On the Gauss–Newton method for solving equations

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Abstract

We use a combination of the center–Lipschitz condition with the Lipschitz condition condition on the Fréchet–derivative of the operator involved to provide a semilocal convergence analysis of the Gauss-Newton method to a solution of an equation. Using more precise estimates on the distances involved, under weaker hypotheses, and under the same computational cost, we provide an analysis of the Gauss–Newton method with the following advantages over the corresponding results in [8]: larger convergence domain; finer error estimates on the distances involved, and an at least as precise information on the location of the solution

1. Introduction

In this study we are concerned with the problem of approximating a solution $x^\star$ of equation

$$(1.1) \qquad\qquad F'(x)^T \; F(x) = 0,$$

where F is a Fréchet–differentiable operator defined on $X = I\!R^n$, with values on $Y = I\!R^m$ $(m \geq n)$.

A large number of problems in applied mathematics and also in engineering are solved by finding the solutions of certain equations. For example, dynamic systems are mathematically modeled by difference or differential equations, and their solutions usually represent the states of the systems. For the sake of simplicity, assume that a time–invariant system is driven by the equation $\dot{x} = T(x)$, for some suitable operator T, where x is the state. Then the equilibrium states are determined by solving equation (1.1). Similar equations are used in the case of discrete systems. The unknowns of engineering equations can be functions (difference, differential, and integral equations), vectors (systems of linear or nonlinear algebraic equations), or real or complex numbers (single algebraic equations with single unknowns). Except in special cases, the most commonly used solution methods are iterative–when starting from one or several initial approximations a sequence is constructed that converges to a solution of the equation. Iteration methods are also applied for solving optimization problems. In such cases, the iteration sequences converge to an optimal solution of the problem at hand. Since all of these methods have the same recursive structure, they can be introduced and discussed in a general framework.

We are seeking least–square solutions of (1.1). That is we solve the minimization problem:

$$(1.2) \qquad\qquad \min_{x \in X} \; \frac{1}{2} \; F(x)^T \; F(x).$$

We use the famous Gauss–Newton method

$$(1.3) \quad x_{k+1} = x_k - (F'(x_k)^T \; F'(x_k))^{-1} \; F'(x_k)^T \; F(x_k) \quad (x_0 \in I\!R^n), \quad (k \geq 0)$$

to generate a sequence approximating a solution $x^\star$ of (1.2).

There is an extensive literature on the local as well as the semilocal convergence analysis of Newton–type methods under various conditions in

the more general setting when X and Y are Banach spaces [1]–[10].

In particular, in the case of Gauss–Newton method (1.3), Li et al. provided a semilocal convergence analysis in [8] using the concept of the Lipschitz condition.

Recently, we have successfully used in [1]–[3] a combination of Lipschitz and center–Lipschitz conditions instead of only Lipschitz conditions as in to provide a finer local and semilocal convergence analysis for Newton–type methods, when F is an isomorphism. The main idea is derived from the observation that more precise upper bounds on the norms $\| F'(x)^{-1} F'(x^\star) \|$ can be obtained if the needed center–Lipschitz condition is used:

$$\| F'(x^\star)^{-1} (F'(x) - F'(x^\star)) \| \leq L_0 \| x - x^\star \|,$$
for all $x \in U(x^\star, r) = \{x \in X : \| x - x^\star \| \leq r\} \subseteq X, \ r > 0, \ L_0 > 0$

instead of

$$\| F'(x^\star)^{-1} (F'(x) - F'(y)) \| \leq L \| x - y \|, \quad \text{for all } x, y \in U(x^\star, r), \ L > 0$$

(which is commonly used in [4]–[10]).

It turns out that these ideas can be used to study the semilocal convergence of the Gauss–Newton method (1.3). In particular, we provide a semilocal convergence analysis with the following advantages over the work by Li et al. [8]:

1. Weaker sufficient convergence conditions;

2. Larger convergence domain;

3. Finer estimates on the distances involved, which implies that fewer iterations are needed to achieve a desired error tolerance.

