

Some separation axioms in L -topological spaces

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Abstract

In this paper, under the idea of L - T_0 or sub- T_0 , we propose a set of new separation axioms in L -topological spaces, namely sub-separation axioms. And some of their properties are studied. In addition, the relation between the sub-separation axioms defined in the paper and other separation axioms is discussed. The results show that the sub-separation axioms in this paper are weaker than other separation axioms that had appeared in literature.

Keywords : L -topology; sub-separation axioms; sub- T_1 ; sub- T_2 ; sub- $T_{2\frac{1}{2}}$; sub- T_3 ; sub- T_4 .

1. Introduction and preliminaries

Since Chang [1] introduced fuzzy theory into topology, Wong, Lowen, Hutton etc., discussed respectively various aspects of fuzzy topology (Wong [17], Lowen [12], Hutton [6]).

Separation is an essential part of fuzzy topology, on which a lot of work have been done [2–19]. In 1983, Liu [9] introduced the sub- T_0 axiom, for underlying lattice L being a completely distributive DeMorgan algebra, in terms of closed sets and proved that the fuzzy real line and the fuzzy unit interval satisfy this axiom. Wuyts and Lowen [18] and Rodabaugh [13] independently gave a more general L - T_0 axioms, the latter for L being a complete lattice, using only open sets and equivalent to the sub- T_0 when L is a completely distributive DeMorgan algebra. The aim of this paper is to study some separation axioms on the basis of the thought of the sub- T_0 and the layer of L -topology.

Now we recall some the concepts required in the sequel.

Throughout this paper, $(L, \vee, \wedge, ')$ is a completely distributive DeMorgan algebra, i.e., a complete and completely distributive lattice with an order-reversing involution $(\quad)'$, and with the smallest element \perp and the largest element \top ($\perp \neq \top$). Obviously, for every nonempty set X , L^X , the family of all L -sets, i.e., all mappings from X to L , is also a complete and completely distributive lattice under the pointwise order. we denote the smallest element and the largest element of L^X by \perp_X and \top_X , respectively. For any $A \in L^X$, the set $\{x : A(x) \neq \perp\}$ is called the support of A and denoted by $\text{supp}A$, i.e., $\text{supp}A = \{x : A(x) \neq \perp\}$

An L -topological space, briefly L -ts, is a pair (L^X, δ) , where δ , called an L -topology on L^X , a subfamily of L^X closed under the operation of finite intersections and arbitrary unions, and $\delta' = \{A' : A \in \delta\}$; the member of δ (resp. δ') is called open (resp., closed) L -sets, and for each $B \in L^X$, the L -set $B^\circ = \bigvee \{U \in \delta : U \leq B\}$ (resp. $B^- = \bigwedge \{C \in \delta' : B \leq C\}$) is called the interior (resp., closure) of B . An element $\lambda \in L$ is called a molecule if $\lambda \neq \perp$ and $\lambda \leq a \vee b$ implies $\lambda \leq a$ or $\lambda \leq b$. The set of all molecules of L (resp., L^X) will be denoted by $M(L)$ (resp., $M(L^X)$); obviously, $M(L^X) = \{x_\lambda : x \in X, \lambda \in M(L)\}$. For any $x_\lambda \in M(L^X)$, a closed L -set $P \in \delta'$ is called a closed remote neighborhood of x_λ if $x_\lambda \not\leq P$. The set of all closed remote neighborhood of x_λ is denoted by $\eta^-(x_\lambda)$. For any $A \in L^X$, a closed L -set $P \in \delta'$ is called a closed remote neighborhood of A if for any $x \in \text{supp}A$ such that $A(x) \not\leq P(x)$. The set of all closed

remote neighborhood of A is denoted by $\eta^-(A)$. For any $A \in L^X$, A is called pseudo-crisp closed set if $\exists a \in L - \{\perp\}$ such that $A(x) > \perp$ if and only if $\forall x \in X, A(x) \geq a$.

Let $f : X \longrightarrow Y$ be an ordinary mapping. Based on $f : X \longrightarrow Y$ define an mapping $f^\rightarrow : L^X \longrightarrow L^Y$ which is called a function of Zadeh's type and its right adjoint mapping $f^\leftarrow : L^Y \longrightarrow L^X$ by

$$\forall A \in L^X, \forall y \in Y, f^\rightarrow(A)(y) = \bigvee \{A(x) : x \in X, f(x) = y\}, \text{ and}$$

$$\forall B \in L^Y, \forall x \in X, f^\leftarrow(B)(x) = B(f(x)), \text{ respectively.}$$

For other undefined notions and symbols in this paper, please refer to Wang [16].

Definition 1.1 (Liu [9]). An L -ts (L^X, δ) is called a sub- T_0 space if for any $x, y \in X$ with $x \neq y$, there exists $\lambda \in M(L)$, either there is $P \in \eta^-(x_\lambda)$ such that $y_\lambda \leq P$ or there is $Q \in \eta^-(y_\lambda)$ such that $x_\lambda \leq Q$.

Definition 1.2 (Chen and Meng [2]). An L -ts (L^X, δ) is called a $T_{2\frac{1}{2}}$ or L -Urysohn space if for any $x_\lambda, y_\mu \in M(L^X)$ with $x \neq y$, there exist $P \in \eta^-(x_\lambda)$ and $Q \in \eta^-(y_\mu)$ such that $P^\circ \vee Q^\circ = \top_X$.

Definition 1.3 (Wang [16]). Let (L^X, δ) be an L -ts. Then,

(1) (L^X, δ) is said to be T_1 if for any $x_\lambda, y_\mu \in M(L^X)$ with $x_\lambda \not\leq y_\mu$, there exists $P \in \eta^-(x_\lambda)$ such that $y_\mu \leq P$.

(2) (L^X, δ) is said to be T_2 (or Hausdorff) if for any $x_\lambda, y_\mu \in M(L^X)$ with $x_\lambda \not\leq y_\mu$, there exist $P \in \eta^-(x_\lambda)$ and $Q \in \eta^-(y_\mu)$ such that $P \vee Q = \top_X$.

(3) (L^X, δ) is said to be regular if for each $x_\lambda \in M(L^X)$ and each nonempty pseudo-crisp closed set A with $x \notin \text{supp}A$, there exist $P \in \eta^-(x_\lambda)$ and $Q \in \eta^-(A)$ such that $P \vee Q = \top_X$. (L^X, δ) is said to be T_3 if it is regular and T_1 .

(4) (L^X, δ) is said to be normal if for each pair of nonempty pseudo-crisp closed set A and B with $\text{supp}A \cap \text{supp}B = \emptyset$, there exist $P \in \eta^-(A)$ and $Q \in \eta^-(B)$ such that $P \vee Q = \top_X$. (L^X, δ) is said to be T_4 if it is normal and T_1 .

Theorem 1.4 (Wang [16], You [19]). $T_i (i = 1, 2, 2\frac{1}{2}, 3, 4)$ is L -good extension in Lowen's sense.

Definition 1.5 (Gu and Zhao [4]). An L -ts (L^X, δ) is said to be layer T_0 if for any $\alpha \in M(L)$, $(X, (\tau_\alpha(\delta'))')$ is T_0 , where $\tau_\alpha(\delta) = \{\tau_\alpha(A) : A \in$

$\delta'\}$, $\tau_\alpha(A) = \{x \in X : A(x) \geq \alpha\}$. In the same way, layer T_i ($i = 1, 2, 3, 4$) and layer regular (completely regular, normal) are defined.

Definition 1.6 (Kubiak [7]). An L -ts (L^X, δ) is said to be

(i) Kubiak- T_1 (or L - T_1) if for all $x, y \in X$ with $x \neq y$, there exist $U, V \in \delta$ such that $U(x) \not\leq U(y)$ and $V(y) \not\leq V(x)$.

(ii) Kubiak- T_2 if for all $x, y \in X$ with $x \neq y$, there exist $U, V \in \delta$ such that $U(x) \not\leq U(y)$, $V(y) \not\leq V(x)$ and $U \leq V'$.

