Some new generalized I-convergent difference sequence spaces defined by a sequence of modulii

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Abstract

In this article we introduce the sequence space $c_0^I(F, p, \Delta^n_v)$ and $\ell^I_{\infty}(F, p, \Delta^n_v)$ for the of sequence of modulii $F = (f_k)$ and given some inclusion relations. These results here proved are analogus to those by M.Aiyub [1](Global Journal of Science Frontier Research Mathematics and Decision Sciences 12(9)(2012),32-36).

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1. Introduction and Preliminaries

Let $\omega, \ell_{\infty}, c_0$ be the set of all sequences of complex numbers, the linear spaces of bounded, convergent and null sequences $x = (x_k)$ with complex terms, respectively, normed by

$$\|x\|_{\infty} = \sup_k |x_k|, \text{ where } k \in \mathbb{N} = \{1, 2, 3, \ldots\}.$$ 

The idea of difference sequence spaces was introduced by H. Kizmaz [17]. In 1981, Kizmaz defined the sequence spaces as follow:

$$\ell_{\infty}(\Delta) = \{x = (x_k) \in \omega : (\Delta x_k) \in \ell_{\infty}\},$$

$$c(\Delta) = \{x = (x_k) \in \omega : (\Delta x_k) \in c\},$$

$$c_0(\Delta) = \{x = (x_k) \in \omega : (\Delta x_k) \in c_0\},$$

where

$$\Delta x = (x_k - x_{k+1}) \text{ and } \Delta^0 x = (x_k),$$

These are Banach space with the norm

$$\|x\|_{\Delta} = |x_1| + \|\Delta x\|_{\infty}.$$ 

Later Çolak and Et [4] defined the sequence spaces:

$$\ell_{\infty}(\Delta^n) = \{x = (x_k) \in \omega : (\Delta^n x_k) \in \ell_{\infty}\},$$

$$c(\Delta^n) = \{x = (x_k) \in \omega : (\Delta^n x_k) \in c\},$$

$$c_0(\Delta^n) = \{x = (x_k) \in \omega : (\Delta^n x_k) \in c_0\},$$

where $n \in \mathbb{N}, \Delta^0 x = (x_k), \Delta x = (x_k - x_{k+1}), \Delta^n x = (\Delta^n x_k) = (\Delta^{n-1} x_k - \Delta^{n-1} x_{k+1})$.
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and this generalized difference notion has the following binomial representation.

\[ \Delta^n x_k = \sum_{v=0}^{n} (-1)^{v} (nv) x_{k+v} \]

and showed that these spaces are Banach space with the norm

\[ \|x\|_\Delta = \sum_{i=1}^{n} |x_i| + \|\Delta^n x\|_\infty \]

Esi and Isik [7] defined sequence spaces:

\[ \ell_\infty(\Delta^n_v, s, p) = \{ x = (x_k) \in \omega : \sup_k k^{-s} |\Delta^n_v x_k|^{p_k} < \infty, s \geq 0 \}, \]

\[ c(\Delta^n_v, s, p) = \{ x = (x_k) \in \omega : k^{-s} |\Delta^n_v x_k - L|^{p_k} \rightarrow 0 (k \rightarrow \infty), s \geq 0, \text{ for some } L \}, \]

\[ c_0(\Delta^n_v, s, p) = \{ x = (x_k) \in \omega : k^{-s} |\Delta^n_v x_k|^{p_k} \rightarrow 0 (k \rightarrow \infty), s \geq 0 \}. \]

Where \( v = (v_k) \) is any fixed sequence of non zero complex numbers \( n \in \mathbb{N} \) is fixed number.

\[ \Delta^n_v x_k = (v_k x_k), \Delta^n_v x_k = (v_k x_k - v_{k+1} x_{k+1}) \text{ and } \Delta^n_v x_k = (\Delta^{n-1}_v x_k - \Delta^{n-1}_v x_{k+1}). \]

And this generalized difference notion has the following binomial representation.