4. An at least as precise information is provided on the uniqueness of the solution.

The above improvements are also obtained under the same computational cost since the computation of the Lipschitz condition with constant ℓ (see (1.4)) requires that of the center–Lipschitz condition with constant ℓ_0 (see (1.5)).

In particular, we assume there exist $\ell > 0$ such that the Lipschitz condition

(1.4)
$$\| F'(x) - F'(y) \| \leq \ell \, \| x - y \|,$$
holds for all $x, y \in D$.

In view (1.4), there exists $\ell_0 > 0$ such that center–Lipschitz condition

(1.5)
$$\| F'(x) - F'(x_0) \| \leq \ell_0 \, \| x - x_0 \|,$$
holds for all $x \in D$.

Note that in [8], the same Lipschitz constant ℓ is used in (1.4) and (1.5). However,

(1.6)
$$a = \frac{\ell_0}{\ell} \leq 1$$

holds in general, and a can be arbitrarily small [1]–[3].

Let us provide a simple example where strict inequality holds in (1.6). Let $n = m = 1$, $x_0 = 1$, and define function F on $D = [\delta, 2 - \delta]$, $\delta \in \left[0, \dfrac{1}{2}\right)$ by

(1.7)
$$F(x) = x^3 - \delta.$$

Using (1.4)–(1.7), we obtain

(1.8)
$$\ell_0(t) = 3 - \delta < 2 \, (2 - \delta) = \ell \quad \text{for all } \delta \in \left[0, \frac{1}{2}\right).$$

Using weaker (1.5) (if $\ell_0 < \ell$) (which is actually needed in the computation of the norms

$$\| (F'(x)^T \, F'(x))^T \, F'(x)^T \| \, \| F'(x) - F'(x_0) \|$$

instead of the stronger hypotheses (1.4) used in [8], leads us to the advantages as stated in the abstract of this study of our approach over the corrssponding ones in [8].

2. Semilocal convergence analysis of the Gauss–Newton method (1.3)

Let $I\!R^{m \times n}$ be the set of all $m \times n$ matrices, and A^+ be the generalized inverse of $A \in I\!R^{m \times n}$. Then, when $m \geq n$, and A is of full rank, we have $A^+ = (A^T \, A)^{-1} \, A^T$.

We need the lemmas: [8], [9], [10] Let $A, E \in I\!\!R^{m \times n}$. Assume $B = A + E$, and $\| A^+ \| \| E \| < 1$. Then, the following hold:

$$\text{rank } (B) \geq \text{rank } (A).$$

If rank $(A) = n$, $m \geq n$, then we have rank $(B) = n$.   [8], [9], [10] Let $A, E \in I\!\!R^{m \times n}$. Assume $B = A + E$, and $\| A^+ \| \| E \| < 1$. Then, the following hold:

$$\| B^+ \| \leq \frac{\| A^+ \|}{1 - \| A^+ \| \| E \|},$$

provided that rank $(B) = $ rank (A).
Moreover,

$$\| B^+ - A^+ \| \leq \frac{\sqrt{2} \, \| A^+ \|^2 \, \| E \|}{1 - \| A^+ \| \| E \|}$$

provided that rank $(A) = $ rank $(B) = \min \{m, n\}$. It is convenient for us to define for each fixed $a \in (0, 1]$, functions f_a and g_a on $[0, 1]$ by

$$
(2.1) \quad
\begin{aligned}
f_a(t) \quad = \quad & \left(\sqrt{2} \, (1 + p_1) - (1 + a \, p) \, (1 + 2 \, a \, p \, q) \right) t^2 + \\
& \left(1 + 2 \, \sqrt{2} + 2 \, q \, (1 + a \, p) + 2 \, a \, p \, q \right) t - 2 \, q,
\end{aligned}
$$

and

$$(2.2) \qquad g_a(t) = (\sqrt{2} - a) \, t^2 - (2 \, a + 5 \, \sqrt{2}) \, t + 3 \, \sqrt{2} - a,$$

where,

$$(2.3) \qquad q = \frac{2 \, a + 5 \, \sqrt{2} - \sqrt{26 + 36 \, a \, \sqrt{2}}}{2 \, (\sqrt{2} - a)},$$

$$(2.4) \qquad q = \frac{1}{1 - q},$$

and

$$(2.5) \qquad p_1 = \frac{q^2}{1 - q^2}.$$

It is simple algebra to show q is the small positive zero of function g_a with $q \in (0, 1)$. We also have $f_a(0) = -2 \, q < 0$, and

$$f_a(1) = \frac{g_a(q^0)}{(1 - q^0)^2 \, (1 + q^0)} \geq 0, \quad \text{for all} \quad q^0 \in [0, q].$$