Lemma 1.7 (Liu and Luo [10]). Let (L^X, δ) be an L -ts, where δ is generated by a classical topology, then for any $A \in L^X$ such that $A^\circ = \bigvee \{\alpha \chi_{[\tau_\alpha(A)]^\circ} : \alpha \in M(L)\}$.

Definition 1.8 (Shi [14]). An L -ts (L^X, δ) is called L - T_2 if for all $x, y \in X$ with $x \neq y$, there exist $P \in \delta'$ and $Q \in \delta$ such that $Q \leq P$ and $Q(x) \not\leq P(y)$.

2. Definitions and characterizations

In this section, we introduce the concept of sub- T_1 , sub- T_2 , sub- $T_{2\frac{1}{2}}$, sub- T_3 and sub- T_4 separation axioms in L -topological spaces and establish the characteristic theorems of these sub-separation axioms. First, some definitions are given as follows:

Definition 2.1. Suppose that (L^X, δ) is an L -ts. Then,

(1) (L^X, δ) is said to be sub- T_1 if for any $x, y \in X$ with $x \neq y$, there exists $\lambda \in M(L)$, both there is $P \in \eta^-(x_\lambda)$ such that $y_\lambda \leq P$ and there is $Q \in \eta^-(y_\lambda)$ such that $x_\lambda \leq Q$.

(2) (L^X, δ) is said to be sub- T_2 if for any $x, y \in X$ with $x \neq y$, there exists $\lambda \in M(L)$, and there are $P \in \eta^-(x_\lambda)$ and $Q \in \eta^-(y_\lambda)$ such that $P \vee Q = \top_X$.

(3) (L^X, δ) is said to be sub- $T_{2\frac{1}{2}}$ if for any $x, y \in X$ with $x \neq y$, there exists $\lambda \in M(L)$, and there are $P \in \eta^-(x_\lambda)$ and $Q \in \eta^-(y_\lambda)$ such that $P^\circ \vee Q^\circ = \top_X$.

(4) (L^X, δ) is said to be sub-regular if for each $x \in X$ and each nonempty pseudo-crisp closed set A with $x \notin \text{supp}A$, there exists $\lambda \in M(L)$, and there are $P \in \eta^-(x_\lambda)$ and $Q \in \eta^-(\lambda A)$ such that $P \vee Q = \top_X$. (L^X, δ) is said to be sub- T_3 if it is sub-regular and sub- T_1 .

(5) (L^X, δ) is said to be sub-normal if for each pair of nonempty pseudo-

crisp closed set A and B with $\text{supp}A \cap \text{supp}B = \emptyset$, there exists $\lambda \in M(L)$, and there are $P \in \eta^-(\lambda A)$ and $Q \in \eta^-(\lambda B)$ such that $P \vee Q = \top_X$, where $\lambda A(x) = \lambda \wedge A(x)$ for any $x \in X$. (L^X, δ) is said to be sub- T_4 if it is sub-normal and sub- T_1 .

By Definition 2.1, we have:

Corollary 2.2. The following implications hold: $T_i \implies \text{sub-}T_i$, where $i = 1, 2, 2\frac{1}{2}, 3, 4$.

Now we introduce the convergence of molecular nets. Let (L^X, δ) be an L -ts, $S = \{S(n) : n \in D\}$ a molecular net and $e \in M(L^X)$, e is said to be a limit point of S , (or S converges to e ; in symbols, $S \rightarrow e$), if for $\forall P \in \eta^-(e)$, $S(n) \not\leq P$ is eventually true, that is there exists $m \in D$ such that $S(n) \not\leq P$ for all $n \in D$ with $n \geq m$. The following results show that the convergence of molecular nets is unique under a certain condition for the sub- T_2 space.

Theorem 2.3. Let (L^X, δ) be a sub- T_2 space, then for each molecular net S such that $|K_S| \leq 1$, where $K_S = \{x \in X : \lim S(x) = \top\}$.

Proof. Let (L^X, δ) be a sub- T_2 space and $S = \{S(n) : n \in D\}$ be a molecular net. Assume that $|K_S| \geq 2$, for any $x, y \in K_S$ with $x \neq y$, since (L^X, δ) is sub- T_2 , there exists $\lambda \in M(L)$, and there are $P \in \eta^-(x_\lambda)$ and $Q \in \eta^-(y_\lambda)$ such that $P \vee Q = \top_X$. Then we have $S \rightarrow x_\lambda$ and $S \rightarrow y_\lambda$ from $x_\lambda \leq \lim S$ and $y_\lambda \leq \lim S$ by Theorem 2.3.4 (Wang [16]). So there exists an $n_1 \in D$ such that $S(n) \not\leq P$ for all $n \in D$ with $n \geq n_1$ and there exists an $n_2 \in D$ such that $S(n) \not\leq Q$ for all $n \in D$ with $n \geq n_2$. Taking $n_3 \in D$ such that $n_3 \geq n_1$ and $n_3 \geq n_2$, hence we have $S(n) \not\leq P \vee Q$ when $n \geq n_3$. This implies that we must have $P \vee Q \neq \top_X$. This is a contradiction. \square

If \top is a molecule, the inverse of Theorem 2.3 is also true.

Theorem 2.4. (L^X, δ) is an L -ts, if for each molecular net S with $|K_S| \leq 1$, where $K_S = \{x \in X : \lim S(x) = \top\}$, then (L^X, δ) is a sub- T_2 space.

Proof. Suppose that (L^X, δ) is not a sub- T_2 space, then there exist $x, y \in X$ satisfying $x \neq y$, $\forall \lambda \in M(L), \forall P \in \eta^-(x_\lambda)$ and $\forall Q \in \eta^-(y_\lambda)$ such that $P \vee Q \neq \top_X$. Let $D(\lambda) = \eta^-(x_\lambda) \times \eta^-(y_\lambda)$ and $D(\lambda)$ be a directed set by

product order. For each $m = (P, Q) \in D(\lambda)$, we can take a molecule $S^\lambda(m)$ such that $S^\lambda(m) \not\leq P \vee Q$. Let $S^\lambda = \{S^\lambda(m) : m \in D(\lambda)\}$, hence it is easy to prove $S^\lambda \rightarrow x_\lambda$ and $S^\lambda \rightarrow y_\lambda$. Therefore, $\lim S^\lambda \geq x_\lambda \vee y_\lambda$. Since \top is a molecule, the standard minimal set $\beta^*(\top)$ is a directed set (Wang [16]). We denote $\beta^*(\top)$ by E , i.e., $E = \beta^*(\top)$. Noticing that $\{x_\lambda\}_{\lambda \in E}$, $\{y_\lambda\}_{\lambda \in E}$ are molecular nets and $\{x_\lambda\}_{\lambda \in E} \rightarrow x_\top$, $\{y_\lambda\}_{\lambda \in E} \rightarrow y_\top$, we can make a molecular net $\bar{S} : E \times \prod_{\lambda \in E} D(\lambda) \rightarrow M(L^X)$ such that

$$\bar{S}(\lambda, f) = S^\lambda(f(\lambda)), \quad \forall (\lambda, f) \in E \times \prod_{\lambda \in E} D(\lambda).$$

Then, $\bar{S} \rightarrow x_\top$, $\bar{S} \rightarrow y_\top$. In fact, for every $P \in \eta^-(x_\top)$, since $\{x_\lambda\}_{\lambda \in E} \rightarrow x_\top$, there exists a $\lambda_0 \in E$ such that $x_\lambda \not\leq P$ for all $\lambda \in E$ with $\lambda \geq \lambda_0$. It follows from $S^\lambda \rightarrow x_\lambda$ for $\lambda \in E$ that there exists $m_\lambda \in D(\lambda)$ s.t. $S^\lambda(m) \not\leq P$ for all $m \in D(\lambda)$ with $m \geq m_\lambda$. We define $f_0 \in \prod_{\lambda \in E} D(\lambda)$ as follows:

$$f_0(\lambda) = \begin{cases} m_\lambda, & \lambda \leq \lambda_0. \\ \text{any fixed element in } D(\lambda), & \lambda \not\leq \lambda_0. \end{cases}$$

Then we can prove for every pair $(\lambda, f) \in E \times \prod_{\lambda \in E} D(\lambda)$ with $(\lambda, f) \geq (\lambda_0, f_0)$ such that $\bar{S}(\lambda, f) \not\leq P$, i.e., \bar{S} is not in any closed remote neighborhood P of x_\top eventually. So we have $\bar{S} \rightarrow x_\top$. Similarly, we can prove $\bar{S} \rightarrow y_\top$. Therefore, $|K_{\bar{S}}| \geq 2$. This contradicts to $|K_{\bar{S}}| \leq 1$. Thus, we conclude that (L^X, δ) is a sub- T_2 space. \square

With Theorem 2.3 and Theorem 2.4, we have:

Corollary 2.5. Let \top be a molecule, then (L^X, δ) is a sub- T_2 space iff for each molecular net S such that $|K_S| \leq 1$, where $K_S = \{x \in X : \lim S(x) = \top\}$.

