde{M}_p(\mathcal{R}^n)} \leq r\}$.
By Lemma 2, $\{f_n\}_{n\in\mathcal{N}}$ is a Cauchy sequence in $L^1_{loc,u}(\mathcal{R}^n)$. Since this space is complete, there exists a function $f \in L^1_{loc,u}(\mathcal{R}^n)$ such that $f_n \to f$ in $L^1_{loc,u}(\mathcal{R}^n)$.

By Fatous's Lemma, we have $f_{\tilde{M}_p(\mathcal{R}^n)} \leq \liminf f_{n\,\tilde{M}_p(\mathcal{R}^n)} \leq r$.
Thus $f \in \overline{B}(0,r)$, which means that $\overline{B}(0,r)$ is complete with respect to the topology generated by $L^1_{loc,u}(\mathcal{R}^n)$ - norm. By Corollary 2 of Proposition 9 in [4, Chapter III $\S\, 3, no.5$] we obtain the assertion. $\square$

LEMMA 4. *If* $1 < p < n$, *then* $M_p(\mathbf{R}^n)$ *is closed in* $\tilde{M}_p(\mathbf{R}^n)$.

Proof. Let us define the map $\varphi : \tilde{M}_p(\mathbf{R}^n) \to [0,\infty)$ by $\varphi(f) = \lim_{r\to 0}\phi_f(r)$ (see definition 1). It is not hard to prove that the family $\{\varphi_r\}_{r>0}$ where $\varphi_r(f) = \phi_f(r)$ is equicontinuous and $\varphi_r \to \varphi$ pointwise as $r \to 0$. Since $M_p(\mathbf{R}^n) = \varphi^{-1}(0)$. We obtain the result. $\square$

Nonlinear Bessel Potentials

In this section, we gather some well-known results concerning Riesz and Bessel potentials (see, e.g., [8]). *Let*

$$\mathrm{G}_\alpha(x) = \frac{1}{(2\pi)^{\frac{n}{2}}2^{\frac{\alpha}{2}}\Gamma(\frac{\alpha}{2})}\int_0^\infty t^{\frac{\alpha-n}{2}} e^{-\frac{|x|^2}{2t}-\frac{t}{2}}\frac{dt}{t},$$

denote the Bessel kernel of order $\alpha > 0$. For more information on the Bessel kernel, we refer the reader to [8], Chapter 5.

DEFINITION 3. *For any* $f \in L^1_{loc}(\mathbf{R}^n)$, *and* $\alpha > 0$, *the function*

$$G_\alpha * (G_\alpha * f)^{\frac{1}{p-1}}$$

is called the nonlinear Bessel potential of f*, (see [1], p. 21).*

The symbol I_α will stand for the Riesz potential kernel which is defined as follows:

$$I_\alpha(x) = \frac{\gamma_\alpha}{|x|^{n-\alpha}}, \tag{2.4}$$

where γ_α is a certain constant (see [8], section V.1). It is Known that

$$I_\alpha(x) = \frac{1}{(2\pi)^{\frac{n}{2}} 2^{\frac{\alpha}{2}} \Gamma(\frac{\alpha}{2})} \int_0^\infty t^{\frac{\alpha-n}{2}} e^{-\frac{\pi|x|^2}{2t}} \frac{dt}{t}, \tag{2.5}$$

where $0 < \alpha < n$. We have from (2.3) and (2.4) that

$$0 < G_\alpha(x) < I_\alpha(x) \text{ for } 0 < \alpha < n. \tag{2.6}$$

It is known that the local behavior of the Bessel potential kernel and the corresponding Riesz potential kernel is the same for $0 < \alpha \leq n$. It is also known that the Bessel potential kernels decay exponentially at infinity. More exactly, the following estimates holds: if $0 < \alpha < n$, then there exist $C_\alpha > 0$ and $\tilde{C}_\alpha > 0$ such that

$$\tilde{C}_\alpha \, |x|^{\alpha-n} \leq G_\alpha(x) \leq C_\alpha \, |x|^{\alpha-n}, \tag{2.7}$$

for all x with $0 < |x| < 1$. On the other hand, for every $\alpha > 0$ we have

$$G_\alpha(x) \leq C_\alpha e^{-c|x|}, \tag{2.8}$$

for all $x \in \mathbf{R}^n$ with $|x| > 1$. We have from (2.7) and (2.8) that for all x with $0 < |x| < \infty$,

$$G_\alpha(x) \leq C_\alpha \Big(\frac{\chi_{B(0,1)}(x)}{|x|^{n-\alpha}} + e^{-c|x|} \Big). \tag{2.9}$$

Main Results

In this section we will give a characterization of the classes $\tilde{M}_p(\mathbf{R}^n)$ and $M_p(\mathbf{R}^n)$ in terms of nonlinear Bessel potentials.