It follows from the intermediate value theorem that there exists a maximal $h_0 \in [0, 1]$ such that

$$(2.6) \qquad\qquad f_a(t) \leq 0 \quad \text{for all} \quad t \in [0, h_0].$$

Define also h_1, b and c by

$$(2.7) \qquad\qquad h_1 = \frac{1 - q}{1 - q + a},$$

$$(2.8) \qquad\qquad b = \| (F'(x_0)^T \ F'(x_0))^{-1} \ F'(x_0)^T \|,$$

and

$$(2.9) \qquad\qquad c = \| F(x_0) \|,$$

for some $x_0 \in D$.

We can state the main semilocal convergence theorem for the Gauss–Newton method (1.3). Let $F : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be continuously Fréchet–differentiable on $U(x^\star, r_0)$, for some $x_0 \in D$ and $r > 0$.
Assume:

Operator $F'(x_0)$ is of full rank and F' satisfies the Lipschitz condition (1.4);

$$(2.10) \qquad\qquad h = b^2 \ c \ \ell = \begin{cases} < h_1 & \text{if} \quad h_1 \leq h_0 \\ \leq h_0 & \text{if} \quad h_0 < h_1 \end{cases} ;$$

$$(2.11) \qquad\qquad \overline{U}(x_0, r_0) \subseteq D,$$

where,

$$(2.12) \qquad\qquad r_0 = \frac{b \ c}{1 - q} \leq r.$$

Then, sequence $\{x_k\}$ $(k \geq 0)$ generated by the Gauss–Newton method (1.3) is well defined, remains in $\overline{U}(x_0, r_0)$ for all $k \geq 0$, and converges to some $x^\star \in U(x_0, r_0)$.
Moreover, the following estimates hold for all $k \geq 1$:

$$\| x_{k+1} - x_k \| \ \leq \ \alpha_k \ \| x_k - x_{k-1} \|^2 + \beta_k \ \| x_k - x_{k-1} \|$$
$$(2.13)$$
$$q_k \ \| x_k - x_{k-1} \| \ \leq \ q \ \| x_k - x_{k-1} \|,$$

where,

$$(2.14) \qquad\qquad \alpha_k = \frac{b \ \ell}{2 \ (1 - a \ p \ h)},$$

$$\beta_k = \frac{\sqrt{2} \; b^2 \; \ell}{(1 - a \; p \; h) \; (1 - a \; p \; h - b \; c \; \| \; x_k - x_{k-1} \; \|)} \left(c + \frac{\ell}{2} \sum_{i=1}^{i=k} \| \; x_i - x_{i-1} \; \|^2 \right),$$

(2.15)

and

(2.16) $$q_k = \alpha_k \; \| \; x_k - x_{k-1} \; \| + \beta_k.$$

Proof of theorem 2. We shall show estimate (2.13), and $x_k \in U(x_0, r_0)$, for all $k \geq 1$. Note that $x_1 \in \overline{U}(x_0, r_0)$ by the definition of c and (2.12). Using (1.3), (1.5), (2.8)–(2.10), we obtain

$$\| \; (F'(x_0)^T \; F'(x_0))^{-1} \; F'(x_0)^T \; \| \; \| \; F'(x_1) - F'(x_0) \; \|$$

$$\leq \quad b \; \ell_0 \; \| \; x_1 - x_0 \; \|$$
$$\leq \quad b^2 \; c \; \ell_0 \leq a \; h < 1.$$

(2.17)

In view of Lemmas 2, 2 respectively, and (2.17), we get $F'(x_1)$ is of full rank,