For the sub- T_2 space, we have the following theorem:

Theorem 2.6. Let (L^X, δ) be a sub- T_2 space and \top be a molecule, then super F-compactness, N-compactness, strongly F-compactness and F-compactness are equivalent.

Proof. The proof is similar to that of Wang's Theorem 6.4.29 in [16].

Theorem 2.7. Suppose that (L^X, δ) is a weakly induced L -ts. If it is a sub- T_2 space, then $(X, [\delta])$ is a T_2 space.

Proof. Let (L^X, δ) be a sub- T_2 space and $x, y \in X$ with $x \neq y$. Then there exists $\lambda \in M(L)$, and there are $P \in \eta^-(x_\lambda)$ and $Q \in \eta^-(y_\lambda)$ such that $P \vee Q = \top_X$. We put

$$U = \{t \in X : P'(t) \not\leq \lambda'\} = \{t \in X : P(t) \not\leq \lambda\}, \text{ and}$$

$$V = \{t \in X : Q'(t) \not\leq \lambda'\} = \{t \in X : Q(t) \not\leq \lambda\}.$$

Then it is easy to know that $\chi_U, \chi_V \in \delta$ i.e., $U, V \in [\delta]$. Obviously $x \in U, y \in V$. Thus it remains only to show that $U \cap V = \emptyset$. In fact, if there were a $z \in U \cap V$, then we have $\lambda \not\leq P(z)$ and $\lambda \not\leq Q(z)$. Hence, $\lambda \not\leq (P \vee Q)(z)$, which contradicts to $P \vee Q = \top_X$. Therefore, $(X, [\delta])$ is a T_2 space.

3. Properties

In this section, we will investigate some nice properties of sub-separation axioms. At first, we show that sub-separation axioms are good extensions in the sense of Lowen.

Theorem 3.1. Let (X, \mathcal{T}) be a crisp topological space. Then $(L^X, \omega_L(\mathcal{T}))$ is a sub- T_i space iff (X, \mathcal{T}) is a T_i space, where $i = 1, 2, 2\frac{1}{2}, 3, 4$.

Proof. Sufficiency. Let (X, \mathcal{T}) be a T_i space ($i=1, 2, 2\frac{1}{2}, 3, 4$). Then $(L^X, \omega_L(\mathcal{T}))$ is a T_i space by Theorem 1.4. Thus from Corollary 2.2, we know that $(L^X, \omega_L(\mathcal{T}))$ is a sub- T_i space.

Necessity. Case $i=1$: Let $(L^X, \omega_L(\mathcal{T}))$ be a sub- T_1 space. For any $x \in X$ and taking $y \in X$ with $x \neq y$, since $(L^X, \omega_L(\mathcal{T}))$ is a sub- T_1 space, there exists $\lambda \in M(L)$, both there is $P \in \eta^-(y_\lambda)$ such that $x_\lambda \leq P$ and there is $Q \in \eta^-(x_\lambda)$ such that $y_\lambda \leq Q$. We put

$$U = \{z \in X : P'(z) \not\leq \lambda'\}.$$

It is clear that $U \in \mathcal{T}, x \notin U$ and $y \in U$. Hence $y \notin \{x\}^-$, where $\{x\}^-$ is the closure of $\{x\}$. Therefore, (X, \mathcal{T}) is a T_1 space.

Case $i = 2$: For any two distinct points $x, y \in X$ with $x \neq y$, since $(L^X, \omega_L(\mathcal{T}))$ is a sub- T_2 space, there exists $\lambda \in M(L)$, and there are $P \in \eta^-(x_\lambda)$ and $Q \in \eta^-(y_\lambda)$ such that $P \vee Q = \top_X$. We put

$$U = \{z \in X : P'(z) \not\leq \lambda'\}, \quad V = \{z \in X : Q'(z) \not\leq \lambda'\}.$$

Noticing that $P', Q' \in \omega_L(\mathcal{T})$ and $x_\lambda \not\leq P, y_\lambda \not\leq Q$, hence $U, V \in \mathcal{T}$ and $x \in U, y \in V$. Thus it remains only to show that $U \cap V = \emptyset$. In fact, if there were a $z \in U \cap V \neq \emptyset$, then we have $\lambda \not\leq P(z)$ and $\lambda \not\leq Q(z)$. Hence $\lambda \not\leq (P \vee Q)(z) = \top$, which contradicts to $\lambda \leq \top$. Therefore, (X, \mathcal{T}) is a T_2 space.

Case $i = 2\frac{1}{2}$: For any $x, y \in X$ with $x \neq y$, since $(L^X, \omega_L(\mathcal{T}))$ is a sub- $T_{2\frac{1}{2}}$ space, there exists $\lambda \in M(L)$, and there are $P \in \eta^-(x_\lambda)$ and $Q \in \eta^-(y_\lambda)$ such that $P^\circ \vee Q^\circ = \top_X$. Clearly, $P \vee Q = \top_X$. From $\lambda \not\leq P(x)$ and $\lambda \not\leq Q(y)$, we know that there exist $\lambda_P \in \beta^*(\lambda)$ and $\lambda_Q \in \beta^*(\lambda)$ such that $\lambda_P \not\leq P(x)$ and $\lambda_Q \not\leq Q(y)$. Since λ is a molecule, the standard minimal set $\beta^*(\lambda)$ is a directed set (Wang [16]). Taking $\gamma \in \beta^*(\lambda)$ such that $\gamma \geq \lambda_P \vee \lambda_Q$. We put

$$E = \tau_\gamma(P) = \{z : P(z) \geq \gamma\}, \quad F = \tau_\gamma(Q) = \{z : Q(z) \geq \gamma\}.$$

It is clear that $E, F \in \mathcal{T}', x \notin E, y \notin F$ and $E \cup F = X$. In order to prove that (X, \mathcal{T}) is a $T_{2\frac{1}{2}}$ space, we need only to verify $E^\circ \cup F^\circ = X$ by the definition of $T_{2\frac{1}{2}}$. For this purpose, we firstly prove that $\tau_\lambda(P^\circ) \subseteq [\tau_\gamma(P)]^\circ$ and $\tau_\lambda(Q^\circ) \subseteq [\tau_\gamma(Q)]^\circ$. In fact, taking $z \in \tau_\lambda(P^\circ)$, from Lemma 1.7, we have

$$\bigvee_{\alpha \in M(L)} \alpha \chi_{[\tau_\alpha(P)]^\circ}(z) = P^\circ(z) \geq \lambda.$$

Therefore, there exists $\alpha \in M(L)$ such that $z \in [\tau_\alpha(P)]^\circ$ and $\alpha \geq \gamma$. i.e., $z \in [\tau_\alpha(P)]^\circ \subseteq [\tau_\gamma(P)]^\circ$. Hence, we obtain that $\tau_\lambda(P^\circ) \subseteq [\tau_\gamma(P)]^\circ$ from the arbitrariness of z . Similarly, we can get $\tau_\lambda(Q^\circ) \subseteq [\tau_\gamma(Q)]^\circ$, as desired. Naturally, we have

$$E^\circ \cup F^\circ = [\tau_\gamma(P)]^\circ \cup [\tau_\gamma(Q)]^\circ \supseteq \tau_\lambda(P^\circ) \cup \tau_\lambda(Q^\circ) = \tau_\lambda(P^\circ \vee Q^\circ) = X,$$

i.e., (X, \mathcal{T}) is a $T_{2\frac{1}{2}}$ space.