\[ \Delta^n_v x_k = \sum_{i=0}^{n} (-1)^{i} (n i) v_{k+i} x_{k+i}, \]

when \( s = 0, m = 1, v = (1, 1, 1, \ldots) \) and \( p_k = 1 \) for all \( k \in \mathbb{N} \), they are just \( \ell_\infty(\Delta), c(\Delta), c_0(\Delta) \), defined by Kizmaz [17]. When \( s = 0 \) and \( p_k = 1 \) for all \( k \in \mathbb{N} \), they are the following sequence spaces defined by Et and Esi [9]

\[ \ell_\infty(\Delta^n_v) = \{ x = (x_k) \in \omega : (\Delta^n_v x_k) \in \ell_\infty \}, \]

\[ c(\Delta^n_v) = \{ x = (x_k) \in \omega : (\Delta^n_v x_k) \in c \}, \]

\[ c_0(\Delta^n_v) = \{ x = (x_k) \in \omega : (\Delta^n_v x_k) \in c_0 \}. \]

For more development about difference sequence spaces we refer to Bektaş and Çolak [2], M.Et[8] and V.A.khan [14-16]
The idea of modulus was defined by Nakano [22] in 1953. A function $f : [0, \infty) \to [0, \infty)$ is called a modulus if

(i) $f(t) = 0$ if and only if $t = 0$,

(ii) $f(t + u) \leq f(t) + f(u)$, for all $t, u \geq 0$,

(iii) $f$ is increasing and

(iv) $f$ is continuous from the right at 0.

Let $X$ be a sequence spaces. Then Ruckle [25-27] defined the sequence space $X(f)$ for a modulus $f$ as

$$X(f) = \{ x = (x_k) \in \omega : (f(|x_k|)) \in X \},$$

Later Kolk [18,19] gave an extension of $X(f)$ by considering a sequence of moduli $F = (f_k)$, that is

$$X(F) = \{ x = (x_k) \in \omega : (f_k(|x_k|)) \in X \}.$$

Gaur and Mursaleen[13] defined the following sequence spaces:

$$\ell_\infty(F, \Delta) = \{ x = (x_k) \in \omega : (\Delta x_k) \in \ell_\infty(F) \},$$

$$c_0(F, \Delta) = \{ x = (x_k) \in \omega : (\Delta x_k) \in c_0(F) \}. $$

After then Ç.Bektas and R.Çolak [2] defined the following sequence spaces:

$$\ell_\infty(F, \Delta^n) = \{ x = (x_k) \in \omega : (\Delta^n x_k) \in \ell_\infty(F) \},$$

$$c_0(F, \Delta^n) = \{ x = (x_k) \in \omega : (\Delta^n x_k) \in c_0(F) \}. $$

Recently Vakeel A Khan [14] defined the following sequence spaces:

$$X(F,p) = \{ x = (x_k) \in \omega : (f(|x_k|)) \in X(p) \},$$

$$\ell_\infty(F,p) = \{ x = (x_k) \in \omega : \sup_{k} f_k(|x_k|^{p_k}) < \infty \},$$

$$c_0(F,p) = \{ x = (x_k) \in \omega : f_k(|x_k|^{p_k}) \to 0 (k \to \infty) \},$$

$$\ell_\infty(F,p, \Delta^n) = \{ x = (x_k) \in \omega : \Delta^n x \in \ell_\infty(F,p) \},$$
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$$c_0(F, p, \Delta^n) = \{x = (x_k) \in \omega : \Delta^n x \in c_0(F, p)\}.$$

For a sequence of moduli $F = (f_k)$ and gave the necessary and sufficient conditions for the inclusion relations between $X(\Delta^n)$ and $Y(F, \Delta^n)$, where $X, Y = \ell_\infty$ or $c_0$. Sequence of moduli have been studied by Ç.A.Bektaş and R. Çolak[2] and many other authors.

The notion of statistical convergence was introduced by H.Fast[10]. Later on it was studied by J.A.Fridy [11,12] from the sequence space point view and linked with the summability theory.