(2.18) $$\| \; (F'(x_1)^T \; F'(x_1))^{-1} \; F'(x_1)^T \; \| \leq \frac{b}{1 - a \; h},$$

and

$$\| \; (F'(x_1)^T \; F'(x_1))^{-1} \; F'(x_1)^T - (F'(x_0)^T \; F'(x_0))^{-1} \; F'(x_0)^T \; \|$$

$$\leq \frac{\sqrt{2} \; b \; \ell_0 \; \| \; x_1 - x_0 \; \|}{1 - a \; h}$$

$$\leq \frac{\sqrt{2} \; a \; h}{1 - a \; h}.$$

(2.19)

By (1.3), we obtain the identity

$$\begin{aligned}
x_2 - x_1 &= x_1 - x_0 - (F'(x_1)^T \; F'(x_1))^{-1} \; F'(x_1)^T \; F(x_1) \\
&\quad + (F'(x_0)^T \; F'(x_0))^{-1} \; F'(x_0)^T \; F(x_0) \\
&= (F'(x_0)^T \; F'(x_0))^{-1} \; F'(x_0)^T \; (F'(x_0) \; (x_1 - x_0) - F(x_1) + F(x_0)) + \\
&\quad \left(F'(x_0)^T \; F'(x_0))^{-1} \; F'(x_0)^T - (F'(x_1)^T \; F'(x_1))^{-1} \; F'(x_1)^T \right) \; F(x_1).
\end{aligned}$$

(2.20)

Using (1.5), we obtain the estimate

$\| F(x_1) \|$
$\leq \| F(x_1) - F(x_0) - F'(x_0) (x_1 - x_0) \| + \| F(x_0) + F'(x_0) (x_1 - x_0) \|$
$\leq \dfrac{\ell_0}{2} \| x_1 - x_0 \|^2 + \| F(x_0) + F'(x_0) (x_1 - x_0) \|$
$\leq \dfrac{b^2 c^2 \ell_0}{2} + c \leq \left(\dfrac{a h}{2} + 1 \right) c.$
(2.21)

By taking norms in (2.20), and using (2.6), (2.18), (2.19), and (2.21), we obtain in turn:

$\| x_2 - x_1 \| \leq \| (F'(x_0)^T F'(x_0))^{-1} F'(x_0)^T \|$
$\| \int_0^1 (F'(x_0) - F'(x_0 + \theta (x_1 - x_0)) (x_1 - x_0) d\theta \| +$
$\| (F'(x_1)^T F'(x_1))^{-1} F'(x_1)^T - (F'(x_0)^T F'(x_0))^{-1} F'(x_0)^T \| \| F(x_1) \|$
$\leq \left(\dfrac{b \ell}{2 (1 - a h)} + \dfrac{\sqrt{2} a h}{1 - a h} (\dfrac{a h}{2} + 1) \right) \| x_1 - x_0 \|$
$\leq \alpha_1 \| x_1 - x_0 \|^2 + \beta_1 \| x_1 - x_0 \| \leq q \| x_1 - x_0 \|$
(2.22)

which shows (2.13) for $k = 1$.

We also have

(2.23)
$\begin{aligned} \| x_2 - x_0 \| &\leq \| x_2 - x_1 \| + \| x_1 - x_0 \| \\ &\leq (1 + q) \| x_1 - x_0 \| = \dfrac{1 - q^2}{1 - q} \| x_1 - x_0 \| \leq r_0, \end{aligned}$

which implies $x_2 \in \overline{U}(x_0, r_0)$.
Let us assume estimate (2.13), and $x_i \in \overline{U}(x_0, r_0)$ hold true for all $i < k$.
Using the induction hypotheses, (1.5), (2.4), (2.8), and (2.10), we get in turn:

(2.24)
$\begin{aligned} &\| (F'(x_0)^T F'(x_0))^{-1} F'(x_0)^T \| \| F'(x_i) - F'(x_0) \| \\ &\leq b \ell_0 \| x_i - x_0 \| \\ &\leq b \ell_0 (\| x_i - x_{i-1} \| + \| x_{i-1} - x_{i-2} \| + \cdots + \| x_1 - x_0 \|) \\ &\leq b \ell_0 (q^{i-1} + q^{i-2} + \cdots + 1) \| x_1 - x_0 \| \leq a p h < 1. \end{aligned}$