Case $i = 3$: Since sub- T_1 separation axiom is an L -good extension, we prove this theorem only for the sub-regular case.

For any $x \in X$, suppose that $E \in \mathcal{T}'$ with $x \notin E$. Clearly, χ_E is a nonempty pseudo-crisp closed set in $(L^X, \omega_L(\mathcal{T}))$ and $x \notin \text{supp}(\chi_E)$. Since

$(L^X, \omega_L(\mathcal{T}))$ is a sub-regular space, there exists $\lambda \in M(L)$, and there are $P \in \eta^-(x_\lambda)$ and $Q \in \eta^-(\lambda_{\chi_E})$ such that $P \vee Q = \top_X$.

Let

$$U = \{z : P'(z) \not\leq \lambda'\}, \quad V = \{z : Q'(z) \not\leq \lambda'\}.$$

It is easy to verify that $x \in U$, $E \subseteq V$ and $U \cap V = \emptyset$ ($U, V \in \mathcal{T}$). Therefore, (X, \mathcal{T}) is a regular space.

Case $i = 4$: the proof is similar to that of the case $i = 3$. \square

Now we consider the heredity of the sub-separation. The following results show that sub- T_i ($i = 1, 2, 2\frac{1}{2}$) separation axioms are hereditary. Firstly, the concept of the extension is introduced. Let $Y \subseteq X, A \in L^Y$. $A^* \in L^X$ is defined as follows: $\forall x \in X$,

$$A^*(x) = \begin{cases} A(x), & x \in Y. \\ 0, & x \notin Y. \end{cases}$$

Then A^* is called the extension of A .

Theorem 3.2. Let (L^X, δ) be L -ts and Y be a nonempty crisp subset of X . If (L^X, δ) is a sub- T_i space, then the subspace $(L^Y, \delta|Y)$ is also a sub- T_i space, where $\delta|Y = \{G|Y : G \in \delta\}$, $i = 1, 2, 2\frac{1}{2}$.

Proof. We only prove the case $i=2$ and $i=2\frac{1}{2}$.

Case $i = 2$: Let $x, y \in Y$ with $x \neq y$. Since (L^X, δ) is a sub- T_2 space, there exists $\lambda \in M(L)$, and there are $P \in \eta^-(x_\lambda^*)$ and $Q \in \eta^-(y_\lambda^*)$ such that $P \vee Q = \top_X$, where x_λ^*, y_λ^* are the extensions of x_λ, y_λ , respectively. Notice that $P \in \eta^-(x_\lambda^*)$ implies that $P|Y \in \eta^-(x_\lambda)$. Similarly, $Q|Y \in \eta^-(y_\lambda)$. Therefore, there exists $\lambda \in M(L)$ and there are $P|Y \in \eta^-(x_\lambda)$ and $Q|Y \in \eta^-(y_\lambda)$ such that $(P|Y) \vee (Q|Y) = \top_Y$, i.e., $(L^Y, \delta|Y)$ is also a sub- T_2 space.

Case $i = 2\frac{1}{2}$: Suppose that $x, y \in Y$ with $x \neq y$. Since (L^X, δ) is a sub- $T_{2\frac{1}{2}}$ space, there exists $\lambda \in M(L)$, and there are $P \in \eta^-(x_\lambda^*)$ and $Q \in \eta^-(y_\lambda^*)$ such that $P^\circ \vee Q^\circ = \top_X$, where x_λ^*, y_λ^* is the extensions of x_λ, y_λ , respectively. Noticing that $P \in \eta^-(x_\lambda^*)$ implies that $P|Y \in \eta^-(x_\lambda)$ and $Q \in \eta^-(y_\lambda^*)$ implies that $Q|Y \in \eta^-(y_\lambda)$, then we have that there exists $\lambda \in M(L)$ and there are $P|Y \in \eta^-(x_\lambda)$ and $Q|Y \in \eta^-(y_\lambda)$. Hence we need only to show that $(P|Y)^\circ \vee (Q|Y)^\circ = \top_Y$. In fact, from $(P|Y)^\circ \geq (P^\circ|Y)$,

we get that

$$(P|Y)^\circ \vee (Q|Y)^\circ \geq (P^\circ|Y) \vee (Q^\circ|Y) = \top_Y.$$

Thus $(L^Y, \delta|Y)$ is also a sub- $T_2^{\frac{1}{2}}$ space. \square

Theorem 3.3. Let (L^X, δ) be L -ts, Y be a nonempty crisp subset of X and $\chi_Y \in \delta'$. If (L^X, δ) is a sub- T_i space, then the subspace $(L^Y, \delta|Y)$ is also a sub- T_i space, where $i = 3, 4$.

Proof. We only prove this theorem only for the case $i = 3$. Since sub- T_1 separation axiom is hereditary, we prove the theorem only for the sub-regular case.

Let B be a nonempty pseudo-crisp closed set in $(L^Y, \delta|Y)$ and $y \in Y$ with $y \notin \text{supp} B$. Since $B \in (\delta|Y)' = \delta'|Y$, there exists $A \in \delta'$ such that $B = A|Y$. And we have $B = B^*|Y$, where B^* is the extension of B . It is easy to prove that $B^* = A \wedge \chi_Y$ and B^* is a nonempty pseudo-crisp closed set. By the sub-regularity of (L^X, δ) and $y \notin \text{supp} B^*$, there exists $\lambda \in M(L)$, and there are $P \in \eta^-(y_\lambda^*)$ and $Q \in \eta^-(\lambda B^*)$ such that $P \vee Q = \top_X$, where y_λ^* is the extension of y_λ . Then we know that $P|Y \in \eta^-(y_\lambda)$ and $Q|Y \in \eta^-(\lambda B)$ such that $(P|Y) \vee (Q|Y) = \top_Y$. This shows $(L^Y, \delta|Y)$ is also a sub-regular space. Therefore, the proof of the theorem is complete. \square

In the end of this section, we show that sub- T_i ($i = 1, 2, 2^{\frac{1}{2}}$) separation axioms are productive. First, a lemma is needed.

Lemma 3.4. Let $(L^X, \delta), (L^Y, \mu)$ be two L -ts, $f^\rightarrow : (L^X, \delta) \rightarrow (L^Y, \mu)$ be a closed bijection and $f^\rightarrow, f^\leftarrow$ be continuous. If (L^X, δ) is a sub- T_i space, then so is (L^Y, μ) , where $i = 1, 2, 2^{\frac{1}{2}}, 3, 4$.

Proof. We prove the theorem only for the case $i = 1$ and $i = 3$.

Case $i = 1$: For any $y, z \in Y$ with $y \neq z$, since f is a closed bijection, there are $u, v \in X$ with $u \neq v$ such that $f(u) = y, f(v) = z$. Since (L^X, δ) is a sub- T_1 space, there exists $\lambda \in M(L)$, both there is $P \in \eta^-(u_\lambda)$ such that $v_\lambda \leq P$ and there is $Q \in \eta^-(v_\lambda)$ such that $u_\lambda \leq Q$. Therefore,

$$y_\lambda = f^\rightarrow(u_\lambda) \not\leq f^\rightarrow(P), \quad z_\lambda = f^\rightarrow(v_\lambda) \leq f^\rightarrow(P), \quad \text{and}$$

$$z_\lambda = f^\rightarrow(v_\lambda) \not\leq f^\rightarrow(Q), \quad y_\lambda = f^\rightarrow(u_\lambda) \leq f^\rightarrow(Q).$$

From $P, Q \in \delta'$ and f^\leftarrow is continuous, we have $f^\rightarrow(P) \in \eta^-(y_\lambda)$ and $f^\rightarrow(Q) \in \eta^-(z_\lambda)$. Hence, (L^Y, μ) is a sub- T_1 space.