The notion of $I$-convergence is a generalization of the statistical convergence. It was studied at initial stage by Kostyrko, Šalat and Wileczynski [20]. Later on it was studied by Šalat [29],Šalat, Tripathy and Ziman [30], Demirci[5]

Let $N$ be a non empty set. Then a family of sets $I \subseteq 2^N$ (power set of $N$) is said to be an ideal if $I$ is additive i.e $(A, B) \in I \Rightarrow (A \cup B) \in I$ and i.e $A \in I, B \subseteq A \Rightarrow B \in I$. A non empty family of sets $\mathcal{L}(I) \subseteq 2^N$ is said to be filter on $N$ if and only if $\Phi \notin \mathcal{L}(I)$ for $A, B \in \mathcal{L}(I)$ we have $(A \cap B) \in \mathcal{L}(I)$ and for each $A \in \mathcal{L}(I)$ and $A \subseteq B$ implies $B \in \mathcal{L}(I)$.

An ideal $I \subseteq 2^N$ is called non trivial if $I \neq 2^N$. A non trivial ideal $I \subseteq 2^N$ is called admissible if $\{(x) : x \in N\} \subseteq I$. A non trivial ideal is maximal if there cannot exist any non-trivial ideal $J \neq I$ containing $I$ as a subset. For each ideal $I$, there exist a filter $\mathcal{L}(I)$ corresponding to $I$, i.e $\mathcal{L}(I) = \{K \subseteq N : K^c \in I\}$,where $K^c = N - K$.

**Definition 1.1.** A sequence $(x_k) \in \omega$ is said to be $I$-convergent to a number $L$ if for every $\epsilon > 0$, \{k \in N : |x_k - L| \geq \epsilon\} \in I$.In this case we write $I - \lim x_k = L$.

**Definition 1.2.** A sequence $(x_k) \in \omega$ is said to be $I$-null if $L=0$. In this case we write $I - \lim x_k = 0$.

**Definition 1.3.** A sequence $(x_k) \in \omega$ is said to be $I$-cauchy if for every $\epsilon > 0$, there exist a number $m = m(\epsilon)$ such that \{k \in N : |x_k - x_m| \geq \epsilon\} \in I$.

**Definition 1.4.** A sequence $(x_k) \in \omega$ is said to be $I$-bounded if there exist $M > 0$ such that \{K \in N : |x_k| \geq M\} \in I$. 
We need the following Lemmas.

**Lemma 1.5.** The condition $\sup_k f_k(t) < \infty, t > 0$ hold if and only if there is a point $t_0 > 0$ such that $\sup_k f_k(t_0) < \infty$ (see [2,13]).

**Lemma 1.6.** The condition $\inf_k f_k(t) > 0$ hold if and only if there exist is a point $t_0 > 0$ such that $\inf_k f_k(t_0) > 0$ (see [2,13]).

**Lemma 1.7.** Let $K \in \mathcal{L}(I)$ and $M \subseteq N$. If $M \neq I$ then $M \cap K \neq I$ (see [29]).

**Lemma 1.8.** If $I \subseteq 2^N$ and $M \subseteq N$. If $M \neq I$ then $M \cap K \neq I$ (see [20]).

2. Main results

Let $F = (f_k)$ be a sequence of moduli, $v = (v_k)$ be any sequence such that $v_k \neq 0$ for all $k$ and $p = (p_k)$ be sequence space of strictly positive real numbers then we define the following sequence spaces.

$$c_0^I(F, p, \Delta^T_v) = \{(x_k) \in \omega : I - \lim_{k} f_k(|\Delta^T_v x_k|) = 0\},$$

$$\ell_\infty^I(F, p, \Delta^T_v) = \{(x_k) \in \omega : I - \sup_k f_k(|\Delta^T_v x_k|) < \infty\}.$$

**Theorem 2.1.** For a sequence $F = (f_k)$ of moduli and for all $v = (v_k)$ and $p = (p_k)$ the following statements are equivalent:

(a) $\ell_\infty^I(\Delta^T_v) \subseteq \ell_\infty^I(F, p, \Delta^T_v)$,

(b) $c_0^I(\Delta^T_v) \subseteq c_0^I(F, p, \Delta^T_v)$,

(c) $\sup_k f_k(t) < \infty, t > 0$.