In view of Lemmas 2, 2, and (2.24), we get $F'(x_i)$ is of full rank, and

(2.25) $\| (F'(x_i)^T F'(x_i))^{-1} F'(x_i)^T \| \leq \dfrac{b}{1 - a p h}.$

Moreover, we have:

$$\| (F'(x_{k-1})^T \ F'(x_{k-1}))^{-1} \ F'(x_{k-1})^T \| \ \ell \ \| x_k - x_{k-1} \|$$

$$\leq$$

(2.26)
$$\frac{b \ \ell \ q^{n-1} \ \| x_1 - x_0 \|}{1 - a \ p \ h}$$

$$\leq$$

$$\frac{h}{1 - a \ p \ h} < 1.$$

Using Lemmas 2, 2, we get

$$\| (F'(x_k)^T \ F'(x_k))^{-1} \ F'(x_k)^T - (F'(x_{k-1})^T \ F'(x_{k-1}))^{-1} \ F'(x_{k-1})^T \|$$

$$\leq \frac{\sqrt{2} \left(\dfrac{b}{1 - a \ p \ h} \right)^2 \ell \ \| x_k - x_{k-1} \|}{1 - \dfrac{b \ \ell \ \| x_k - x_{k-1} \|}{a \ p \ h}}.$$

(2.27)

By (1.3) as in (2.20), we obtain the identity:

$$x_{k+1} - x_k =$$

$$(F'(x_{k-1})^T \ F'(x_{k-1}))^{-1} \ F'(x_{k-1})^T \ \int_0^1 (F(x_{k-1}) - F(x_{k-1} + \theta \ (x_k - x_{k-1})))$$

$$(x_k - x_{k-1}) \ d\theta + \left(F'(x_k)^T \ F'(x_k)^{-1} \ F'(x_k)^T - (F'(x_{k-1})^T \ F'(x_{k-1}))^{-1} \right.$$

$$\left. F'(x_{k-1})^T \right) \ F(x_k).$$

(2.28)

We also need the estimate

$$\| F(x_k) \| \leq \| F(x_k) - F(x_{k-1}) - F'(x_{k-1}) \ (x_k - x_{k-1}) \| +$$

$$\| F(x_k) + F'(x_{k-1}) \ (x_k - x_{k-1}) \|$$

(2.29)
$$\leq \frac{\ell}{2} \| x_k - x_{k-1} \|^2 + \| F(x_k) + F'(x_{k-1}) \ (x_k - x_{k-1}) \|$$

$$\leq \| F(x_0) \| + \frac{\ell}{2} \sum_{i=1}^{i=k} \| x_i - x_{i-1} \|^2 .$$

By (2.26)–(2.29), we get in turn

$$\| x_{k+1} - x_k \| \leq \| (F'(x_{k-1})^T \ F'(x_{k-1}))^{-1} \ F'(x_{k-1})^T \|$$

$$\| \int_0^1 (F'(x_{k-1}) - F'(x_{k-1} + \theta \ (x_k - x_{k-1}))) \ (x_k - x_{k-1}) \ d\theta \| +$$

$$\| (F'(x_k)^T \ F'(x_k))^{-1} \ F'(x_k)^T - (F'(x_{k-1})^T \ F'(x_{k-1}))^{-1} \ F'(x_{k-1})^T \|$$

$$\| F(x_k) \|$$
$$\leq \alpha_k \| x_k - x_{k-1} \|^2 + \beta_k \| x_k - x_{k-1} \| .$$
(2.30)

We also have by the definition of q and (2.6):

$$q_k \leq \frac{b^2 \ c \ \ell \ q^{k-1}}{2 \ (1 - a \ p \ h)} + \frac{\sqrt{2} \left(\dfrac{b}{1 - a \ p \ h} \right)^2 \ell}{1 - \dfrac{b \ \ell \ q^{n-1} \ \| x_1 - x_0 \|}{1 - a \ p \ h}}$$

(2.31)
$$\left(c + \frac{c \ h}{2} + \frac{\ell}{2} \sum_{i=1}^{i=k-1} q^{2 \ i} \ \| x_1 - x_0 \|^2 \right)$$

$$\leq \frac{h}{2 \ (1 - a \ p \ h)} + \frac{\sqrt{2} \ h \left(1 + \dfrac{1}{2} \ h \ (1 + p_1) \right)}{(1 - a \ p \ h) \ (1 - a \ p \ h - h)} \leq q,$$

which completes the induction for (2.13).