Case $i=3$: For any $y \in Y$ and nonempty pseudo-crisp closed set $A \in \mu'$ with $y \notin \text{supp}A$. Since f is closed bijection, there exist $x \in X, B \in \delta'$ such that $f(x) = y, f(B) = A$ i.e., $x = f^{-1}(y), B = f^{\leftarrow}(A)$. We have $B \in \delta'$ from f^{\rightarrow} is continuous. It is easy to prove that B is a nonzero pseudo-crisp closed set. From $y \notin \text{supp}A$, we have:

$$\begin{aligned} y \notin \text{supp}A &\Rightarrow A(y) = \perp \Rightarrow A(f(x)) = \perp \quad (f(x) = y) \\ &\Rightarrow f^{\leftarrow}(A)(x) = \perp \quad (\text{by the definition of } f^{\leftarrow}) \\ &\Rightarrow x \notin \text{supp}f^{\leftarrow}(A) = \text{supp}B. \end{aligned}$$

For $x \in X$ and $B \in \delta'$ with $x \notin \text{supp}B$, since (L^X, δ) is a sub- T_3 space, there exists $\lambda \in M(L)$, and there are $P \in \eta^-(x_\lambda)$ and $Q \in \eta^-(\lambda B)$ such that $P \vee Q = \top_X$.

Therefore, $y_\lambda = f^{\rightarrow}(x_\lambda) \not\leq f^{\rightarrow}(P)$, $\lambda A = \lambda f^{\rightarrow}(B) \not\leq f^{\rightarrow}(Q)$, $f^{\rightarrow}(P) \vee f^{\rightarrow}(Q) = \top_Y$. From $P \in \delta'$ and f^{\leftarrow} is continuous, we have $f^{\rightarrow}(P) \in \eta^-(y_\lambda)$ and $f^{\rightarrow}(Q) \in \eta^-(\lambda A)$. Thus, (L^Y, μ) is a sub- T_3 space. \square

Theorem 3.5. Let $\{(L^{X_t}, \delta_t)\}_{t \in T}$ be a family of L -ts and (L^X, δ) be a product topological space. If for any $t \in T$, (L^{X_t}, δ_t) is a sub- T_i space, then so is (L^X, δ) . If (L^X, δ) is a sub- T_i space and (L^{X_t}, δ_t) is a fully stratified space, then so is (L^{X_t}, δ_t) , where $i=1, 2, 2\frac{1}{2}$.

Proof. We only prove the case $i=2$, other cases are obtained in the similar way.

Necessity. Suppose that $\{(L^{X_t}, \delta_t)\}_{t \in T}$ is a family of sub- T_2 space. Let $\forall x = \{x^t\}_{t \in T}, y = \{y^t\}_{t \in T} \in X$ with $x \neq y$, then there exists a $r \in T$ such that $x^r \neq y^r$. Since (L^{X_r}, δ_r) is a sub- T_2 space, there exists $\lambda \in M(L)$, and there are $B_r \in \eta^-(x_\lambda^r), C_r \in \eta^-(y_\lambda^r)$ such that $B_r \vee C_r = \top_{X_r}$. Clearly, $P_r^{\leftarrow}(B_r), P_r^{\leftarrow}(C_r) \in \delta'$, $P_r^{\leftarrow}(B_r)(x) = B_r(x^r) \not\leq \lambda$ and $P_r^{\leftarrow}(C_r)(y) = C_r(y^r) \not\leq \lambda$. Furthermore, $x_\lambda \not\leq P_r^{\leftarrow}(B_r), y_\lambda \not\leq P_r^{\leftarrow}(C_r)$ and $P_r^{\leftarrow}(B_r) \vee P_r^{\leftarrow}(C_r) = \top_X$. Hence, we prove that (L^X, δ) is a sub- T_2 space.

Sufficiency. Let (L^X, δ) be a sub- T_2 space and (L^{X_r}, δ_r) be a fully stratified space, where $r \in T$. For any $x = \{x^t\}_{t \in T} \in X$, from Theorem 2.8.9 (Wang [16]), $(L^{\tilde{X}_r}, \delta|_{\tilde{X}_r})$ which is parallel to (L^{X_r}, δ_r) through x is homeomorphic to (L^{X_r}, δ_r) . Since $(L^{\tilde{X}_r}, \delta|_{\tilde{X}_r})$ is a sub- T_2 space as a subspace of (L^X, δ) , (L^{X_r}, δ_r) is a sub- T_2 space from Lemma 3.4. \square

The next result follows from the above Theorem.

Corollary 3.6. Let $\{(L^{X_t}, \omega_L(\mathcal{T}_t))\}_{t \in T}$ be a family of L -ts topologically generated by a family of topological spaces $\{(X_t, \mathcal{T}_t)\}_{t \in T}$ and $(L^X, \omega_L(\mathcal{T}))$ be a product L -ts of $\{(L^{X_t}, \omega_L(\mathcal{T}_t))\}_{t \in T}$. Then $(L^X, \omega_L(\mathcal{T}))$ is a sub- T_i space iff for $\forall t \in T$, $(L^{X_t}, \omega_L(\mathcal{T}_t))$ is a sub- T_i space, where $\mathcal{T} = \prod_{t \in T} \mathcal{T}_t$, $i = 1, 2, 2\frac{1}{2}$.

4. The relations with respect to other separation axioms.

In this section, we make a comparison between separation axioms defined in this paper and those presented by Chen and Meng [2], Fang and Ren [3], Gu and Zhao [4], Ganguly and Saha [5], Kubiak [7], Kandil and El-Shafee [8], Shi [14], Shi and Chen [15] and Wang [16], and offer a lot of examples to show the relations between them. At first, we show that the sub-separation axioms defined in this paper are harmonious.

From Definition 2.1, the following theorem is obvious .

Theorem 4.1. Let (L^X, δ) be an L -ts. Then the following implications hold:

- (1) $\text{sub-}T_1 \Rightarrow \text{sub-}T_0$
- (2) $\text{sub-}T_4 \Rightarrow \text{sub-}T_3 \Rightarrow \text{sub-}T_2$

Theorem 4.2. Let (L^X, δ) be an L -ts. Then $\text{sub-}T_2 \Rightarrow \text{sub-}T_1$.

Proof. Suppose that (L^X, δ) is a sub- T_2 space. For any $x, y \in X$ with $x \neq y$, since (L^X, δ) is a sub- T_2 space, there exists $\lambda \in M(L)$, and there are $P \in \eta^-(x_\lambda)$ and $Q \in \eta^-(y_\lambda)$ such that $P \vee Q = \top_X$. From $y_\lambda \leq P \vee Q$ and $y_\lambda \not\leq Q$, we have $y_\lambda \leq P$. Similarly, on account of $x_\lambda \leq P \vee Q$ and $x_\lambda \not\leq P$, then $x_\lambda \leq Q$. Hence, for any $x \in X$ with $x \neq y$, there exists $\lambda \in M(L)$, both there is $P \in \eta^-(x_\lambda)$ such that $y_\lambda \leq P$ and there is $Q \in \eta^-(y_\lambda)$ such that $x_\lambda \leq Q$, i.e., (L^X, δ) is a sub- T_1 space. \square

From Theorem 4.1 and 4.2, we obtain the following result that shows the sub-separation axioms are harmonious.

Corollary 4.3. $\text{sub-}T_4 \Rightarrow \text{sub-}T_3 \Rightarrow \text{sub-}T_2 \Rightarrow \text{sub-}T_1 \Rightarrow \text{sub-}T_0$.

Theorem 4.4. Let (L^X, δ) be an L -ts. Then $\text{sub-}T_{2\frac{1}{2}} \Rightarrow \text{sub-}T_2$.