**Proof.** (a) implies (b) is obvious.
(b) implies (c). Let $c_0^I(\Delta^n_v) \subseteq c_0^I(F, p, \Delta^n_v)$. Suppose that (c) is not true. Then by Lemma (1.5)

$$\sup_k f_k(t) = \infty, \text{ for all } t > 0,$$

and therefore there is a sequence $(k_i)$ of positive integers such that

$$f_{k_i}\left(\frac{1}{i}\right) > i, \text{ for each } i = 1, 2, 3, \ldots$$

(1)

Define $x = (x_k)$ as follow

$$x_k = \begin{cases} \frac{1}{i}, & \text{if } k = k_i, i = 1, 2, 3, \ldots; \\ 0, & \text{otherwise.} \end{cases}$$

Then $x \in c_0^I(\Delta^n_v)$ but by (1), $x \notin \ell^I_\infty(F, p, \Delta^n_v)$ for $v_k = p_k$ and $k \in \mathbb{N}$ which contradicts (b). Hence (c) must hold.

(c) implies (a). Let (c) be satisfied and $x \in c_0^I(\Delta^n_v)$. If we suppose that $x \notin \ell^I_\infty(F, p, \Delta^n_v)$. Then

$$\sup_k f_k(|\Delta^n_v x_k|^{p_k}) = \infty \text{ for } \Delta^n_v x \in \ell^I_\infty.$$

If we take $t = |\Delta^n_v x_k|^{p_k}$. Then $\sup_k f_k(t) = \infty$ which contradicts (c). Hence $\ell^I_\infty(\Delta^n_v) \subseteq \ell^I_\infty(F, p, \Delta^n_v)$.

**Theorem 2.2.** For a sequence $F = (f_k)$ is a sequence of moduli and for all $v = (v_k)$ and $p = (p_k)$ the following statements are equivalent:

(a) $c_0^I(F, p, \Delta^n_v) \subseteq c_0^I(\Delta^n_v)$,

(b) $c_0^I(F, p, \Delta^n_v) \subseteq \ell^I_\infty(\Delta^n_v)$,
(c) \( \inf_k f_k(t) > 0, (t > 0) \).

**Proof.** (a) implies (b) is obvious.

(b) implies (c). Let \( c_0^I(F, p, \Delta^n_v) \subseteq \ell^I_\infty(\Delta^n_v) \). Suppose that (c) is not true. Then by Lemma (1.6)

\[
\inf_k f_k(t) = 0, \quad (t > 0)
\]

and therefore there is a sequence \((k_i)\) of positive integers such that

\[
f_{k_i}(i^2) < \frac{1}{i} \quad \text{for each} \quad i = 1, 2, 3, \ldots.
\]

Define \( x = (x_k) \) as follow

\[
x_k = \begin{cases} 
i^2, & \text{if} \quad k = k_i \quad i = 1, 2, 3, \ldots; \\
0 & \text{otherwise}.
\end{cases}
\]

By (2) \( x \in c_0^I(F, p, \Delta^n_v) \) but \( x \notin \ell^I_\infty(\Delta^n_v) \) for \( v_k = (p_k) \) and \( k \in \mathbb{N} \) which contradicts (b). Hence (c) must hold.

(c) implies (a). Let (c) be satisfied and \( x \in c_0^I(F, p, \Delta^n_v) \) that is

\[
I - \lim_k f_k(|\Delta^n_v x_k|) = 0.
\]

Suppose that \( x \notin c_0^I(\Delta^n_v) \). Then for some number \( \epsilon_0 > 0 \) and positive integer \( k_0 \) we have \( |\Delta^n_v x_k| < \epsilon_0 \) for \( k \geq k_0 \). Therefore \( f_k(\epsilon_0) \geq f_k(|\Delta^n_v x_k|) \) for \( k \geq k_0 \) and hence \( \lim_k f_k(\epsilon_0) = 0 \), which contradicts our assumption that \( x \notin c_0^I(\Delta^n_v) \).