REMARK 3. *It is not hard to prove that the following are equivalent*

(a) $f \in M_p(\mathbf{R}^n)$,

(b) $\lim_{r\to 0^+} \sup_{x\in\mathbf{R}^n} \int_{|y-x|\leq r} \frac{1}{|x-y|^{n-1}} \left(\int_{|y-x|\leq 1} \frac{|f(z)|}{|y-z|^{n-1}} dz \right)^{\frac{1}{p-1}} dy = 0,$

(c) $\lim_{r\to 0^+} \sup_{x\in\mathbf{R}^n} \int_{|y-x|\leq r} \frac{1}{|x-y|^{n-1}} \left(\int_{\mathbf{R}^n} G_1(y-z)|f(z)|dz \right)^{\frac{1}{p-1}} = 0.$

THEOREM 1. *Let* $f \in L^1_{loc}(\mathbf{R}^n)$, *and* $1 < p < n$. *Then* $f \in \tilde{M}_p(\mathbf{R}^n)$ *if and only if* $\sup_{x\in\mathbf{R}^n}\{G_1 * (G_1 * |f|)^{\frac{1}{p-1}}\} < \infty$.

Proof. Let $f \in \tilde{M}_p(\mathbf{R}^n)$, $G_{in} = \chi_{B(0,1)}G_1$ and $G_{out} = \chi_{\mathbf{R}^n|B(0,1)}G_1$. Since $G_1 = G_{in} + G_{out}$ and using (2.9), we have $\sup_{x\in\mathbf{R}^n}\{G_1 * (G_1 * |f|)^{\frac{1}{p-1}}(x)\}$

$$
\begin{aligned}
&= \sup_{x\in\mathbf{R}^n}\{(G_{in}+G_{out}) * (G_1 * |f|)^{\frac{1}{p-1}}(x)\}\\
&\leq \sup_{x\in\mathbf{R}^n}\{G_{in} * (G_1 * |f|)^{\frac{1}{p-1}}(x)\}\\
&+ \sup_{x\in\mathbf{R}^n}\{G_{out} * (G_1 * |f|)^{\frac{1}{p-1}}(x)\}\\
&= \sup_{x\in\mathbf{R}^n}\{G_{in} * [(G_{in}+G_{out}) * |f|]^{\frac{1}{p-1}}(x)\}\\
&+ \sup_{x\in\mathbf{R}^n}\{G_{out} * (G_1 * |f|)^{\frac{1}{p-1}}(x)\}\\
&\leq \sup_{x\in\mathbf{R}^n}\{G_{in} * (G_{in} * |f|)^{\frac{1}{p-1}}(x)\}\\
&+ \sup_{x\in\mathbf{R}^n}\{G_{out} * (G_1 * |f|)^{\frac{1}{p-1}}(x)\}\\
&= \sup_{x\in\mathbf{R}^n}\int_{\mathbf{R}^n} G_{in}(x-y)\left(\int_{\mathbf{R}^n} G_{in}(y-z)|f(z)|dz\right)^{\frac{1}{p-1}} dy\\
&+ \sup_{x\in\mathbf{R}^n}\int_{\mathbf{R}^n} G_{in}(x-y)\left(\int_{\mathbf{R}^n} G_{out}(y-z)|f(z)|dz\right)^{\frac{1}{p-1}} dy\\
&+ \sup_{x\in\mathbf{R}^n}\int_{\mathbf{R}^n} G_{out}(x-y)\left(\int_{\mathbf{R}^n} G_1(y-z)|f(z)|dz\right)^{\frac{1}{p-1}} dy\\
&= \sup_{x\in\mathbf{R}^n}\int_{\mathbf{R}^n} \chi_{B(0,1)}G_1(x-y)\left(\int_{\mathbf{R}^n} \chi_{B(0,1)}G_1(x-y)|f(z)|dz\right)^{\frac{1}{p-1}} dy\\
&+\sup_{x\in\mathbf{R}^n}\int_{\mathbf{R}^n} \chi_{B(0,1)}G_1(x-y)\left(\int_{\mathbf{R}^n} \chi_{\mathbf{R}^n|B(0,1)}(y-z)G_1(y-z)|f(z)|dz\right)^{\frac{1}{p-1}} dy\\
&+ \sup_{x\in\mathbf{R}^n}\int_{\mathbf{R}^n} \chi_{\mathbf{R}^n|B(0,1)}(x-y)G_1(x-y)\left(\int_{\mathbf{R}^n} G_1(y-z)|f(z)|dz\right)^{\frac{1}{p-1}} dy\\
&= \sup_{x\in\mathbf{R}^n}\int_{B(x,1)} G_1(x-y)\left(\int_{B(y,1)} G_1(y-z)|f(z)|dz\right)^{\frac{1}{p-1}} dy\\
&+ \sup_{x\in\mathbf{R}^n}\int_{B(x,1)} G_1(x-y)\left(\int_{\mathbf{R}^n|B(y,1)} G_1(y-z)|f(z)|dz\right)^{\frac{1}{p-1}} dy\\
&+ \sup_{x\in\mathbf{R}^n}\int_{\mathbf{R}^n|B(x,1)} G_1(x-y)\left(\int_{\mathbf{R}^n} G_1(y-z)|f(z)|dz\right)^{\frac{1}{p-1}} dy
\end{aligned}
$$