Proof. Let (L^X, δ) be a $\text{sub-}T_{2\frac{1}{2}}$ space. For any $x, y \in X$ with $x \neq y$, since (L^X, δ) is a $\text{sub-}T_{2\frac{1}{2}}$ space, there exists $\lambda \in M(L)$, and there are $P \in \eta^-(x_\lambda)$ and $Q \in \eta^-(y_\lambda)$ such that $P^\circ \vee Q^\circ = \top_X$. Noticing that $P^\circ \vee Q^\circ \leq P \vee Q$, we get that $P \vee Q = \top_X$. Therefore, (L^X, δ) is a $\text{sub-}T_2$ space. \square

The following example shows that the L -unit interval $I(L)$ need not satisfy the $\text{sub-}T_1$ axiom.

Example 4.5. The $[0,1]$ -real line $[0,1](I)$ does not satisfy $\text{sub-}T_1$ axiom. In fact, take $x, y \in [0,1](I)$ satisfying $\forall t \in R$

$$x(t) = \begin{cases} 1, & t \in (-\infty, 0), \\ 0.5, & t \in [0, 1], \\ 0, & t \in (1, +\infty), \end{cases} \quad y(t) = \begin{cases} 1, & t \in (-\infty, 0), \\ 0.6, & t \in [0, 1], \\ 0, & t \in (1, +\infty). \end{cases}$$

For convenience, we only consider P or Q which has the forms of $R'_s \vee L'_t$.
Case I: When $0 < \lambda \leq 0.4$, we have that

$$\begin{aligned} \varepsilon(x_\lambda) &= \vee\{t \mid x_\lambda \leq L'_t\} = \vee\{t \mid \lambda \leq x(t-)\} = 1; \\ \sigma(x_\lambda) &= \wedge\{s \mid x_\lambda \leq R'_s\} = \wedge\{s \mid \lambda \leq x(s+)\}' = 0. \end{aligned}$$

Hence,

$$x_\lambda \not\leq L'_t \vee R'_s \Rightarrow t > 1, s < 0.$$

Then we get that

$$(L'_t \vee R'_s)(y) = 0 \quad \text{from } t > 1, s < 0.$$

Naturally, we obtain that $y_\lambda \not\leq L'_t \vee R'_s$.

Case II: When $0.4 < \lambda \leq 0.5$, we have that

$$\varepsilon(x_\lambda) = \vee\{t \mid x_\lambda \leq L'_t\} = 1; \quad \sigma(x_\lambda) = \wedge\{s \mid x_\lambda \leq R'_s\} = 0.$$

Hence,

$$x_\lambda \not\leq L'_t \vee R'_s \Rightarrow t > 1, s < 0.$$

Then we get that

$$(L'_t \vee R'_s)(y) = 0 \text{ from } t > 1, s < 0 \Rightarrow y_\lambda \not\leq L'_t \vee R'_s.$$

Case III: When $0.5 < \lambda \leq 0.6$, we have that

$$\varepsilon(y_\lambda) = \vee\{t \mid x_\lambda \leq L'_t\} = 1; \quad \sigma(y_\lambda) = \wedge\{s \mid y_\lambda \leq R'_s\} = 1.$$

Hence,

$$y_\lambda \not\leq L'_t \vee R'_s \Rightarrow t > 1, s < 1.$$

Then we get that

$$(L'_t \vee R'_s)(x) \leq 0 \vee 0.5 = 0.5 \text{ from } t > 1, s < 1 \Rightarrow x_\lambda \not\leq L'_t \vee R'_s.$$

Case IV: When $0.6 < \lambda \leq 1$, we have that

$$\varepsilon(x_\lambda) = \vee\{t \mid x_\lambda \leq L'_t\} = 0; \quad \sigma(x_\lambda) = \wedge\{s \mid x_\lambda \leq R'_s\} = 1.$$

Hence,

$$x_\lambda \not\leq L'_t \vee R'_s \Rightarrow t > 0, s < 1.$$

Then we get that

$$(L'_t \vee R'_s)(y) \leq 0.5 \vee 0.4 = 0.5 \text{ from } t > 0, s < 1 \Rightarrow y_\lambda \not\leq L'_t \vee R'_s.$$

From case I, II, III and IV, we have that $[0, 1](I)$ does not satisfy the sub- T_1 axiom. \square

Remark 4.6. From the above example, we know that the L -unit interval need not satisfy the sub- T_1 axiom. So the L -unit interval is not compatible with the sub-separation axioms proposed in this paper.

Next we make a comparison between the sub-separation axioms and those presented by Kubiak [7].

Theorem 4.7. Let (L^X, δ) be an L -ts. Then sub- $T_1 \Rightarrow$ Kubiak- T_1 .

Proof. Let (L^X, δ) be sub- T_1 . In order to prove that (L^X, δ) is Kubiak- T_1 , take $x, y \in X$ with $x \neq y$. Then there exists $\lambda \in M(L)$, and there are $P \in \eta^-(x_\lambda)$ and $Q \in \eta^-(y_\lambda)$ such that $y_\lambda \leq P$, $x_\lambda \leq Q$, respectively. Taking $U = P'$, $V = Q'$, we have that $U, V \in \delta$, $U(x) \not\leq U(y)$ and

$V(y) \not\leq V(x)$. Therefore, (L^X, δ) is Kubiak- T_1 . \square

In general, Kubiak- T_1 need not imply our sub- T_1 . This can be seen from the following example.

Example 4.8. Let $L = \{\perp, a, b, \top\}$ satisfy $a \vee b = \top$, $a \wedge b = \perp$, $a' = b$ and $X = \{x, y\}$ with $x \neq y$. Take $\delta = \{\perp_X, \top_X, x_a, y_b, x_a \vee y_b\}$, then $\delta' = \{\perp_X, \top_X, M, N, R\}$, where M, N and R are defined as follows:

$$M(x) = b, M(y) = \top; N(x) = \top, N(y) = a; R(x) = b, R(y) = a.$$

We can prove that (L^X, δ) is not sub- T_1 , but it is Kubiak- T_1 . Now we show that (L^X, δ) is not sub- T_1 . We need to show that $\forall \lambda \in M(L)$, $\forall P \in \eta^-(x_\lambda)$ such that $y_\lambda \not\leq P$, or $\forall Q \in \eta^-(y_\lambda)$ such that $x_\lambda \not\leq Q$. In fact, we have that $P \in \eta^-(x_\lambda) = \{\perp_X, M, R\}$, $Q \in \eta^-(y_\lambda) = \{\perp_X\}$ when $\lambda = a$. Hence we get that $x_\lambda \not\leq Q$. Similarly, we have that $P \in \eta^-(x_\lambda) = \{\perp_X\}$, $Q \in \eta^-(y_\lambda) = \{\perp_X, N, R\}$ when $\lambda = b$. Then we obtain that $y_\lambda \not\leq P$. Therefore (L^X, δ) is not sub- T_1 . Next we show that (L^X, δ) is Kubiak- T_1 . Taking $U = x_a, V = y_b$, we get that $U, V \in \delta$, $U(x) \not\leq U(y)$ and $V(y) \not\leq V(x)$. \square

Theorem 4.9. Let (L^X, δ) be an L -ts and \top be a molecule. Then sub- $T_2 \Rightarrow$ Kubiak- T_2 .

Proof. For any $x, y \in X$ with $x \neq y$, since (L^X, δ) is a sub- T_2 space, there exists $\lambda \in M(L)$, there are $P \in \eta^-(x_\lambda)$ and $Q \in \eta^-(y_\lambda)$ such that $P \vee Q = \top_X$. Taking $U = P', V = Q'$, we have that $U, V \in \delta$, $U(x) \not\leq U(y)$ and $V(y) \not\leq V(x)$. Now we only need to prove that $U \leq V'$ i.e. $P' \leq Q$. In fact, since $P \vee Q = \top_X$ and \top is a molecule, we have that $P(x) = \top$ or $Q(x) = \top$ for $\forall x \in X$. Easily we get $P' \leq Q$ i.e. $U \leq V'$. Therefore (L^X, δ) is a Kubiak- T_2 space.