Thus \( c_0^I(F, p, \Delta^n_v) \subseteq c_0^I(\Delta^n_v) \).

**Theorem 2.3.** The inclusion \( \ell^I_\infty(F, p, \Delta^n_v) \subseteq c_0^I(\Delta^n_v) \) holds if and only if
\[ \lim_{k} f_k(t) = \infty, \quad \text{for} \quad t > 0. \quad (3) \]

**Proof.** Let \( \ell_\infty^I(F, p, \Delta^n_v) \subseteq c_0^I(\Delta^n_v) \) such that \( \lim_k f_k(t) = \infty \) for \( t > 0 \) doesn’t hold. Then there is a number \( t_0 > 0 \) and a sequence \((k_i)\) of positive integer such that

\[ f_{k_i}(t_0) \leq M < \infty. \quad (4) \]

Define the sequence \( x = (x_k) \) by

\[
x_k = \begin{cases} 
  t_0, & \text{if} \quad k = k_i \quad i = 1, 2, 3, \ldots; \\
  0, & \text{otherwise}.
\end{cases}
\]

Thus \( x \in \ell_\infty^I(F, p, \Delta^n_v) \) by (4). But \( x \notin c_0^I(\Delta^n_v) \), for \( v_k = p_k \) and \( k \in \mathbb{N} \) so that (3) must hold. If \( \ell_\infty^I(F, p, \Delta^n_v) \subseteq c_0^I(\Delta^n_v) \). Conversely, let (3) hold. If \( x \in \ell_\infty^I(F, p, \Delta^n_v) \), then

\[ \lim_k f_k(|\Delta^n_v x_k|p_k) \leq \lim_k M < \infty, \quad \text{for} \quad k = 1, 2, 3, \ldots \]

Suppose that \( x \notin c_0^I(\Delta^n_v) \). Then for some number \( \epsilon_0 > 0 \) and positive integer \( k_0 \) we have \( |\Delta^n_v x_k| < \epsilon_0 \) for \( k \geq k_0 \). Therefore \( f_k(\epsilon_0) \leq f_k(|\Delta^n_v x_k|p_k) \leq M \) for \( k \geq k_0 \), which contradicts (3). Hence \( x \in c_0^I(\Delta^n_v) \).

**Theorem 2.4.** The inclusion \( \ell_\infty^I(\Delta^n_v) \subseteq c_0^I(F, p, \Delta^n_v) \) holds if and only if

\[ \lim_{k} f_k(t) = 0, \quad \text{for} \quad t > 0. \quad (5) \]

**Proof.** Suppose that \( \ell_\infty^I(\Delta^n_v) \subseteq c_0^I(F, p, \Delta^n_v) \) but (5) doesn’t hold,

Then
\[ \lim_{k} f_k(t_0) = l \neq 0, \text{ for some } t_0 > 0 \quad (6). \]

Define the sequence \( x = (x_k) \) by
\[
x_k = t_0 \sum_{v=0}^{k-n} (-1)^n (n + k - v - 1k - v)
\]
for \( k = 1, 2, 3, \ldots \). Then \( x \in c_0^I(F, p, \Delta^n_v) \) by (6) for \( v_k = p_k \) and \( k \in \mathbb{N} \). Hence (5) must hold.

Conversely, let \( x \in \ell^I_\infty(\Delta^n_v) \) and suppose that (5) holds. Then \( |\Delta^n_v x_k| \leq M < \infty \) for \( k = 1, 2, 3, \ldots \). There for \( f_k(|\Delta^n_v x_k|) \leq f_k(M) \) for \( k = 1, 2, 3, \ldots \) and \( \lim_{k} f_k(|\Delta^n_v x_k|) \leq \lim_{k} f_k(M) = 0 \) by (5). Hence \( x \in c_0^I(F, p, \Delta^n_v) \).

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