by (2.9) we have

$$
\begin{aligned}
&\sup_{x\in\mathbf{R}^n}\{G_1 * (G_1 * |f|)^{\frac{1}{p-1}}(x)\}\\
&\leq \sup_{x\in\mathbf{R}^n}\int_{B(x,1)} \frac{1}{|x-y|^{n-1}}\left(\int_{B(y,1)} \frac{|f(z)|}{|y-z|^{n-1}}dz\right)^{\frac{1}{p-1}}\\
&+ \sup_{x\in\mathbf{R}^n}\int_{B(x,1)} \frac{1}{|x-y|^{n-1}}\left(\int_{\mathbf{R}^n|B(y,1)} e^{-|y-z|}|f(z)|dz\right)^{\frac{1}{p-1}} dy\\
&+ \sup_{x\in\mathbf{R}^n}\int_{\mathbf{R}^n|B(x,1)} e^{-|y-z|}\left(\int_{\mathbf{R}^n} G_1(y-z)|f(z)|dz\right)^{\frac{1}{p-1}} dy\\
&\leq \sup_{x\in\mathbf{R}^n}\int_{B(x,1)} \frac{1}{|x-y|^{n-1}}\left(\int_{\mathbf{R}^n|B(y,1)} \frac{|f(z)|}{|y-z|^{n-1}}dz\right)^{\frac{1}{p-1}} dy
\end{aligned}
$$

$+ e^{-1} \sup_{x\in\mathbf{R}^n} \int_{B(x,1)} \frac{1}{|x-y|^{n-1}} \left(\int_{\mathbf{R}^n|B(y,1)} |f(z)|dz\right)^{\frac{1}{p-1}} dy$
$+ \sup_{x\in\mathbf{R}^n} \int_{\mathbf{R}^n|B(x,1)} e^{-|x-y|} \left(\int_{\mathbf{R}^n} |f(z)|dz\right)^{\frac{1}{p-1}} dy$
$< \infty.$

To prove the sufficiency in Theorem 1, let us assume that $\sup_{x\in\mathbf{R}^n} G_1 * (G_1 * |f|)^{\frac{1}{p-1}}(x) < \infty$ and $0 < r < 1$. Then (2.7) gives

$$\mathrm{C}_p \sup_{x\in\mathbf{R}^n} \int_{\mathbf{R}^n} \frac{\chi_{B(0,r)}(x-y)}{|x-y|^{n-1}} \left(\int_{\mathbf{R}^n} \frac{\chi_{B(0,r)}(y-z)\, f(z)}{|y-z|^{n-1}} dz\right)^{\frac{1}{p-1}} dy$$

$$\leq \sup_{x\in\mathbf{R}^n} \int_{\mathbf{R}^n} G_1(x-y)\left(\int_{\mathbf{R}^n} G_1(y-z)\ f(z)\ dz\right)^{\frac{1}{p-1}} dy < \infty,$$

therefore, $f \in \tilde{M}_p(\mathbf{R}^n)$.

This completes the proof of Theorem 1.

□

THEOREM 2. *For* $1 < p < n$, *then* $f \in M_p(\mathbf{R}^n)$ *if and only if* $G_1 * (G_1 * |f|)^{\frac{1}{p-1}} \in BUC$.

Proof. Let $f \in M_p(\mathbf{R}^n)$, and φ be any function in $C_c(\mathbf{R}^n)$ such that

$$\varphi(x) = \begin{cases} 1 & \text{if } |x| \leq 1/2, \\ 0 & \text{if } |x| \geq 1 \end{cases} \tag{2.10}$$

with $0 \leq \varphi(x) \leq 1$ for all $x \in \mathbf{R}^n$ and $spt\, \varphi \subseteq B(0,1)$.