Remark 4.10. If \top is not a molecule, is Theorem 4.9 true? We leave it as an open problem. Generally, Kubiak- T_2 need not imply sub- T_2 (seeing Example 4.11 below). \square

Example 4.11. Let $L = [0, 1]$ and $X = \{x, y\}$. Take $\delta = \{\perp_X, \top_X, A, B, 0.3^*, 0.7^*\}$, where A, B are defined as follows:

$$A(x) = 0.7, A(y) = 0.3; B(x) = 0.3, B(y) = 0.7$$

Then we get that (L^X, δ) is a Kubiak- T_2 space. In fact, taking $U = A, V = B$, we get that $U(x) \not\leq U(y)$, $V(y) \not\leq V(x)$ and $U \leq V'$. It is easy to check that (L^X, δ) is not a sub- T_2 space. For $\forall \lambda \in M(L)$, we get that $\eta^-(x_\lambda), \eta^-(y_\lambda) \subseteq \{\perp_X, A, B, 0.3^*, 0.7^*\}$. Therefore for $\forall P \in \eta^-(x_\lambda), \forall Q \in \eta^-(y_\lambda)$, we have $P \vee Q \leq 0.7^*$. Hence (L^X, δ) is not a sub- T_2 space. \square

Now, we discuss the relation between the sub-separation axioms and other separation axioms presented by Shi[14], Wang[16], Gu and Zhao[4]. The following two examples show that sub- T_2 need not imply L - T_2 and L - T_2 also need not imply sub- T_2 .

Example 4.12. Let $L=[0,1]$ and $X = \{x, y\}$. Take $\delta = \{\perp_X, \top_X, C_1, C_2, C_1 \vee C_2\}$, where C_i is defined as follows:

$$C_1(x) = 0.5, C_1(y) = 0; C_2(x) = 0, C_2(y) = 0.5.$$

Easily we get that (L^X, δ) is a sub- T_2 space. In fact, taking $\lambda = \frac{2}{3}, P = C_1'$ and $Q = C_2'$, we get that $P \in \eta^-(x_\lambda), Q \in \eta^-(y_\lambda)$ and $P \vee Q = \top_X$. It is easy to check that (L^X, δ) is not L - T_2 . \square

Example 4.13. Let L, X and δ be defined as in Example 4.11. From Example 4.11, we know that (L^X, δ) is not a sub- T_2 space. Next we prove that (L^X, δ) is L - T_2 . Take $Q = A, P = B'$, then $Q \in \delta, P \in \delta', Q \leq P$ and $Q(x) \not\leq P(y)$. \square

Lemma 4.14 (Wang [16]). If (L^X, δ) is N-compact and T_2 , then it is T_4 . \square

Obviously, we have the following result.

Theorem 4.15. If (L^X, δ) is N-compact and T_2 , then it is sub- T_i , where $i = 1, 2, 2\frac{1}{2}, 3, 4$. \square

Lemma 4.16 (Gu and Zhao [4]). Let (L^X, δ) be an L -ts. Then,

- (1) (L^X, δ) is Layer T_0 iff for any $x_\lambda, y_\lambda \in M(L^X)$ with $x \neq y$, there exists $P \in \delta'$ such that $x_\lambda \not\leq P$ and $y_\lambda \leq P$ or $x_\lambda \leq P$ and $y_\lambda \not\leq P$.
- (2) (L^X, δ) is Layer T_1 iff for any $x_\lambda, y_\lambda \in M(L^X)$ with $x \neq y$, there exists $P \in \delta'$ such that $x_\lambda \not\leq P$ and $y_\lambda \leq P$.

(3) (L^X, δ) is Layer T_2 iff for any $x_\lambda, y_\lambda \in M(L^X)$ with $x \neq y$, there exist $P, Q \in \delta'$ such that $P \in \eta^-(x_\lambda)$, $Q \in \eta^-(y_\lambda)$ and $P \vee Q \geq [\lambda]$. \square

By Lemma 4.16, we have the following conclusion.

Theorem 4.17. Let (L^X, δ) be an L -ts. If (L^X, δ) is a layer T_i space, then it is a sub- T_i space, where $i = 0, 1$. \square

Lemma 4.18 (Gu and Zhao [4]). Let (L^X, δ) be an L -ts. If (L^X, δ) is a T_i space, then it is a layer T_i space, where $i = 0, 1, 2$. \square

Therefore, by Theorem 4.17, Lemmas 4.18, 4.16 (3), the following results hold.

Corollary 4.19. (1) $T_i \Rightarrow \text{layer } T_i \Rightarrow \text{sub-}T_i$, where $i = 0, 1$.

(2) layer $T_2 \Rightarrow \text{sub-}T_2$ whenever the largest element \top is a molecule. \square

In the following, we give an example showing that a sub- T_2 space need not be a layer T_2 space, to say nothing of being T_2 .

Example 4.20. Let L, X and δ be defined as in Example 4.12. From Example 4.12, we know that (L^X, δ) is a sub- T_2 space. But (L^X, δ) is not a layer T_2 space. In fact, taking $\alpha = \frac{1}{3}$, then we get that $(X, (\tau_\alpha(\delta'))') = \{\emptyset, X\}$. Clearly, $(X, (\tau_\alpha(\delta'))')$ is not a T_2 space. From Definition 1.5, we know that (L^X, δ) is not a layer T_2 space. It is easy to prove that (L^X, δ) is not a T_2 space. \square

From all of examples above, we find that, in general, all sub-separation axioms in this paper are weaker than other separation axioms that had appeared in literature. Indeed, there are many L -topological spaces which satisfy sub-separation axioms, but doesn't fulfill other separation axioms. This is one of differences between sub-separation axioms and other separation axioms. For examples, there are good work on separation axioms of L -topological spaces in [2], [3], [5] and [8]. In the following, we will offer more examples to show that our sub-separation axioms is very different from separation axioms established in these papers. For simplicity, we only consider $T_{2\frac{1}{2}}$ and T_2 separation axiom therein. Recall the definition of WT_2 in [3] as follows.

Definition 4.21. An L -ts (L^X, δ) is called a WT_2 -space if for any $x_\lambda, y_\mu \in M(L^X)$ with $x \neq y$, there are $P \in \eta^-(x_\lambda)$ and $Q \in \eta^-(y_\mu)$ such that $P \vee Q \geq (\lambda \vee \mu)^*$. \square

Remarks 4.22. (1) For an L -ts (L^X, δ) , it is easy to check its $T_{2\frac{1}{2}}$ [2] (seeing Definition 1.2 introduced in the paper) means sub- $T_{2\frac{1}{2}}$, but Example 4.23 below shows that the converse needn't be true.

(2) For an L -ts (L^X, δ) , it is easy to check its WT_2 implies sub- T_2 whenever the largest element \top is a molecule, and Example 4.23 below proves that the converse needn't be true. \square

Example 4.23. Let L be the completely distributive De Morgan algebra having four elements: \perp, a, b, \top satisfying $a \vee b = \top$, $a \wedge b = \perp$, $a' = b$, and $X = \{x, y\}$ ($x \neq y$). Take $\delta' = \{\perp_X, \top_X, x_a \vee y_b, x_b \vee y_a\}$. We will show the following conclusions.

(i) The L -ts (L^X, δ) is sub- T_2 . In fact, now $M(L) = \{a, b\}$. For $x, y \in X$ with $x \neq y$, there is $\lambda = a$, and also $P = x_b \vee y_a \in \eta^-(x_\lambda)$, $Q = x_a \vee y_b \in \eta^-(y_\lambda)$ such that $P \vee Q = \top_X$. Thus (L^X, δ) is sub- T_2 , as desired.

(ii) The L -ts (L^X, δ) is sub- T_0 and also sub- T_1 . These can be obtained by (i) and Theorems 4.1(1), 4.2.