Moreover, we have:

$$
\begin{aligned}
\| x_{k+1} - x_0 \| &\leq \| x_{k+1} - x_k \| + \| x_k - x_{k-1} \| + \cdots + \| x_1 - x_0 \| \\
&\leq (q^k + q^{k-1} + \cdots + 1) \| x_1 - x_0 \| \\
&= \frac{1 - q^{k+1}}{1 - q} \| x_1 - x_0 \| \leq r_0,
\end{aligned}
$$

(2.32)

which implies $x_{k+1} \in \overline{U}(x_0, r_0)$.

Let $j \geq 0$. We get

$$
\begin{aligned}
\| x_{k+j} - x_k \| &\leq \| x_{k+j} - x_{k+j-1} \| + \| x_{k+j-1} - x_{k+j-2} \| + \cdots + \| x_{k+1} - x_k \| \\
&\leq (q^{j-1} + q^{j-2} + \cdots + 1) \| x_{k+1} - x_k \| \\
&= \frac{1 - q^j}{1 - q} q^k \| x_1 - x_0 \| \leq r_0,
\end{aligned}
$$

(2.33)

which by letting $j \longrightarrow \infty$ implies $\lim_{k \to \infty} x_k = x^\star \in \overline{U}(x_0, r_0)$ (since $\overline{U}(x_0, r_0)$ is a closed set), and

$$
(2.34) \qquad \| x_k - x^\star \| \leq \frac{q^k}{1 - q} \| x_1 - x_0 \| .
$$

That completes the proof of Theorem 2. $\qquad \diamondsuit$

Case $\ell = \ell_0$

Using (2.1)–(2.7), we get

$$
\begin{aligned}
&q = .363504384, \quad p = 1.57110273, \quad p_1 = 1.52126262, \\
&f_1(t) = -1.307374992\, t^2 + 6.83984613\, t - .727008768, \\
&h_0 = .108542119 \quad and \quad h_1 = .388938173.
\end{aligned}
$$

Condition (2.10) becomes

$$
(2.35) \qquad h \leq .108542119 = h_2.
$$

The corresponding $\overline{q}, \overline{h}$ given by Li and Zhang in [8] are

$$
\overline{q} = .5 > q
$$

and
$$\overline{h} > h.$$

Hence, we have expanded with applicability of Gauss–Newton method (1.3) under the same computational cost and with a smaller ratio than in [8].

In the next case, we take advantage of the case $a < 1$.

Case $a = .5$

As in the previous case we obtain

$q = .491020341, \quad p = 1.964715057, \quad p_1 = .31528735,$
$f_1(t) = 2.034670476\ t^2 - 6.739897921\ t + .982040682 \geq 0,$
$h_0 = .15274924, \quad h_1 = .504449871 \quad and \quad h = h_0 \leq .15274924 = h_3,$

which extends further the applicability of Gauss–Newton method, since

$$h_3 > h_2$$

where h_2 is given by (2.35).

Finally, note that if $a \longrightarrow 0^+$, we get by (2.1)–(2.7) that $q \longrightarrow .69722436$, $h_1 \longrightarrow 1$, and $h_0 \longrightarrow .246590672$. That is the largest possible upper bound on h given in (2.10) is $h_0 = .246590672$.

The advantages obtained here can also be extended to the case of the Newton–type method

$$x_{n+1} = x_n - F'(x_n)^+\ F(x_n) \quad (n \geq 0)$$

analyzed by Ben–Israel in [4]. However, we leave the details to the motivated reader.

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