Let us define $G_{in,\alpha} = \varphi(\frac{1}{\alpha}\cdot)G_1$ and $G_{out,\alpha} = (1-\varphi(\frac{1}{\alpha}\cdot))G_1$. Observe that $G_{out,\alpha}$ is a continuous function. We claim that $G_{out,\alpha} * (G_1 * |f|)^{\frac{1}{p-1}} \in BUC$ for $1 < p < n$, to prove this let us consider

$$\sup_{x\in\mathbf{R}^n} \left|G_{out,\alpha} * (G_1 * |f|)^{\frac{1}{p-1}}(x+h) - G_{out,\alpha} * (G_1 * |f|)^{\frac{1}{p-1}}(x)\right|$$
$$= \sup_{x\in\mathbf{R}^n} \left|\int_{\mathbf{R}^n} [G_{out,\alpha}(x+h-y) - G_{out,\alpha}(x-y)]\, (G_1 * |f|)^{\frac{1}{p-1}}\,(y)dy\right| = I$$

by Lemma 2 we obtain

$$\mathrm{I} \leq \sup_{x\in\mathbf{R}^n} \left|\int_{\mathbf{R}^n} (G_{out,\alpha}(x+h-y) - G_{out,\alpha}(x-y))\right|$$

$$\left(\sup_{x\in\mathbf{R}^n} \int_{B(x,1)} |f(z)|\right)^{\frac{1}{p-1}}$$

$$\leq \left[\sup_{x\in\mathbf{R}^n} \int_{\mathbf{R}^n} |G_1(x+h) - G_1(x)| + \sup_{x\in\mathbf{R}^n} \int_{\mathbf{R}^n} |G_{in,\alpha}(x+h) - G_{in,\alpha}(x)|\, dx\right] \to 0$$

as $h \to 0$, and the claim is proved.

Next we want to show that $G_1 * (G_1 * |f|)^{\frac{1}{p-1}}$ can be approximate by $G_{out,\alpha} * (G_1 * |f|)^{\frac{1}{p-1}}$. Since we get $G_1 = G_{in}, \alpha + G_{out,\alpha}$ we have

$\sup_{x\in\mathbf{R}^n} \left|G_1 * (G_1 * |f|)^{\frac{1}{p-1}}(x) - G_{out,\alpha} * (G_1 * |f|)^{\frac{1}{p-1}}(x)\right|$
$= \sup_{x\in\mathbf{R}^n} \left|\int_{\mathbf{R}^n} G_{in,\alpha}(x-y) \left(\int_{\mathbf{R}^n} G_1(y-z)|f(x)|dz\right)^{\frac{1}{p-1}} dy\right|$
$= \sup_{x\in\mathbf{R}^n} \int_{|x-y|\leq\alpha/2} G_1(x-y) \left(\int_{\mathbf{R}^n} G_1(y-z)|f(z)|dz\right)^{\frac{1}{p-1}} dy$

by hypothesis and Remark 3 we have

$\leq \sup_x \int_{|x-y|\leq\alpha/2} \frac{1}{|x-y|^{n-1}} \left(\int_{\mathbf{R}^n} G_1(y-z)|f(z)|dz\right)^{\frac{1}{p-1}} dy \to 0$
as $\alpha \to 0$.

Next, assume that $G_1 * (G_1 * |f|)^{\frac{1}{p-1}} \in BUC$. Then by theorem 1 in [5]

$\lim_{\alpha\to 0} \alpha \sup_{x\in\mathbf{R}^n} \left(\int_{\mathbf{R}^n} G_1(x-y) \left(\int_{\mathbf{R}^n} G_1(\alpha y - z)|f(z)|dz\right)^{\frac{1}{p-1}} dy\right) = 0,$
using (2.7), we see that

$\alpha \sup_{x\in\mathbf{R}^n} \left(\int_{\mathbf{R}^n} G_1(x-\alpha y) \left(\int_{\mathbf{R}^n} G_1(\alpha y - z)|f(z)|dz\right)^{\frac{1}{p-1}} dy\right)$
$\leq \alpha^{1-n} \sup_{x\in\mathbf{R}^n} \int_{\mathbf{R}^n} \frac{\chi_{B(0,1)}\left(\frac{x-u}{\alpha}\right)}{\alpha^{1-n}|x-u|^{n-1}} \left(\int_{\mathbf{R}^n} \frac{\chi_{B(0,1)}(u-z)}{|u-z|^{n-1}}|f(z)|dz\right)^{\frac{1}{p-1}} du$
$= \sup_{x\in\mathbf{R}^n} \int_{B(x,\alpha)} \frac{1}{|x-u|^{n-1}} \left(\int_{B(x,\alpha)} \frac{|f(z)|}{|u-z|^{n-1}} dz\right)^{\frac{1}{p-1}} du \to 0,$
as $\alpha \to 0$. Applying Theorem 1 in [5] again, we get $f \in M_p(\mathbf{R}^n)$.

This completes the proof of Theorem 2

□

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