(iii) The L -ts (L^X, δ) isn't WT_2 in the sense of [3]. Letting $x_a, y_b \in M(L^X)$, it is easy to check $\eta^-(x_a) = \{\perp_X, x_b \vee y_a\}$ and $\eta^-(y_b) = \{\perp_X, x_b \vee y_a\}$. Hence for any $P \in \eta^-(x_a)$ and $Q \in \eta^-(y_b)$, it cannot be true that

$$P \vee Q \geq (a \vee b)^* = \top_X,$$

where $(a \vee b)^*$ is the constant L -set with its value $(a \vee b)$ in the sense of [3]. In fact, for any $P \in \eta^-(x_a)$ and $Q \in \eta^-(y_b)$, $P \vee Q \leq x_b \vee y_a$, but $x_b \vee y_a \leq \top_X$ and $x_b \vee y_a \neq \top_X$. By Definition 4.21 of WT_2 , the L -ts (L^X, δ) isn't WT_2 .

(iv) The L -ts (L^X, δ) is sub- $T_{2\frac{1}{2}}$. Let $P = x_b \vee y_a$ and $Q = x_a \vee y_b$. It is observed that both P and Q are open since $P' = Q$ and $Q' = P$. For the unique pair of x and y with $x \neq y$, putting $\lambda = a \in M(L)$, there are $P \in \eta^-(x_\lambda)$ and $Q \in \eta^-(y_\lambda)$ such that

$$P^\circ \vee Q^\circ = P \vee Q = \top_X.$$

Thus, (L^X, δ) is sub- $T_{2\frac{1}{2}}$ by Definition 2.1 (3).

(v) The (L^X, δ) isn't $T_{2\frac{1}{2}}$ in the sense of [2] (see Definition 1.2 also).

Indeed, $\eta^-(x_a) = \{x_b \vee y_a\}$ and $\eta^-(y_b) = \{x_b \vee y_a\}$. We find that there are two points $x_a, y_b \in M(L^X)$ with $x \neq y$ such that for any $P \in \eta^-(x_a) = \{x_b \vee y_a\}$ and $Q \in \eta^-(y_b) = \{x_b \vee y_a\}$ (it must be $P = x_b \vee y_a$ and $Q = x_b \vee y_a$)

$$P^\circ \vee Q^\circ = P \vee Q = x_b \vee y_a \neq \top_X.$$

It implies that (L^X, δ) isn't $T_{2\frac{1}{2}}$ by Definition 1.2, as desired. \square

In [15], Shi and Chen redefined Urysohn in L -topology, where it was called Shi-Urysohn. Now we consider the relation between the Shi-Urysohn axiom and our sub- $T_{2\frac{1}{2}}$ axiom. Recall the definition of Shi-Urysohn axiom in [15] as follows.

Definition 4.24. (Shi and Chen [15]).

An L -ts (L^X, δ) is said to be Urysohn if any $x_\lambda, y_\mu \in M(L^X)$ with $x_\lambda \not\leq y_\mu$, there exist $P \in \eta^-(x_\lambda)$ and $Q \in \aleph^\circ(y_\mu)$ such that $P^\circ \geq Q^-$, where $\aleph^\circ(y_\mu) = \{V : y_\mu \leq V, V \in \delta\}$. \square

The following example shows that our sub- $T_{2\frac{1}{2}}$ need not imply Shi-Urysohn.

Example 4.25. Let $X = L = [0, 1]$, and $\delta = \{\chi_E : E \subset X\}$, where χ_E is the characteristic function of E . Then δ is a $[0, 1]$ -topology on X . It is easy to check that (L^X, δ) is $T_{2\frac{1}{2}}$ (Urysohn), then it is sub- $T_{2\frac{1}{2}}$. But it is not Shi-Urysohn. In fact, for any $x \in X$ and any $P \in \eta^-(x_\top)$, it follows that $P^\circ(x) = \perp$. But there is no $Q \in \aleph^\circ(x_{0.5})$ such that $P^\circ \geq Q^-$. \square

Remark 4.26. Does the Shi-Urysohn axiom imply our sub- $T_{2\frac{1}{2}}$? We can't solve it. So we leave it as an open problem. \square

To discuss the relation between sub- T_2 separation axiom and other T_2 separation axiom introduced in [5] and [8]. Note that we consider the case of $L = I = [0, 1]$, the unit interval, so that the conclusions is available for the membership valued lattice using in the published papers [6] and [9]. We introduce some definitions for the convenience of readers.

Definition 4.27. (Liu and Luo[11]). Let $x_\lambda \in M(I^X)$ and $A, B \in I^X$. We say x_λ quasi-coincides with A , or say x_λ quasi-coincident with A , denoted

by $x_\lambda q A$, if $A(x) + \lambda > \top$; say A quasi-coincides with B at $x \in X$, or say A is quasi-coincident with B at x , AqB at x for short, if $A(x) + B(x) > \top$; say A quasi-coincides with B , or say A is quasi-coincident with B , denoted by AqB , if A quasi-coincides with B at some point $x \in X$. Relation “does not quasi-coincides with” or “is not quasi-coincident with” is denoted by \bar{q} . \square

Definitio 4.28. (Liu [9]). Let (I^X, δ) be I -ts and $x_\lambda \in M(I^X)$. A fuzzy set U is called a quasi-coincident neighborhood (q-nbd, for short) of x_λ if there exists $V \in \delta$ such that $x_\lambda q V$ and $V \leq U$. \square

Definition 4.29. (C.K. Wong [16]). Let (I^X, δ) be an I -ts, $A \in I^X$ and $x_\lambda \in M(I^X)$.

A is said to be a neighborhood (nbd, in short) in (X, δ) iff there is a $B \in \delta$ such that $x_\lambda \leq B \leq A$. Therefore, an open set $U \in \delta$ is the nbd of each of its points. \square

Definition 4.30. (S. Ganguly and S. Saha [5]). An I -ts (I^X, δ) is $GS-T_2$ (Originally, T_2) iff for any two distinct points x_λ and y_μ :

Case *I*. When $x \neq y$, x_λ and y_μ have open nbds which are not quasi-coincident.

Case *II*. When $x = y$ and $\lambda < \mu$, then y_μ has an open q -nbd V and x_λ has an open nbd U such that $V \bar{q} U$. \square

Definition 4.31. (A. Kandil and M.E. El-Shafee [8]). An I -ts (I^X, δ) is FT_2 if $\forall x_\lambda, y_\mu \in M(I^X)$ with $x_\lambda \bar{q} y_\mu$, there exist $Q_{x_\lambda} \in \delta$ and $Q_{y_\mu} \in \delta$ such that

$$\lambda \leq Q_{x_\lambda}(x), \mu \leq Q_{y_\mu}(y) \text{ and } Q_{x_\lambda} \bar{q} Q_{y_\mu}. \square$$

Remark 4.32. The following Example 4.33 shows that our sub- T_2 needn't be $GS-T_2$ in the sense of Definition 4.30. \square

Example 4.33. Let (I^X, δ) be the I -ts defined in Example 4.12. We have showed that (I^X, δ) is a sub- T_2 space in Example 4.12. Now we assert that (I^X, δ) isn't $GS-T_2$ in the sense of Definition 4.30. Taking $\lambda = \frac{1}{8}$ and $\mu = \frac{1}{4}$, then x_λ and x_μ are different points with $\lambda < \mu$. Moreover, \top_X is the unique open Q -neighborhood of x_μ and the set of open neighborhood of x_λ is $A = \{C_1, C_1 \vee C_2, \top_X\}$. Obviously, for each $V \in A$, we cannot have $V \bar{q} \top_X$. Thus, (I^X, δ) isn't $GS-T_2$ in the sense of Definition 4.30. \square

Remark 4.34. The following Example 4.35 shows that our sub- T_2 needn't be FT_2 in the sense of Definition 4.31. \square

Example 4.35. Let (I^X, δ) be the I -ts defined in Example 4.12. We have showed that (I^X, δ) is a sub- T_2 space in Example 4.12. Now we assert that (I^X, δ) isn't FT_2 . For the $x, y \in X$ with $x \neq y$, taking $\lambda = \frac{1}{3}$, then $x_\lambda \bar{q} y_\lambda$. The unique neighborhood of x_λ and y_λ is \top_X , moreover $\top_X \bar{q} \top_X$ never is true. Hence (I^X, δ) isn't FT_2 . \square

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